Preliminary evidence for a stable 2-sphere in the Yang-Mills flow for $SU(3)$ gauge fields on S^4

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The Yang-Mills flow

G a compact Lie group, M a riemannian 4-manifold

B a principle G-bundle over M

A a connection, D the covariant derivative, F its curvature 2-form

The Yang-Mills action

$$S_{YM}(A) = \int_M \mathrm{dvol}(x)\, ||F(x)||^2 = \int_M *F \wedge F$$

Its gradient flow

$$\frac{dA}{dt} = *D*F(A) \qquad \frac{d}{dt} A_\mu^a(x) = \partial^\nu \partial_\nu A_\mu^a(x) + \cdots$$

The Y-M flow on the space of 2-spheres of connections

$\tilde{\mathcal{A}}$ = the space of connections in the G-bundle B

$\mathcal{G}$ = the group of automorphisms of B (gauge transformations)

$\mathcal{A} = \tilde{\mathcal{A}}/\mathcal{G}$ (the gauge equivalence classes of connections)

The Y-M flow is $\mathcal{G}$-invariant, so it acts:

- on $\mathcal{A}$,
- pointwise on $Maps(S^2 \to \mathcal{A})$,
- on $\mathcal{S} = Maps(S^2 \to \mathcal{A})/Diff_0(S^2)$

What is the generic long-time behavior of the Y-M flow on $\mathcal{S}$?

Are there stable 2-spheres for all elements in $\pi_0\mathcal{S} = \pi_2\mathcal{A}$?

Personal motivation: a very hypothetical application in a very speculative physics theory [DF: JHEP 0310:063,2003].

Here, I will describe some very preliminary, elementary investigations still in progress. Any advice or help would be much appreciated.

A more specific setting

M = round S^4 (or euclidean $\mathbb{R}^4 \cup \infty$)

B = the trivial G-bundle over S^4

then the gauge transformations are $\mathcal{G} = Maps(S^4 \to G)$

$\pi_2 \mathcal{A} = \pi_5 G$

$\tilde{\mathcal{A}}$ is an affine space, so contractible,
so the long exact sequence for $\tilde{\mathcal{A}} \to \tilde{\mathcal{A}}/\mathcal{G} = \mathcal{A}$ gives $\pi_2 \mathcal{A} = \pi_1 \mathcal{G}$
and $\pi_1 Maps(S^4 \to G) = \pi_5 G$, so $\pi_2 \mathcal{A} = \pi_5 G$.

$\pi_5 SU(2) = \mathbb{Z}_2 \qquad \pi_5 SU(3) = \mathbb{Z}$

- $\pi_5 SU(3) = \mathbb{Z}$ classifies the $SU(3)$ bundles over S^6.
- $S^6 = G_2/SU(3)$ is a geometric model for the generator.
- Is there a geometric model for the generator of $\pi_5 SU(2)$?

Non-trivial 2-spheres of riemannian metrics?

$$\pi_1 Diff(S^4) = \pi_5 S^4 = \mathbb{Z}_2$$

but

$$O(4) \subset Diff(S^4)$$

$$\pi_1 O(4) = \pi_1 Diff(S^4)$$

so

- no non-trivial 2-spheres in the metrics on S^4,
- nor in the asymptotically euclidean metrics on $\mathbb{R}^4$

because any 2-sphere can be contracted to the equivalence class of the round/euclidean metric?

Still?

$G_2/SU(3) = S^6$

Defining representation on $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$

- $g_2 = su(3) \oplus \mathbb{C}^3$
- $\mathbf{v} : (z_0, \mathbf{z}) \mapsto (-\bar{\mathbf{v}} \cdot \mathbf{z} - \mathbf{v} \cdot \bar{\mathbf{z}},\ 2\mathbf{v}z_0 + \bar{\mathbf{v}} \times \bar{\mathbf{z}})$

$S^6 = \{(z_0, z_1, z_2, z_3),\ z_0^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$

- $SU(3)$ is the isotropy group of the N-pole $(1, 0, 0, 0)$

$\exp\left(\frac{\pi}{2} D^6\right)$ (N-pole) $= S^6 - \{\text{S-pole}\}$

- trivializes the $SU(3)$ bundle over $S^6 - \{\text{S-pole}\}$
- makes explicit patching $S_5 \to SU(3)$ which generates $\pi_5 SU(3)$
  $SU(3)/SU(2) = S^5$, $\pi_5 SU(3) \to \pi_5 S^5$ [Chaves&Rigas, 1996]

Pull back along $f : S^4 \times D^2 \to S^6$

$$f(S^4 \times \partial D^2) = f(\{\text{S-pole}\} \times D^2) = \{\text{S-pole}\} \subset S^6$$

∂D^2 being identified to a point, the S-pole $\in S^2$

Pull back $G_2 \to S^6$ along f to get a map $S^2 \to \mathcal{A}$ which sends the S-pole $\in S^2$ to the flat connections. This is a generator of $\pi_2 \mathcal{A}$.

For explicit calculation: use the exponential map to trivialize G_2 over $S^6 - \{\text{S-pole}\}$, then pull back along f to get a 2-disk of connections with $\partial D^2 = S^1$ going to a loop in the flat connections.

What happens to this 2-sphere under the Y-M flow?

I want to turn this into a question that I can address with elementary tools.

A $U(2)$-invariant $f : S^4 \times D^2 \to S^6$

$S^4 \times D^2 \subset \mathbb{R} \oplus \mathbb{C}^3$

$$\left\{(u_0, u_1, u_2, u_3) : u_0^2 + |u_1|^2 + |u_2|^2 = 1,\ |u_3| \leq 1\right\}$$

$U(2) \subset SU(3)$ preserves $S^4 \times D^2$

$$U \in U(2) \mapsto \begin{pmatrix} U & \\ & (\det U)^{-1} \end{pmatrix} \in SU(3)$$

A $U(2)$-invariant f

$$r = \left(1 - |u_3|^2\right)^{\frac{1}{2}}$$

$$f(u_0, u_1, u_2, u_3) = (-|u_3|^2, 0, 0, ru_3) + r\frac{(ru_0, ru_1, ru_2, u_0 u_3)}{(r^2 + |u_0 u_3|^2)^{\frac{1}{2}}}$$

giving, for each $u_3 \in D^2$, an $S^4 \subset S^6$ of radius r.

The $U(2)$ symmetry

$$S^4 = \{u_0^2 + |u_1|^2 + |u_2|^2 = 1\} \quad D^2 = \{|u_3| \le 1\} \quad r = \left(1 - |u_3|^2\right)^{\frac{1}{2}}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto U \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad u_3 \mapsto (\det U)^{-1} u_3$$

- $u_3 \in D^2 \mapsto A(u_3)$ a connection over S^4
- $SU(2)$ fixes every u_3, so all the $A(u_3)$ are $SU(2)$-invariant, so u_0 is the only dependent variable on S^4.
- $U(2)$ fixes the origin $u_3 = 0$, so $A(0)$ is $U(2)$-invariant.
- The Y-M flow preserves $U(2)$ invariance.

Naive conjectures on the $U(2)$-invariant flow

(1) The connection at $u_3 = 0$ flows to a $U(2)$-invariant fixed point.

(2) The unstable manifold of the fixed point is 2-dimensional.

(3) The unstable manifold is a stable 2-sphere, $U(2)$-invariant, with S-pole at the flat connections.

An unlucky alternative (but consistent with $U(2)$ invariance): $u_3 = 0$ could flow to the flat connection.

I'll show some computer calculations that support (1) and that suggest a likely candidate for the $U(2)$-invariant fixed point.

Additional $\mathbb{Z}_2$ symmetry at $u_3 = 0$

Let R be the element in G_2

$$R : (u_0, u_1, u_2, u_3) \mapsto (-u_0, \bar{u}_1, \bar{u}_2, -\bar{u}_3)$$

$$R = \exp(0, 0, -\pi/2) \circ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

R fixes $u_0 = 0$, so is a symmetry of the connection at $u_0 = 0$.

R acts on the maximal S^4 (reversing orientation) by

$$R : (u_0, u_1, u_2, 0) \mapsto (-u_0, \bar{u}_1, \bar{u}_2, 0)$$

The symmetry of the flow is now $U(2) \rtimes \mathbb{Z}_2$.

The general $U(2)\rtimes\mathbb{Z}_2$-invariant connection on S^4

Use $SU(2)$ to restrict to a longitude

$$(u_0, u_1, u_2, 0) = (u_0, \sqrt{1 - {u_0}^2}, 0, 0) \qquad -1 \leq u_0 \leq 1$$

and trivialize along this longitude.

The covariant derivative is now $D = du_0 \partial_{u_0} + D^{\perp}(u_0)$ where $D^{\perp}$ is the $SU(2)$-invariant covariant derivative along the $SU(2)$ orbit.

Write $D^{\perp} = D^{0,\perp} + A^{\perp}(u_0)$ where $D^{0,\perp}$ is the covariant derivative for the flat connection along the orbits and $A^{\perp}(u_0)$ is an $su(3)$-valued 1-form on the orbit.

The remaining $U(1)$ symmetry, $u_2 \to e^{i\theta} u_2$, implies

$$A^{\perp} = \begin{pmatrix} i(f_1 - f_2)\omega^3 & -\bar{g}_1\omega^- & -\bar{g}_2\omega^+ \\ g_1\omega^+ & -if_1\omega^3 & 0 \\ g_2\omega^- & 0 & if_2\omega^3 \end{pmatrix} \qquad f_{1,2} = \bar{f}_{1,2}$$

where $\omega^3, \omega^\pm$ are the Maurer-Cartan forms on the $SU(2)$ orbit. At the longitudinal slice, $\omega^3 = d\,\mathrm{Im}\,u_1$, $\omega^+ = du_2$, $\omega^- = d\bar{u}_2$.

Invariance under R, which takes $u_0 \to -u_0$, implies

$$f_2(u_0) = f_1(-u_0) \qquad g_2(u_0) = g_1(-u_0)$$

Continuity at the poles implies the boundary conditions

$$f_1(1) = 1 \qquad f_1(-1) = 0 \qquad g_1(1) = 1 \qquad g_1(-1) = 0$$

S_{YM} (in instanton units)

$$S_{YM} = \int_{-\infty}^{\infty} dx \, \Big[\frac{1}{4}(\partial_x f_1)^2 + (f_1 - |g_1|^2)^2 + \frac{1}{4}(\partial_x f_2)^2 + (f_2 - |g_2|^2)^2$$
$$+ \frac{1}{4}(\partial_x f_1 - \partial_x f_2)^2 + (f_1 - |g_1|^2 - f_2 + |g_2|^2)^2$$
$$+ |\partial_x g_1|^2 + (-2f_1 + f_2 + 2)^2 |g_1|^2$$
$$+ |\partial_x g_2|^2 + (-2f_2 + f_1 + 2)^2 |g_2|^2 \Big]$$

$$x = \frac{1}{2} \ln \left(\frac{1 - u_0}{1 + u_0} \right) \qquad 2e^x = \text{stereographic radius.}$$

For the initial connection $A(0)$:

$$f_1^{(0)}(u_0) = \frac{1 + u_0}{2} \qquad g_1^{(0)}(u_0) = \left(\frac{1 + u_0}{2} \right)^{\frac{3}{2}} \qquad S_{\mathrm{YM}} = 2.4$$

A simple-minded computer investigation

Explore a finite dimensional submanifold of connections:

$$f_1(u_1) = \left(\frac{1+u_1}{2}\right) + (1-u_1^2)\sum_{k=0}^{N-1} F_k u_1^k$$

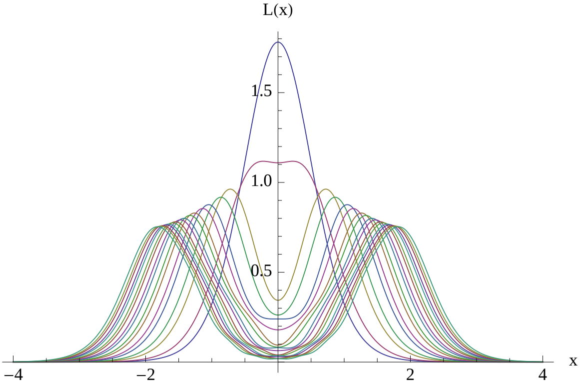
$$g_1(u_1) = \left(\frac{1+u_1}{2}\right)^{\frac{1}{2}} \left[\left(\frac{1+u_1}{2}\right) + (1-u_1^2)\sum_{k=0}^{N-1} G_k u_1^k\right]$$

S_{YM} is now a quartic polynomial in the $2N$ real variables F_k, G_k (the G_k being real because g_1 stays real under the flow).

Minimize S_{YM} as a function of the F_k, G_k (using Sage, Mathematica, Maple).

$$S_{YM} = \int_{-\infty}^{\infty} dx\; L(x)$$

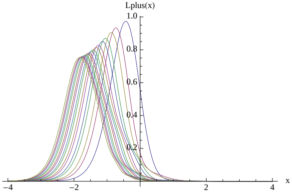
N	min S_{YM}
0	2.40000
2	2.08672
4	2.03528
6	2.01789
8	2.01028
10	2.00642
12	2.00426
14	2.00296
16	2.00211

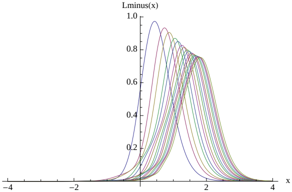


The programs start behaving badly around $n = 16$. The method of approximation is not well-suited to the problem.

(anti-)self-duality

$$S = S_+ + S_- \qquad S_\pm = \int \left(\frac{F \pm *F}{2} \right)^2 = \int_{-\infty}^{\infty} dx\, L_\pm(x)$$





$(+)$ instanton (self-dual solution of Y-M equation)

$$f_1 = g_1 = \frac{1}{1 + e^{2(x+x_0)}} \quad f_2 = g_2 = 0 \quad L_+(x) = 0.75 \cosh^{-4}(x + x_0)$$

(-) anti-instanton (anti-self-dual, R-reflection of instanton)

$$f_2 = g_2 = \frac{1}{1 + e^{-2(x-x_0)}} \quad f_1 = g_1 = 0 \quad L_-(x) = .75 \cosh^{-4}(x - x_0)$$

Patching the instanton and anti-instanton solutions

- Want: $x < 0$ instanton solution, $x > 0$ anti-instanton.
- Can only patch these together at $x = 0$ if $x_0 \to \infty$.
- Try patching $x \leq -\delta$ instanton with $x \geq \delta$ anti-instanton, using some interpolation for $-\delta \leq x \leq \delta$. Always get $S_{YM} > 2$ (but I have not been able to prove this).
- Can get $S_{YM} \to 2$ for such configurations, by taking $x_0 \to \infty$.
- The limit is an instanton of zero size at the N-pole and an anti-instanton of zero size at the S-pole.
- It seems likely that the $U(2)\rtimes\mathbb{Z}_2$-invariant connection will flow to this zero-size instanton-anti-instanton pair.

Questions about the $U(2)\rtimes\mathbb{Z}_2$-invariant flow

- Prove that $S_{YM} \geq 2$ for the $U(2)\rtimes\mathbb{Z}_2$ invariant connections?
- Prove that $S_{YM} = 2$ *only* for the zero-size instanton-anti-instanton pair?
- Is there a stable $U(2)\rtimes\mathbb{Z}_2$-invariant 2-sphere of connections that contains this zero-size instanton-anti-instanton as unstable fixed point at the N-pole?

Conjectures (guesses?)

- The conformal group of S^4 acts on this fixed point, giving a family of fixed points: zero-size instanton-anti-instantons at arbitrary locations in S^4.
- Might guess that each of these fixed points has as its unstable manifold a stable 2-sphere of connections, such that any nearby 2-sphere ends on one of these stable 2-spheres.
- Might even guess that this family of 2-spheres is globally stable.

I've shown only that the minimum over 2-spheres of connections of the maximum of S_{YM} over the 2-sphere is ≤ 2.

(This is in the language of Sibner, Sibner and Uhlenbeck, 1989, who studied nontrivial loops of connections.)

Plan

- Consider all instanton-anti-instanton pairs of asymptotically small size.
- Patch the self-dual and anti-self-dual solutions in some fixed (arbitrary) way.
- Separate variables into the zero-modes and the rest.
- The zero modes are: location, size, orientation within $SU(3)$ for each of the pair. All other variables are irrelevant, because instantons are stable under the flow.
- The flow should act on the zero mode variables.
- The hope is that the asymptotic behavior of the flow will be insensitive to the method of patching, in the limit of asymptotically small sizes, that it will be possible to identify the fixed points and their instabilities unambiguously.

- Somehow, perhaps for some relative orientations in $SU(3)$, the sizes will have to increase under the flow, because the flow cannot end at the flat connection without the topological charges cancelling.

- If this picture turns out to be correct, I'd be especially interested in the stable 2-sphere emerging from a configuration of a zero-size instanton-anti-instanton pair at finite separation in euclidean $\mathbb{R}^4$.

  I would especialy want to know how the flow along the stable 2-sphere approaches the flat connection. There might be interesting spectral information.

Final comment

M. Carfora remarked to me in a private coversation a year ago in Banff that stable surfaces for a geometric flow would fit into a program of using the geometric flow for Morse theory on the space of geometries.

In that spirit, one might want to look at configurations of any number of zero-size instantons and anti-instantons. They might have instabilities providing stable submanifolds of the Y-M flow.

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Addendum July 6, 2009

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At the workshop *Geometric Flows in Mathematics and Theoretical Physics*, Pisa, June 24, 2009, I described some an ongoing attempt to understand the long time behavior of the Yang-Mills flow acting on topologically non-trivial 2-spheres of $SU(3)$ connections on S^4 [1].

This was very much a report on work in progress. I sketched a half-baked plan for calculating the Yang-Mills flow near configurations of an instanton/anti-instanton pair, both asymptotically small. After further thought, I think I can half-bake the plan a bit more.

The goal is to find a stable 2-sphere by finding a two-dimensional unstable manifold of small instanton/small anti-instanton pairs. Computer calculations suggested that such an unstable manifold might well exist.

The instanton and the anti-instanton are separately stable under the flow, so the flow will concentrate on the zero-mode space: the moduli space of the small-instanton/small-anti-instanton pair. The zero-modes are the locations and sizes of the instanton and the anti-instanton, and the internal moduli that describe their relative orientation within $SU(3)$.

Plan of calculation

Write ξ^I for the moduli. Let $A_\pm(\xi)$ be the self-dual solution on one hemisphere and the anti-self-dual solution on the other hemisphere. The two solutions do not quite fit together at the boundaries of the hemispheres, so they must be corrected slightly

$$A(\xi) = A_\pm(\xi) + \delta A_\pm(\xi) \tag{1}$$

to get a connection that is continuous and differentiable at the common boundary of the two hemispheres.

This family of connections, $A(\xi)$, is supposed to be closed under the Y-M flow. That is, there is supposed to be a flow $\xi(t)$ on the zero-modes such that $A(\xi(t))$ is the Y-M flow:

$$\frac{d}{dt}A(\xi(t)) = \dot{\xi}^I \frac{\partial A}{\partial \xi_I} = *d*F(\xi(t))\,. \tag{2}$$

It should be possible to solve perturbatively for the $\delta A_\pm(\xi)$, because the flow comes to a stop when the sizes vanish.

The plan is to solve for the $\delta A_\pm(\xi)$ separately on each hemisphere, then choose among all possible solutions the pair that make $A(\xi)$ continuous and differentiable at the common boundary of the hemispheres. The choice should be unique.

The linearized equation is

$$\dot{\xi}^I \frac{\partial A_\pm}{\partial \xi_I} = \Box_{YM}\delta A_\pm(\xi) \tag{3}$$

On each hemisphere, choose specific solutions $B_{I\pm}$ of

$$\Box_{YM} B_{I\pm} = \frac{\partial A_\pm}{\partial \xi_I} \tag{4}$$

The most general solution to the linearized equation on the separate hemispheres is

$$\delta A_\pm = \dot{\xi}^I B_{I\pm} + N_\pm \tag{5}$$

for some $N_\pm$ satifying

$$\Box_{YM} N_\pm = 0\,. \tag{6}$$

Now require that $A_\pm + \delta A_\pm$ be continuous and differentiable at the common boundary, with $\delta A_\pm$ given by equation 5. This should determine the velocities $\dot{\xi}^I$ uniquely.

For this to work, the space of boundary values of solutions of $\Box_{YM} N_\pm = 0$ in each hemisphere must have have co-dimension N in the space of boundary conditions (N being the number of zero-modes in each hemisphere).

Atiyah, Hitchen and Singer [2] showed that there exist irreducible $SU(n)$ instantons on S^4 of degree k iff $k \geq \frac{1}{2}n$. So all $SU(3)$ instantons of degree $k = 1$ on S^4 live in an $SU(2)$ subgroup. The linearized equations in each hemisphere are thus in the background of a small $SU(2)$ instanton. There are explicit formulas for Green's functions in the $SU(2)$ instanton background, so the calculation is probably doable.

Speculation

One might guess that the instanton and anti-instanton have to be lined up perfectly in order to merge together so that their topological charges can eventually cancel. It is tempting to speculate that there is only a single flow line along which the small-instanton and the small-anti-instanton would grow larger, eventually merging to reach the flat connection.

The stable 2-sphere would then lie entirely within the moduli space of zero-modes, except for an infinitesimally thin tube flowing from the S-pole of the 2-sphere down to the flat connection.

The internal moduli space for a small $SU(2)$ instanton and a small $SU(2)$ anti-instanton in $SU(3)$ appears to be essentially $\mathbb{CP}^2$, and $\pi_2\mathbb{CP}^2 = \mathbb{Z}$, so this scenario might be feasible.

SU(2)

I'm still puzzled by the $SU(2)$ flow. The only model I've seen for the generator of $\pi_5 SU(2)$ is the map $S(S(H)) \circ S(H) : S^5 \to S^3$, where $H : S^3 \to S^2$ is the Hopf map, $S(H) : S^4 \to S^3$ its suspension, and $S(S(H)) : S^5 \to S^4$ its double suspension. This gives the non-trivial $SU(2)$ bundle over S^6, but I have not been able to find a useful construction. I have no idea what a stable 2-sphere for the $SU(2)$ flow might look like.

[1] D. Friedan, *Preliminary evidence for a stable 2-sphere ...*, Pisa, June 24, 2009, http://www.crm.sns.it/download/corsi/2121/Friedan.pdf or http://www.physics.rutgers.edu/pages/friedan/talks/flows/Friedan_2009.06.24_Pisa.pdf.

[2] M. F. Atiyah, N. J. Hitchin and I. M. Singer, *Self-Duality in Four-Dimensional Riemannian Geometry*, Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences, Vol. 362, No. 1711 (Sep. 12, 1978), pp. 425-461.