Existence of extremals for a Fourier restriction inequality

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(joint work with Shuanglin Shao)

The celebrated Fourier restriction inequality of Tomas and Stein asserts, for dimension three, that the Fourier transform maps $\mathrm{L}^2(S^2) to L^4(R^3)$.

We show that there exist functions which extremize the associated inequality, and that any extremizing sequence of nonnegative functions has a convergent subsequence. This was previously known for paraboloids, where all extremizers are Gaussians and vice versa.

Complex extremizers and extremizing sequences are related to nonnegative ones in a simple way. All critical points of the associated nonlinear functional are infinitely differentiable. Constant functions are local extremizers, but we do not know whether they are global extremizers, nor whether extremizers are unique modulo symmetries of the problem.

The proofs involve concentration compactness ideas, inequalities for convolutions, facts about Fourier integral operators, symmetrization, a characterization of approximate characters, a perhaps nonstandard regularity theorem, an idea from additive combinatorics, facts about spherical harmonics and Gegenbauer polynomials, and several explicit computations.