Topological obstructions to totally skew embeddings

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Configuration Spaces: Geometry, Combinatorics and Topology

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Introduction

Introduction

- M. Ghomi, S. Tabachnikov, 2007

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Given a manifold M^n, what is the smallest dimension $N(M^n)$ such that M^n admits a totally skew embedding in $\mathbb{R}^N$?

Introduction

Definition 1. *Two lines in an affine space are called* skew *if their affine span has dimension* 3. *More generally a collection of affine subspaces* $U_1, \ldots, U_l$ *of* $\mathbb{R}^N$ *are called* skew *if their affine span has dimension* $\dim(U_1) + \ldots + \dim(U_l) + l - 1$.

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Definition 2. *For a given smooth n-dimensional manifold M^n, an embedding $f : M^n \longrightarrow \mathbb{R}^N$ is called* totaly skew *if for each two distinct points $x, y \in M^n$ the affine subspaces $df(T_xM)$ and $df(T_yM)$ of $\mathbf{R}^N$ are skew. Let $N(M^n)$ be the minimum N such that there exists a skew embedding of M^n into $\mathbf{R}^N$.*

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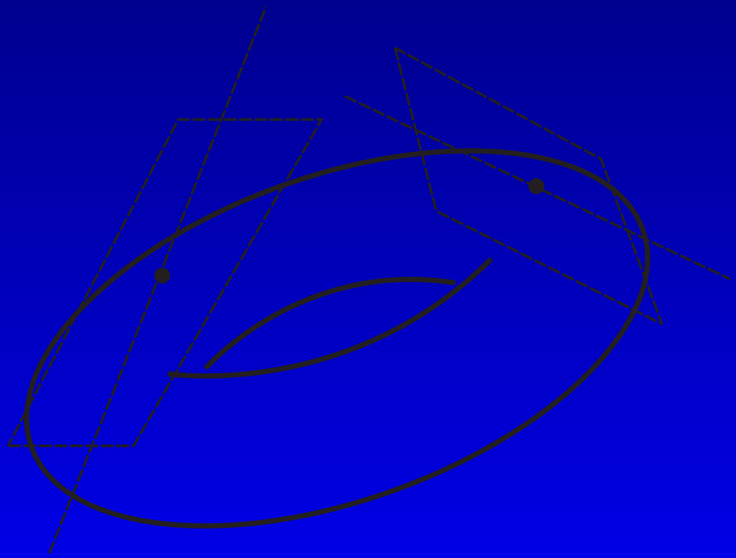

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Example 2. $\mathrm{R} \hookrightarrow \mathrm{R}^3 \qquad t \to (t, t^2, t^3)$

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Theorem 1. *For any manifold* M^n,

$$2n + 1 \leq N(M^n) \leq 4n + 1.$$

Indeed, generically any submanifold $M^n \subset \mathbf{R}^{4n+1}$ *is totally skew. Further, if* M^n *is closed, then* $N(M^n) \geq 2n + 2$.

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Theorem 2. $N(S^n) \leq 3n + 2.$

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- CGTA team: G. Stojanović, S. Vrećica, R. Živaljević, Đ. Baralić

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Vector bundle reduction

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Let $F_2(M) := M^2 \setminus \Delta_M$ be the configuration space (manifold) of all distinct ordered pairs of points in M. The tangent bundle $T(F_2(M))$ admits a splitting

$$T(F_2(M)) \cong \pi_1^* TM \oplus \pi_2^* TM \qquad (1)$$

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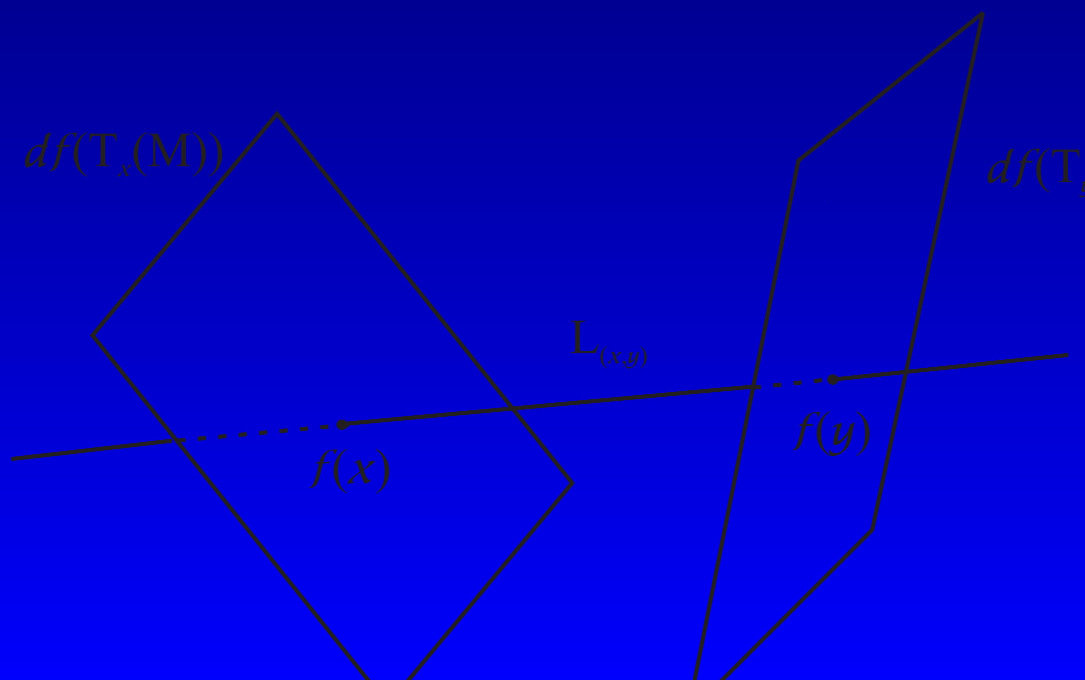

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If $f : M^n \to \mathbb{R}^N$ is a totally skew embedding, then there arises a monomorphism of vector bundles

$$\Phi = \Phi^{(1)} \oplus \Phi^{(2)} : T(F_2(M)) \oplus \varepsilon^1 \longrightarrow F_2(M) \times \mathbb{R}^N$$

where $\Phi^{(1)}_{(x,y)} : T_x(M) \oplus T_y(M) \to \mathbb{R}^N$ is the map defined by $\Phi_{(x,y)}(u, v) = df_x(u) + df_y(v)$ and $\Phi^{(2)}$, defined by $\Phi^{(2)}(\lambda) = \lambda(f(y) - f(x))$, maps the trivial line bundle ε^1 to L.

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In this case the trivial N-dimensional bundle ε^N over $F_2(M)$ splits

$$\varepsilon^N \cong T(F_2(M)) \oplus \varepsilon^1 \oplus \nu$$

Vector bundle reduction

Vector bundle reduction

Proposition 1. If the dual Stiefel-Whitney class

$$\overline{w}_k(T(F_2(M))) := w_k(\nu) \in H^k(F_2(M))$$

is non-zero, then $2n + k + 1 \leq N$. In particular, $N(M) \geq 2n + k + 1$.

Characteristic classes of $T(F_2(M))$

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We are interested in the (dual) Stiefel-Whitney classes so by naturality, in order to check non-triviality of $\overline{w}_k(T(F_2(M)))$, it is sufficient to check if the class $\overline{w}_k(T(M^2))$ is in the image of the map α.

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The image $A := \mathrm{Image}(\alpha)$ of α is generated, as a $H^*(M)$-module, by the "diagonal cohomology class"

$$u'' = \sum_{i=1}^{r} b_i \times b_i^{\sharp}$$

where $\{b_i\}_{i=1}^{r}$ is an additive basis of $H^*(M)$ and $b_i^{\sharp}$ the class dual to b_i.

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$$A = \text{Image}(\alpha) = H^*(M) \cdot u''$$

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Proposition 3. Let M be an n-dimensional manifold, let $w_k \in H^k(M; \mathbb{Z}/2)$ be its highest non-trivial Stiefel-Whitney class, and let $k \leq n-1$. Then $w_k w'_k \notin Im(\alpha)$.

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Theorem.

$$N(P^n) \geq 4 \cdot 2^{[\log_2 n]} - 1 \qquad (2)$$

Other results

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- Grassmannians

Thank you for attention!