Spaces of homomorphisms, and spaces of representations

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joint work with Alex Adem, Enrique Torres-Giese, and José Gomez

The setting and the problems

- Let π and G be topological groups.
- Define

$$Hom(\pi, G) = \{f\colon \pi \to G \mid f \text{ is a homomorphism}\}.$$

- Define

$$Rep(\pi, G) = Hom(\pi, G)/G^{\mathsf{ad}}$$

the quotient of the natural action of G on

$$Hom(\pi, G)$$

via conjugation.

The problems here concern features of the spaces $Hom(\pi, G)$ and $Rep(\pi, G)$.

These problems as well as some of the results below address

- fundamental groups,
- cohomology groups,
- decompositions, some of which arise after suspending spaces which then informs on cohomology and other properties,
- whether certain spaces below are $K(\pi, 1)$'s, and
- the structure of natural representations of the group $Aut(\pi)$.

Motivation

- The original motivation was to try to understand features of subgroups of the pure braid groups by considering topological properties of functions out of the pure braid groups.
- Why braid groups ?

Why braid groups again ?

- Braid groups admit natural sub-quotients which are naturally giving the homotopy groups of spheres.
- The original motivation was to try to understand features of subgroups of the pure braid groups by considering topological properties of functions out of the pure braid groups.
- Earlier joint work with Alex Adem and Dan Cohen arose from bundles obtained from real, orthogonal representations of the pure braid group on n-strands P_n given by homomorphisms $f : P_n \to G$ with $G = O$ the stable, real orthogonal group.
- All such non-trivial bundles were detected by elements in the moduli space of isomorphism classes of flat G-bundles over the n-torus $(S^1)^n$. These appear in other contexts such as work A. Borel, R. Friedman, J. Morgan as well as many others.

Examples, and definitions

- **Example 1:** Let

$$\pi = F_n = F[x_1, \cdots, x_n]$$

denote a free group of rank n. Consider the map

$$e : Hom(F_n, G) \to G^n$$

which evaluates a homomorphism on the choice of 'basis' $x_1, \cdots, x_n$. This map is a bijection of sets. Thus topologize $Hom(F_n, G)$ with the topology of G^n.

Examples, and definitions

- **Example 2:** Let

$$\pi = \oplus_n \mathbb{Z}$$

a quotient of F_n given by $(F_n/[F_n, F_n]) = \oplus_n \mathbb{Z}$ where $[F_n, F_n]$ is the commutator subgroup of F_n. Consider the natural quotient map

$$F_n \to F_n/[F_n, F_n]$$

together with the induced map

$$Hom(\oplus_n \mathbb{Z}, G) \to Hom(F_n, G).$$

Since this last map is a monomorphism of sets, topologize $Hom(\oplus_n \mathbb{Z}, G)$ as a subspace of G^n.

Examples, and definitions

- **Example 3:** Let Γ^q denote the q-th stage of the descending central series of F_n, the subgroup generated by commutators of the form
$$[\cdots[v_1, v_2], v_3]\cdots], v_t]$$
where $v_i \in F_n$ with $t \geq q$, and
$$[v, w] = vwyv^{-1}w^{-1}.$$
Consider
$$Hom(F_n/\Gamma^q, G)$$
to obtain a non-decreasing family of spaces
$$Hom(F_n/\Gamma^2, G) \subset Hom(F_n/\Gamma^3, G) \subset \cdots \subset G^n.$$

Examples, and definitions

- **Example 4**

  Let Γ_p^q denote the q-th stage of the mod-p descending central series of F_n, the subgroup generated by commutators of the form

  $$[\cdots[v_1, v_2], v_3]\cdots], v_t]^{p^r}$$

  where $v_i \in F_n$ with $t \cdot p^r \geq q$. Consider

  $$Hom(F_n/\Gamma_p^q, G)$$

  to give a non-decreasing family of spaces

  $$Hom(F_n/\Gamma_p^2, G) \subset Hom(F_n/\Gamma_p^3, G) \subset \cdots \subset G^n.$$

Examples, and definitions

- **Example 5:** Let $\widehat{F_n}$ denote the pro-finite completion of F_n filtered by the associated descending central series. There is an induced filtration with useful properties

$$Hom(\widehat{F_n}/\Gamma^2, G) \subset \cdots \subset Hom(\widehat{F_n}, G).$$

Simplicial spaces

- Fix an integer $q \geq 2$. This section is a description of how the spaces
$$Hom(F_n/\Gamma^q, G)$$
assemble into a simplicial space for each fixed q and for all non-negative integers n.
- There are $(n+1)$ natural maps
$$d_i : Hom(F_n, G) \to Hom(F_{n-1}, G)$$
as well as
$$s_j : Hom(F_n, G) \to Hom(F_{n+1}, G)$$
defined as follows.

Simplicial spaces continued

- Regard a homomorphism

$$f : F_n \to G$$

as an ordered n-tuple of points $g_1, \cdots, g_n \in G$.
- Define functions (face operations for a simplicial space) $d_i : Hom(F_n, G) \to Hom(F_{n-1}, G)$
- $d_i((g_1, \cdots, g_n))i = (g_2, \cdots, g_n)$ if $i = 0$,
- $d_i((g_1, \cdots, g_n)) = (g_1, \cdots, g_{i-1}, g_i g_{i+1}, \cdots, g_n)$ if $0 < i < n$, and
- $d_i((g_1, \cdots, g_n)) = (g_1, \cdots, g_{n-1})$ if $i = n$.

Simplicial spaces continued

- There are additional functions (degeneracies)

$$s_i : Hom(F_n, G) \to Hom(F_{n+1}, G)$$

defined by the formula

$$s_i(g_1, \cdots, g_n)$$

given by

$$(g_1, \cdots, g_{i-1}, g_i, e, g_{i+1}, \cdots, g_n),$$

if $0 \leq i \leq n$.

Simplicial spaces continued

- The functions d_i and s_j satisfy the simplicial identities, as well as restrict to analogous functions on the level of

$$Hom(F_n/\Gamma^q, G).$$

Geometric realization

- Given any simplicial space, there are associated topological spaces given by the geometric realization described roughly by considering the disjoint union of n-simplices, one for each point in $Hom(F_n, G)$, and making identifications according to the face and degeneracies.

Formal definition of geometric realization

- The geometric realization of a simplicial space Z_* is the following topological space $|Z_*| := \coprod_{n\geq 0} Z_n \times \Delta[n]/\sim$ where $\Delta[n]$ denotes the n–simplex.
- the disjoint union of n-simplices, one for each point in

$$Z_n = Hom(F_n, G),$$

and

- making identifications according to the face and degeneracies.

Identifications in the geometric realization

- The equivalence relation $\sim$ is defined as follows.

  Identify
  $$(x, \delta_i t) \in X_n \times \Delta[n]$$
  with
  $$(d_i x, t) \in X_{n-1} \times \Delta[n-1]$$
  for any $x \in X_n$, $t \in \Delta[n-1]$ and
  $$(x, \sigma_j t) \in X_n \times \Delta[n]$$
  with
  $$(s_j x, t) \in X_{n+1} \times \Delta[n+1]$$
  for any $x \in X_{n-1}$ and $t \in \Delta[n+1]$.

Remarks & Theorems

- The inclusions

$$Hom(F_n/\Gamma^2, G) \subset Hom(F_n/\Gamma^3, G) \subset \cdots \subset G^n$$

induce morphisms of simplicial spaces. Furthermore, the following properties are satisfied.

- The geometric realization obtained from the simplicial space $Hom(F_n, G)$ is precisely Milgram's construction of the classifying space BG.

Remarks & Theorems continued

- The geometric realization obtained from the simplicial space for any fixed integer $q \geq 1$, denoted $B(q, G)$, is a subspace of BG.

- The spaces $B(q, G)$ give a filtration of BG

$$B(2, G) \subset B(3, G) \subset \cdots \subset BG.$$

- Analogous properties are satisfied for the 'other filtrations' above.

Remarks & Theorems continued

- Let

$$E(q, G)$$

denote the homotopy of the natural inclusion

$$B(q, G) \subset BG.$$

- Thus if G is a discrete group, then the natural map

$$E(q, G) \to B(q, G)$$

is a regular G-covering space.

The first homework problem as an introduction to $B(2, G)$

- Give informative properties of the fundamental group $\pi_1(B(2,G))$ and the induced map

$$\pi_1(E(2,G)) \to \pi_1(B(2,G))$$

  in the case where G is discrete,.

- Show that if G is discrete and finite of odd order, then the map

$$H_1(E(2,G)) \to H_1(B(2,G))$$

  is not an epimorphism.

One unacceptable partial solution:

- "The dog ate my homework."

A second unacceptable partial solution:

- Let G be a discrete, finite group of odd order. The following two statements are equivalent.
  1. The map
  $$H_1(E(2,G)) \to H_1(B(2,G))$$
  is not an epimorphism.
  2. The group G is solvable (the odd order Theorem of Feit-Thompson).
- A solution of quoting the odd order theorem is not allowed.

More remarks

- An acceptable solution is to see whether the topology of the covering space
$$E(q,G) \to B(q,G)$$
with
$$B(2,G) = E(2,G)/G$$
informs on this, as yet open, problem. **Caution: It is far from clear whether this approach is informative.**
- Observe that the regular covering space $E(2,G) \to B(2,G)$ gives an induced homomorphism
$$\rho : G \to Out(\pi_1(E(2,G)).$$
What features of this homomorphism inform on qualitative features of $H_1(B(2,G))$?

Conjecture

- If G is of odd order, then

$$\pi_1(E(2,G))$$

is a finitely generated free group.

- If G is of odd order, then

$$\pi_1(E(2,G)) \to \pi_1(B(2,G))$$

is not onto (This statement is actually a theorem.).

Counting homomorphisms

- This problem addressed in this section is to describe the cardinality of $Hom(\oplus_n \mathbb{Z}, G)$ for finite groups G.

- The methods are to use the natural topological properties of $B(2, G)$. Some of the methods are specializations of general methods associated to simplicial spaces.

Stable decompositions of $Hom(F_n/\Gamma^q, G)$, and filtrations of $B(q, G)$

- Define a subspace of

$$Hom(F_n/\Gamma^q, G)$$

given by

$$S_n(q, G) = \cup_{0 \leq i \leq n} s_i(Hom(F_{n-1}/\Gamma^q, G)).$$

Stable decompositions of $Hom(F_n/\Gamma^q, G)$, and filtrations of $B(q,G)$

- If G is a closed subgroup of $GL(m,\mathbb{R})$, then the spaces

$$\Sigma Hom(F_n/\Gamma^q, G),$$

and

$$\bigvee_{1\leq k\leq n} \Sigma \bigvee_{\binom{n}{k}} Hom(F_k/\Gamma^q, G)/S_k(q,G)$$

are naturally homotopy equivalent.

Stable decompositions of $Hom(F_n/\Gamma^q, G)$, and filtrations of $B(q,G)$

- If G is a closed subgroup of $GL(m,\mathbb{R})$, then the natural filtration quotients

$$E_k^0(B(q,G)) = F_kB(q,G)/F_{k-1}B(q,G)$$

of the geometric realization $B(q,G)$ are stably homotopy equivalent to

$$\Sigma^k(Hom(F_k/\Gamma^q, G)/S_k(q,G)).$$

Thus the following spaces are naturally stably homotopy equivalent.

1. $Hom(F_n/\Gamma^q, G)$
2. $\bigvee_{1\leq k\leq n}\bigvee_{\binom{n}{k}}\Sigma^{-k}(E_k^0(B(q,G)))$

Counting the cardinality of $Hom(F_n/\Gamma^q, G)$

- Let G denote a finite group.
- The integer $\lambda_n(q, G)$ is defined as the cardinality of $Hom(F_n/\Gamma^q, G)$.
- The integer $\mu_k(q, G)$ is defined as the rank of $H_k(E_k^0(B(q, G)); \mathbb{Z})$.

Stable decompositions of $Hom(F_n/\Gamma^q, G)$, and filtrations of $B(q, G)$

- An immediate consequence of earlier features is that the cardinality of the set of homomorphisms $Hom(F_n/\Gamma^q, G)$ is given in terms of homology.

- If G is a finite group, then

$$\lambda_n(q, G) = 1 + \sum_{1 \leq k \leq n} \binom{n}{k} \mu_k(q, G).$$

$B(q, G)$ and the $K(\pi, 1)$ problem

- Assume that G is a finite, discrete group.
- Give conditions which guarantee that $B(q, G))$ is a $K(\pi, 1)$.

$B(q,G)$ and the $K(\pi,1)$ problem: Examples

- A finite group G is said to be a TC group (*transitively commutative*) if $[g,h]=1=[h,k]$, then $[g,k]=1$ for all g,h,k outside of the centralizer of G.

- If G is a TC group with trivial center, then $B(2,G)$ is a $K(\pi,1)$.

- Examples of TC groups are dihedral groups, generalized quaternion groups, and $SL(2,\mathbf{F}_{2^n})$.

$B(q,G)$ and the $K(\pi,1)$ problem: Examples

- Let $C_G(a)$ denote the centralizer of a in G with $Z(G)$ the center of G. Let $a_1,\ldots,a_k \in G - Z(G)$ be a set of representatives such that

$$G = \bigcup_{1 \le i \le k} C_G(a_i)$$

  and no smaller number of centralizers covers G.
- If G is a TC group with trivial center, then $B(2,G)$ is homotopy equivalent to $\bigvee_{1\le i\le k}\left(\prod_{p||C_G(a_i)|} BP\right)$ where $P \in Syl_p(G)$ with $G = \cup_{1\le i\le k} C(a_i)$ where no smaller subset of the $C(a_i)$ covers G.

$B(q, G)$ and the $K(\pi, 1)$ problem: Examples

- Let G denote a finite group and let $\mathcal{P}_q(G)$ be the category with objects the set $\{M_\alpha, M_\alpha \cap M_\beta \mid M_\alpha, M_\beta \in \mathcal{M}_q(G)\}$ where $\mathcal{M}_q(G)$ denotes the set of maximal subgroups in G of class $< q$, and the morphisms are the set of inclusions of the form $M_\alpha \cap M_\beta \to M_\alpha$.

- Let $q \geq 2$ and G be a finite group such that $\mathcal{P}_q(G)$ is a tree. Then the space $B(q, G)$ is a $K(\pi, 1)$.

$B(q, G)$ and the $K(\pi, 1)$ problem: Examples

- Wild guess: If G is of finite of odd order, then $E(2, G)$ is a $K(\pi, 1)$ where π is free and finitely generated.

- Describe

  $$H_1(B(2, G))$$

  in case G is discrete of odd order.

- How does the geometry of the covering space

  $$E(2, G) \to B(2, G)$$

  inform on the group theory of G ?

Thank you very much.

- Please remember to hand in the homework !

References

A partial list of references is appended below. Start with some classics on classifying spaces.

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