An explicit Kontsevich integral for welded braids

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Reference

Barbu Berceanu, Stefan Papadima

Universal representations of braid and braid-permutation groups

J. Knot Theory Ramif., 2009

1. Braid groups and welded braid groups
2. Universal representations of welded braids
3. Drinfeld representations of braids
4. Main results for welded braids
5. About proofs

Artin braid groups

$$1 \to PB_n \longrightarrow B_n \xrightarrow{\pi} \Sigma_n \to 1$$

- Artin's braid group $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ (E. Artin, Ann. Math. 1947)
- symmetric group $\Sigma_n = \langle s_1, \ldots, s_{n-1} \rangle$; $s_i = \pi(\sigma_i)$ transposition $(i, i+1)$
- pure braid group $PB_n = \pi_1 F(n, \mathbb{C})$ arrangement group

Theorem (Orlik-Solomon, Invent. Math. 1980)

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Welded braid groups

$\mathbb{F}_n$ free group on $\{x_1, \ldots, x_n\}$

Welded braid group W_n (Fenn-Rimanyi-Rourke, Topology 1997):

$a \in \mathrm{Aut}(\mathbb{F}_n)$ s.t. $a(x_i) = y_i^{-1} x_{s(i)} y_i$ for some $y_i \in \mathbb{F}_n$, $s \in \Sigma_n$

Theorem (Artin)

$B_n = \{a \in W_n \mid a(x_1 \cdots x_n) = x_1 \cdots x_n\}$

- $1 \to PW_n \longrightarrow W_n \xrightarrow{\pi} \Sigma_n \to 1$ splits
- generators $\{a_{ij} \mid 1 \leq i \neq j \leq n\}$ for PW_n given by $x_i \mapsto x_j^{-1} x_i x_j$; $x_k \mapsto x_k$ $(k \neq i)$
- $s a_{ij} s^{-1} = a_{s(i),s(j)}$, for $s \in \Sigma_n$
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Oriented Artin algebra $\mathcal{O}_n^*$

H^* connected graded algebra

- filtration $F_k H = H^{\geq k}$; $F_k H \cdot F_r H \subseteq F_{k+r} H$
- completion $\widehat{H}$: $\sum_{k \geq 0} h_k$, $h_k \in H^k$
- complete filtration $F_k \widehat{H} = \widehat{H}^{\geq k}$; $F_k \widehat{H} \cdot F_r \widehat{H} \subseteq F_{k+r} \widehat{H}$
- $1 + F_1 \widehat{H} \subseteq \widehat{H}^{\times}$ (units): $(1-h)^{-1} = \sum_{k \geq 0} h^k$
- tensor algebra $\mathbb{Q}\langle v \rangle$; completion $\mathbb{Q}\langle\langle v \rangle\rangle$ (Hopf algebras)

Definition

$\mathcal{O}_n^*$ is the quotient of $\mathbb{Q}\langle v_{ij} \mid 1 \leq i \neq j \leq n \rangle$ by

$$\begin{array}{lll} (I) & [v_{ik}, v_{jk}] & 1 \leq i \neq j \neq k \leq n \\ (II) & [v_{ij}, v_{ik} + v_{jk}] & 1 \leq i \neq j \neq k \leq n \\ (III) & [v_{ij}, v_{kl}] & 1 \leq i \neq j \neq k \neq l \leq n \end{array} \tag{1}$$

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The representation R_n

Theorem (McCool, Canadian J. 1986)

Defining relations of PW_n:

$$\begin{array}{lll} (I) & (a_{ik}, a_{jk}) & 1 \leq i \neq j \neq k \leq n \\ (II) & (a_{ij}, a_{ik} \cdot a_{jk}) & 1 \leq i \neq j \neq k \leq n \\ (III) & (a_{ij}, a_{kl}) & 1 \leq i \neq j \neq k \neq l \leq n \end{array} \qquad (2)$$

where $(a, b) = aba^{-1}b^{-1}$.

Lemma

$a_{ij} \mapsto \exp(v_{ij}) \in \widehat{\mathcal{O}_n}^{\times}$ *gives representation* $R_n : \mathbb{Q}[PW_n] \longrightarrow \widehat{\mathcal{O}_n}$

Proof.

$\exp(u) \cdot \exp(v) = \exp(u + v)$, if $uv = vu$ □

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Extension to the full welded braid group

• semidirect product $A \rtimes \Sigma$ (where the group Σ acts on the algebra A): $A \otimes \mathbb{Q}[\Sigma]$, with multiplication

$$(a \otimes s) \cdot (b \otimes t) = a \cdot {}^{s}b \otimes st$$

Definition

Oriented diagram algebras: $\widehat{\mathcal{O}_n} \rtimes \Sigma_n$

Definition

Universal finite-type representations of welded braid groups

$$R_n \otimes \mathrm{id} : \mathbb{Q}[W_n] \longrightarrow \widehat{\mathcal{O}_n} \rtimes \Sigma_n$$

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$$R_n \otimes \mathrm{id} \colon \mathbb{Q}[W_n] \longrightarrow \widehat{\mathcal{O}_n} \rtimes \Sigma_n$$

The construction (Drinfeld, Leningrad J. 1991)

- associator $\Phi \in \mathbb{Q}\langle\langle A, B\rangle\rangle$: axioms (hexagons, pentagon, ...)
- Artin algebra $\mathcal{A}_n^*$ is the quotient of $\mathbb{Q}\langle t_{ij} = t_{ji} \mid 1 \leq i \neq j \leq n\rangle$ by

$$\begin{array}{lll} (II) & [t_{ij}, t_{ik} + t_{jk}] & 1 \leq i \neq j \neq k \leq n \\ (III) & [t_{ij}, t_{kl}] & 1 \leq i \neq j \neq k \neq l \leq n \end{array} \tag{3}$$

- diagram algebra $\widehat{\mathcal{A}_n} \rtimes \Sigma_n$
- Drinfeld representations $\rho_n \colon \mathbb{Q}[B_n] \longrightarrow \widehat{\mathcal{A}_n} \rtimes \Sigma_n$

$$\begin{cases} \sigma_1 & \mapsto & \exp(\frac{t_{12}}{2}) \otimes s_1\ ; \\ \sigma_{i>1} & \mapsto & \Phi(\sum_{j<i} t_{ji}, t_{i,i+1})^{-1} \cdot (\exp(\frac{t_{i,i+1}}{2}) \otimes s_i) \cdot \Phi(\sum_{j<i} t_{ji}, t_{i,i+1}) \end{cases} \tag{4}$$

Filtrations

- multiplicative degree filtration $\{F_k\}$: $\widehat{\mathcal{A}_n}^{\geq k} \otimes \mathbb{Q}[\Sigma_n]$, $\widehat{\mathcal{O}_n}^{\geq k} \otimes \mathbb{Q}[\Sigma_n]$
- associated graded ring gr_F^*: $\mathcal{A}_n^* \rtimes \Sigma_n$, $\mathcal{O}_n^* \rtimes \Sigma_n$

Vassiliev-type filtrations: iteration of local move

- for B_n: exchange negative and positive crossings; $V^k \subseteq \mathbb{Q}[B_n]$
  V ideal generated by $\sigma_i - \sigma_i^{-1}$
- for W_n: replace weld by positive crossing; $J^k \subseteq \mathbb{Q}[W_n]$
  J ideal generated by $\sigma_i - s_i$
- $V^k \subseteq J^k$

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Universal finite-type invariant for braids

Theorem (Drinfeld)

$\rho_n \colon \mathbb{Q}[B_n] \longrightarrow \widehat{\mathcal{A}_n} \rtimes \Sigma_n$ *induces* $\mathrm{gr}^*_V \mathbb{Q}[B_n] \xrightarrow{\simeq} \mathcal{A}^*_n \rtimes \Sigma_n$

• Kontsevich, Adv. Soviet Mat. 1993: $\mathrm{gr}^*_V \mathbb{Q}[\mathcal{L}] \xrightarrow{\simeq} \mathcal{C}^*$
• similar results, over $\mathbb{Z}$: Papadima, Topology Appl. 2002 (for braid groups of $\mathbb{C}$); Meneses-Paris, Trans. AMS 2004 (for braid groups of aspherical compact Riemann surfaces)

Theorem (Kohno, Contemp. Math. 1994)

ρ_n *faithful*

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Statement of main results

Theorem

$R_n \otimes \mathrm{id}$ *induces* $\mathrm{gr}^*_J \mathbb{Q}[W_n] \xrightarrow{\simeq} \mathcal{O}^*_n \rtimes \Sigma_n$

Theorem

$R_n \otimes \mathrm{id} \colon \mathbb{Q}[W_n] \hookrightarrow \widehat{\mathcal{O}_n} \rtimes \Sigma_n$

Key tools

Lower central series of a group G

- l.c.s. $\Gamma_1 G = G$, $\Gamma_2 G = G'$, $\ldots, \Gamma_{k+1} G = (G, \Gamma_k G), \ldots; \Gamma_\infty G = \cap \Gamma_k G$
- associated graded Lie algebras $\mathrm{gr}^*_\Gamma G$, $\mathrm{gr}^*_\Gamma G \otimes \mathbb{Q}$ (generated by gr^1)

I-adic filtration

- augmentation ideal $I_G = \ker\{\varepsilon\colon \mathbb{Q}[G] \to \mathbb{Q}\}$; $\varepsilon(g) = 1$ for $g \in G$
- multiplicative filtration $\{I_G^k\}$; $I_G^\infty = \cap I_G^k$

Theorem (Quillen, Ann. Math. 1969)

$\mathrm{gr}^*_{I_G} \mathbb{Q}[G] = U \mathrm{gr}^*_\Gamma G \otimes \mathbb{Q}$, *as Hopf algebras*

Malcev Lie algebras and exponential groups

- Malcev Lie algebra: complete descending filtration $\{F_k M\}$ of $\mathbb{Q}$-Lie algebra M, with $F_1 M = M$; $[F_k, F_r] \subseteq F_{k+r}$; $\mathrm{gr}^*_F M$ generated by gr^1
- exponential group $E(M) = M$: $x \cdot y = x + y + \frac{1}{2}[x, y] + \ldots$

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Malcev completion (Quillen)

$G \rightsquigarrow M_G$ Malcev Lie algebra and $G \to E(M_G)$ group homomorphism, inducing Lie algebra isomorphism

$$\mathrm{gr}^*_\Gamma G \otimes \mathbb{Q} \xrightarrow{\simeq} \mathrm{gr}^*_F M_G$$

• free (Malcev) Lie algebra: $\mathrm{Prim}\, \mathbb{Q}\langle v \rangle = \mathbb{L}^*(v)$, $\mathrm{Prim}\, \mathbb{Q}\langle\langle v \rangle\rangle = \widehat{\mathbb{L}}(v)$
• quadratic (Malcev) Lie algebra: $L^* = \mathbb{L}^*(v)$ modulo relations $\subseteq \mathbb{L}^2(v)$, $\widehat{L} = \widehat{\mathbb{L}}(v)$ modulo same relations

X connected CW-complex with $X^{(1)}$ finite; $G = \pi_1 X$

Theorem (D. Sullivan, Publ. IHES 1977)

X formal space implies G is 1-formal group, i.e., M_G quadratic

• example: $X = F(n, \mathbb{C})$, $G = PB_n$

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Reduction to pure welded braids

Theorem (Berceanu-Papadima)

$M_{PW_n} = \widehat{\mathbb{L}}(v_{ij} \mid 1 \leq i \neq j \leq n)$, modulo relations $(I) - (III)$. In particular, PW_n is 1-formal.

Set $R = \mathbb{Q}[W_n]$, $I \subseteq \mathbb{Q}[PW_n]$ augmentation ideal

Lemma

$RI^k = I^k R = RI^k R = J^k$

- subgroup PW_n normal in $W_n \Rightarrow RI^k = I^k R = RI^k R$
- subgroup PW_n normal in $W_n \Rightarrow J \subseteq RIR = \ker\{\pi : \mathbb{Q}[W_n] \twoheadrightarrow \mathbb{Q}[\Sigma_n]\}$
- enough: $a_{ij} \equiv 1$ modulo J; $a_{ij} = s a_{i,i+1} s^{-1} = s(\sigma_i s_i^{-1}) s^{-1} \equiv 1$ □

Note $R = \mathbb{Q}[PW_n] \otimes \mathbb{Q}[\Sigma_n]$, additively

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$J^k = I^k \otimes \mathbb{Q}[\Sigma_n]$

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Drinfeld's theorem for PW_n

- $R_n(a_{ij} - 1) = \exp(v_{ij}) - 1 \in \widehat{\mathcal{O}_n}^{\geq 1}$
- $R_n : \mathbb{Q}[PW_n] \longrightarrow \widehat{\mathcal{O}_n}$ sends I^k into $\widehat{\mathcal{O}_n}^{\geq k}$
- $\mathrm{gr}_I^* \mathbb{Q}[PW_n] = U \mathrm{gr}_\Gamma^* PW_n \otimes \mathbb{Q} = U \mathrm{gr}_F^* M_{PW_n} = \mathcal{O}_n^*$ □

Using this, Kohno's theorem for PW_n follows from

$$I^\infty = 0$$

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I-adic filtration and residual properties

• residually torsion-free nilpotent group G: any $1 \neq g \in G$ is detected in a residually torsion-free nilpotent quotient (inherited by subgroups)

Proposition (K.T.-Chen, Advances 1977)

G finitely generated is residually torsion-free nilpotent iff $I_G^\infty = 0$

• Torelli group $T(G) = \{a \in \mathrm{Aut}(G) \mid H_1(a) = \mathrm{id}\}$
• note that $PW_n \subseteq T(\mathbb{F}_n)$

Proposition (Hain, Journal AMS 1997)

If $\Gamma_\infty G = 1$ and $\mathrm{gr}^*_\Gamma G$ is torsion-free, then $T(G)$ is residually torsion-free nilpotent

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• (Magnus) $\Gamma_\infty \mathbb{F}_n = 1$ and $\mathrm{gr}^*_\Gamma \mathbb{F}_n$ is torsion-free

• since $PW_n \subseteq T(\mathbb{F}_n)$ is residually torsion-free nilpotent, $I^\infty_{PW_n} = 0$ $\square$