Resonance varieties of arrangement complements

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Cohomology algebra as a cochain complex

Definition

Let M be a topological space and $A = H^*(M)$ its graded cohomology algebra (the coefficients can be apriori in an arbitrary field but for later use we assume they are in $\mathbb{C}$). We write $A^p = H^p(M)$.

For every $x \in A^1$ recall that $x^2 = 0$ whence the multiplication by x defines the differential $A \to A$ of degree +1, i.e., converts A into the cochain complex that we denote by (A, x).

Resonance variety

The main object of the talk is the cohomology $H^p(A, x)$ as a function of x.

From now on we assume that the linear spaces A^p are finite dimensional.

Definition

A *p*-th resonance variety $R^p = R^p(M)$ is the (determinantal) subvariety of A^1 defined as $R^p = \{x \in A^1 | H^p(A, x) \neq 0\}$ (cohomology jumping loci).

Very short history of the beginning

As far as I know the cohomology $H^*(A, x)$ appeared first for compact manifolds M as the first sheet of so-called Farber - Novikov spectral sequence converging to the cohomology of M with local coefficients.

The first (vanishing and comparison) results about this cohomology for arrangement complements appeared in 1991-1995; soon later M.Falk coined the term.

History

At the beginning, the resonance varieties of arrangement complements were mainly considered due to their connections with the jumping loci for the cohomology with local coefficients; the most recent results about these connections have been obtained by DImca. Now resonance variety appear in many areas of arrangement theory. For instance, their most recent appearance is in a couple of papers (authors: Budur-Dimca-Saito and Budur-Saito-Y) where these varieties have been used for results on the Milnor fiber cohomolgy and roots of *b*-functions.

There are also several recent papers (see this week lectures) where some properties of resonance varieties for arrangement complements were analyzed from the point of view which properties of the spaces imply them. This has allowed the authors to generalize them to other topological spaces.

Arrangement complements

The rest of the paper will be devoted to the case where *M* is the complement in a finite dimensional linear space *V* ($V \approx \mathbb{C}^\ell$) to an arrangement $\mathcal{A}$ of several hyperplanes. We will always assume that $\mathcal{A}$ is *essential*, i.e., $\bigcap_{H \in \mathcal{A}} H = 0$. If we consider several arrangements at the same time we will use the symbol $A(\mathcal{A})$ for *A*.

The cohomology of such an M is determined by theorems of Arnold and Brieskorn.

We suppose that the arrangement $\mathcal{A}$ is linear. For each hyperplane $H \in \mathcal{A}$ fix a linear form α_H with $\ker \alpha_H = H$. Then A can be identified with the subalgbera of the algebra of all the (holomorphic) differential forms on M generated by the logarithmic forms $\frac{d\alpha_H}{\alpha_H}$ $(H \in \mathcal{A})$.

The classes e_H of these forms form a canonical basis of A^1 whence for every $x \in A^1$ we have $x = \sum_{H \in \mathcal{A}} x_H e_H$ for some (unique)$x_H \in \mathbb{C}$.

If a linear order is fixed on $\mathcal{A}$ then we can identify x with the n-tuple $(x_1, \ldots x_n)$ $(x_i \in \mathbb{C})$.

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Relations

Suppose $S \subset \bar{n}$ and $e_S = \prod_{i \in S} e_i$ (in some fixed order) is a 'monomial' in A. If the set $\{\alpha_i | i \in S\}$ is linearly dependent then the following relation for e_i is easy to check

$$\prod_{i=2,\dots,|S|} (e_1 - e_i) = 0.$$

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Properties of $R^p(M)$

There are several properties of varieties R^p for arrangement complements that hold for all p, $0 \leq p \leq \ell = \dim A$.

(i) (linearity of components) R^p is almost always reducible. Its irreducible components are linear subspaces of A^1.

(ii) (*sufficient condition for vanishing)* If $\sum_{H \in \mathcal{A}} x_H \neq 0$ then $R^p = 0$ for all p (we will write $R^* = 0$).

(iii) (*equivalent condition for vanishing)* $R^* \neq 0$ if and only if $\sum_{H \in \mathcal{A}_j} x_H = 0$ for every $j = 1, \ldots, r$ where $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_r$ and every $\mathcal{A}_j$ is not a product of non-empty arrangements (the product $\mathcal{A}_1 \times \mathcal{A}_2$ of two arrangements is defined naturally; in particular $A(\mathcal{A}_1 \times \mathcal{A}_2) = A(\mathcal{A}_1) \otimes_{\mathbb{C}} A(\mathcal{A}_2)$).

(iv)(propagation of cohomology) If $H^p(A, x) \neq 0$ for some p then $H^q(A, x) \neq 0$ (i.e., $R^p \subset R^q$) for every q, $p \leq q \leq \ell$.

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About proofs

The known proofs of the properties are of different difficulties. The easiest property is (ii). It is a consequence of the existence of a differential $\partial : A \to A$ of degree -1 that satisfies the signed Leibniz formula and is normed by the condition $\partial(e_H) = 1$ for every $H \in \mathcal{A}$. Then $\frac{1}{\sum_H x_H}\partial$ is a contracting homotopy for (A, x). If $\mathcal{A}$ is not a product of non-empty arrangements than the converse of (ii) follows from the non-vanishing of the Euler characteristic of $\ker \partial$. This and the fact $sing(\mathcal{A}_1 \times \mathcal{A}_2) = sing(\mathcal{A}_1) \times sing(\mathcal{A}_2)$ implies (iii).

Fur property (i) the idea used in at least two papers is to view the cohomology of (A, x) as a linearization of cohomology of another complex.

Propagation

Finally, for property (iv) has been proved as a corollary of the linearity of the minimal injective resolution for A viewed as an E-module.
A direct linear algebra proof of the propagation for $p = 1$ has been given by J.V. Pereira and goes as follows.

Proof.

Assume that $a \in A^1$ satisfies $H^1(A, a) \neq 0$ and $H^p(A, a) = 0$, for some $1 < p \leq \ell$. Let $b \in A^1 \setminus \mathbb{C}a$ with $ab = 0$. Since a and b can be identified with some rational one-forms on $\mathbb{C}^\ell$, there exists a rational function h such that $b = ha$. This immediately implies that the cocycle spaces for a and b in all degrees r coincide: $Z_a^r = Z_b^r$. In particular, this is true for $r = p - 1$ and $r = p$. Since by assumption $B_a^p = Z_a^p$, where B_a^p is the space of coboundaries of degree p for a, we obtain that
$\dim B_b^p = \dim B_a^p = \dim Z_a^p = \dim Z_b^p$ whence $B_b^p = B_a^p$. $\square$

Propagation

Proof.

Continued. Now consider the isomorphisms $\phi_a, \phi_b \colon C = A^{p-1}/Z_a^{p-1} \to B_a^p$, given by multiplication by a and b, respectively, and the automorphism $\phi = \phi_a^{-1}\phi_b$ of C. If $c \in A^{p-1}$ is such that its projection to C is an eigenvector of ϕ, then we have $bc = \lambda ac$ for some $\lambda \in \mathbb{C}^*$. But since also $bc = hac$, we have $h = \lambda$, which contradicts the choice of b. □

No proof of this kind is known for arbitrary p.

The first resonance variety

For the first resonance variety R^1 more results are known.

First we projectivize the linear space and study an arrangement of projective hyperplanes in the complex projective space. The cohomology algebra of the porjectivized complement (that we still denote by A and call the Orlik-Solomon algebra) is the graded subalgebra of the cohomology algebra of M generated by $e_i - e_j$ for $i \neq j$.

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Local components

Here we describe the simplest components of R^1 (trivial in some sense).

Suppose $P \in \mathbb{P}^2$ is a point where k lines of $\mathcal{A}$ intersect, $k \geq 3$. Denote by $e_1, \ldots, e_k$ the cohomology classes of the respective logarithmic forms. Then the linear subspace

$$V_P = \{x = \sum_{i=1}^{k} x_i e_i | \sum x_i = 0\}$$

of A^1 is a component (of dimension $k-1$) of R^1.

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Nets in $\mathbb{P}^2$

Now we study the non-local components. Our goal is to state the theorem that gives at least two different characterizations of them. For that we need to define the terms to be used.

First we discuss some special configurations of lines and points in $\mathbb{P}^2$.

Definition

A finite set L of lines partitioned in k blocks $L = \cup_{j=1}^{k} L_j$ is a *k-net* if for every point P which is the intersection of lines from different blocks there is a precisely one line from each block passing through P.

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Numerical properties of nets

The following numerical properties of a k-net are almost obvious. In order to state them we denote by $\mathcal{X}$ the set of all points of intersection of lines from different blocks.

(i) For arbitrary $1 \leq i, j \leq k$ and $\ell \in L$ we have
$|L_i| = |L_j| = \mathcal{X} \cap \ell$.
The latter integer is denoted by d and the net is called
(k, d)-net.
(ii) $|L| = kd$.
(iii) $|\mathcal{X}| = d^2$.

Examples of nets

Nets can be defined purely combinatorially using an incidence relation. Then after identifying two blocks of a (k, d) -net with each other, every other block gives a Latin square of size d and these squares are pairwise orthogonal. If $k = 3$ identifying all blocks gives a multiplication table of a quasi-group.

A $(k, 1)$-net consists of k lines passing through a point with each block consisting of one line . This case is considered to be trivial. Clearly $(k, 1)$ nets correspond to local components of R^1.

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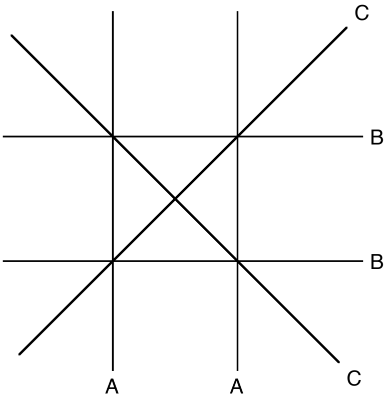
The combinatorial nets that can be realized in $\mathbb{P}^2$ form a very special class (e.g., see the restrictions on k below). However there are plenty of examples of 3-nets. The simplest nontrivial one is given by all the reflection lines of the Coxeter group of type A_3. In appropriate coordinates the blocks can be described by the equation

$$[(x^2 - y^2)][(x^2 - z^2)][(y^2 - z^2)] = 0.$$

This is essentially the only example of a $(3,2)$-net.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Algebraic nets.

There is a general way to construct examples of $(3, d)$-nets for every d.

If H is a group of order d. We say that a $(3, d)$-net realizes H if there is a way to identify all the blocks of the net to obtain the multiplication table of H.

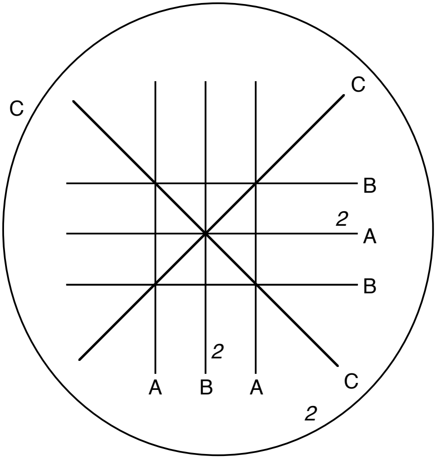
Theorem

Let H be a finite subgroup of a two dimensional torus. Then there exists a 3-net in $\mathbb{P}^2$ realizing H.

A net is called algebraic if the points that are dual to the lines of the net lie on a plane cubic.

Theorem

Let H is a finite Abelian group that is either cyclic or it has at least one element of order greater than 9. Then every realization of H by a 3-net in $\mathbb{P}^2$ is algebraic.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Multinets

Definition

(1) A multi-arrangement of lines is an arrangement L of lines together with a multiplicity function $m : L \to \mathbb{Z}_{>o}$. For every $\ell \in L$ the integer $m(\ell)$ is the multiplicity of ℓ.

(2) Let k be an integer, $k \geq 3$. A (k, d)- multinet is a multi-arrangement (L, m) with L partitioned into k blocks $L_1, \dots, L_k$ subject to the following conditions:

(i) Let $\mathcal{X}$ be (as for nets) the set of the intersections of lines from different blocks; then for each $P \in \mathcal{X}$ the number $n(P) = \sum_{\ell \in L_i, P \in \ell} m(\ell)$ is independent on i;
This number is called the multiplicity of P;

(ii) For every $1 \leq i \leq k$ and $\ell, \ell' \in L_i$ there exists a sequence $\ell_0 = \ell, \ell_1, \dots, \ell_r = \ell'$ such that $\ell_{j-1} \cap \ell_j \notin \mathcal{X}$, for all $j = 1, 2, \dots, r$.

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(2) The more technical axiom (ii) guarantees that the partition is the finest for given (L, m) and $\mathcal{X}$. Due to this axiom, the blocks L_i are the equivalent classes of the transitive hull of the relation $\ell \sim \ell'$ if $\ell \cap \ell' \notin \mathcal{X}$.

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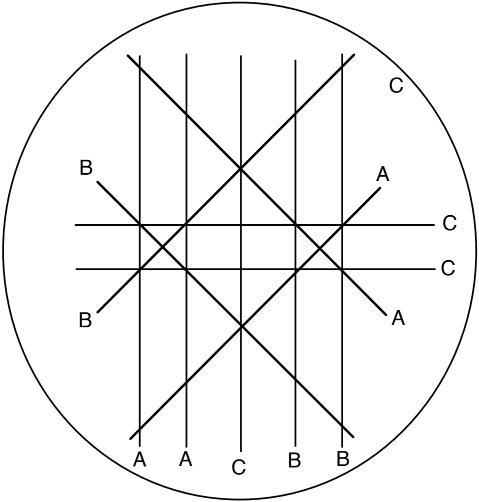
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Equalities and one example

Similarly to nets multinets satisfy simple numerical equalities.
(1) $\sum_{\ell \in L_i} m(\ell) = d$ independently of i;
(2) $\sum_{\ell \in L} m(\ell) = dk$;
(3) $\sum_{P \in \mathcal{X}} n(P)^2 = d^2$;
(4) $\sum_{P \in \mathcal{X} \cap \ell} n(P) = d$ for every $\ell \in L$.

The simplest and motivating example of a multinet that is not a net consists of all reflection lines of the Coxeter group of the type B_3. It can be given by the equation

$$[x^2(y^2 - z^2)][y^2(x^2 - y^2)][z^2(x^2 - y^2)] = 0$$

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Pencils of plane curves

Now we briefly discuss the second ingredient for the characterization of R^1 - pencils of plane algebraic curves.

We will identify a homogeneous polynomial in three variables with the projective plane curve it defines and abusing the language call it either a polynomial or a curve.

A pencil of plane curves is a line in the projective space of homogeneous polynomials of some fixed degree. Thus any two distinct curves of the same degree generate a pencil, and conversely a pencil is determined by any two of its curves C_1, C_2. An arbitrary curve in the pencil (called a fiber) is then $aC_1 + bC_2,\ [a:b] \in \mathbb{P}^1$.

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Pencils of Ceva type

Every two fibers in a pencil intersect in the same set of points $\mathcal{X} = C_1 \cap C_2$, called the base of the pencil. If fibers do not have a common component, e.g., if they are irreducible, then the base is a finite set of points.

A curve of the form $\prod_{i=1}^{q} \alpha_i^{m_i}$, where α_i are different linear forms and m_i iare positive integer, for $1 \leq i \leq q$, will be called completely reducible. A pencil is called connected if the proper transform of its every fiber stays connected after blowup at $\mathcal{X}$. We are interested in connected pencils with at least three completely reducible fibers. For brevity we say that such a pencil is of Ceva type.

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Main theorem

Now we can give a characterization of the resonance variety R^1,

Theorem

Let (L, m) be a multi-arrangement of lines in $\mathbb{P}^2$ and fix for each line a defining linear polynomial α_ℓ. The following are equivalent:

(i) There exists a partition on (L, m) that gives a (k, d)-multinet;

(ii) L is the union of all the factors of k completely reducible fibers of a pencil of Ceva type of degree d;

(iii) There is an irreducible component of R^1 of dimension $k - 1$

.

Details for the theorem

More precisely
(i)⇔(ii) For the partition from (i) let $L_1, \dots L_k$ be its blocks. Then the curves $C_i = \prod_{\ell \in L_i} \alpha_\ell^{m(\ell)}$ are fibers of the pencil from (ii) (generated by any two of them);

(ii)⇒(iii) The cohomology classes in A^1 of the logarithmic forms $\frac{dC_i}{C_i} - \frac{dC_1}{C_1}$, $i = 2, 3, \dots, k$ (see above), form a basis of the component of R^1 from (iii).

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Examples

We start with the example of a multinet (of type B_3) above and change the exponent 2 to an arbitrary $d \geq 2$. What we obtain are examples of $(3, d)$-multinets that we denote by N_d. This can be seen easier if we consider first the respective pencil of Ceva type.
Indeed the curves $x^d - y^d$, and $y^d - z^d$, generate the pencil (the Fermat pencil) of Ceva type with the third completely rerducible fiber $x^d - z^d$.

The other classical example of a pencil of Ceva type is the Hesse pencil of cubics generated by $x^3 + y^3 + z^3$ and xyz. This pencil has four completely reducible fibers, each of which is the product of three distinct lines. The resulting $(4, 3)$-net has 12 lines and $|\mathcal{X}| = 9$. $\mathcal{X}$ can be realized as the set of all inflection points of a smooth irreducible cubic. It is the only known example of a $(4, d)$-net for any d (see below).

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Upper bound of k

Theorem

A Ceva pencil of degree $d > 1$ cannot have more than four completely reducible fibers.

Due to the main theorem this result can be also formulated in at least two other equivalent ways.

(i) For a (k, d) -multinet in $\mathbb{P}^2$ if $d > 1$ then $k < 5$.

(ii) Every non-local irreducible resonance component has dimension either two or three.

Again while there are plenty of examples with $k = 3$, the Hesse pencil is the only example of a Ceva pencil with 4 completely reducible fibers whence the only known $(4, d)$-multinet. The only possible examples would be nets with $d \geq 7$.

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Another piece of history

This or weaker inequalities have been proved by different teams of authors. The first result $k \leq 5$ was proved in [A.Libgober and Y, 2000] for nets (not explicitly defined there) using the pencil corresponding to a components of R^1.

The same result for multinets with all $m(\ell) = 1$ was proved directly in [Falk and Y, 2007]. Then this inequality was proved in general case in [Pereira and Y, 2007].

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Open problems

1. Give a direct proof of the propagation property of the resonance varieties R^p.

2. Give a more constructive description of irreducible components of R^p for $p > 1$. In particular, how are they related to linear systems of subsurfaces?

3. Describe all the groups that can be represented by a 3-net in $\mathbb{P}^2$.

Conjecture. These are all the finite subgroups of $PGL(2, \mathbb{C})$.

4. Find other 4-nets besides the Hesse configuration.
Conjecture. They do not exist.
The conjecture has been proved for d=4,5,6.

5. Is every multinet the limit of a family of nets (in $\mathbb{P}^2$)?
For instance, $(3, 2d)$-multinet N_d ($d = 2, 3, \ldots$) considered above has this property. The easiest way to see this is to consider an arrangement of planes in $\mathbb{P}^3$ given by the polynomial

$$(x_0^d - x_1^d)(x_2^d - x_3^d)(x_0^d - x_2^d)(x_1^d - x_3^d)(x_0^d - x_3^d)(x_1^d - x_2^d).$$

Intersection of this arrangement with a generic (projective) plane gives a $(3, 2d)$-net in $\mathbb{P}^2$. Moving this plane to $x_3 = 0$ makes the intersection approach the multinet N_d.