Generalized Moment-Angle Complexes, Lecture 2

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joint work with Tony Bahri, Martin Bendersky, and Sam Gitler

Outline of the lecture:

This lecture addresses the following points.

- Sketches of proofs for some of the decompositions in Lecture 1 are given.
- Homological consequences of these decompositions are given.
- A short synopsis of the development of this subject (especially since the referee took special care with this information).

Ingredients:

- Let $(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)_{i=1}^{m}\}$ denote a set of triples of CW–complexes with base-point x_i in A_i.
- Let K denote an abstract simplicial complex with m vertices labeled by the set

$$[m] = \{1, 2, \dots, m\}.$$

- The main subject of these lectures is the structure of generalized moment-angle complexes

$$Z(K; (\underline{X}, \underline{A})).$$

The special case of $K = \Delta[m-1]$ the $(m-1)$-simplex:

- Recall that the generalized moment-angle complex is the union

$$Z(K;(\underline{X},\underline{A})) = \bigcup_{\sigma \in K} D(\sigma) = \text{colim } D(\sigma)$$

where D_σ is defined next.

- For every σ in K, let

$$D(\sigma) = \prod_{i=1}^{m} Y_i, \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if} \quad i \in \sigma \\ A_i & \text{if} \quad i \in [m] - \sigma \end{cases}$$

with $D(\emptyset) = A_1 \times \cdots \times A_m$.

- Thus if $K = \Delta[m-1]$, then by definition

$$Z(K;(\underline{X},\underline{A})) = \bigcup_{\sigma \in K} D(\sigma) = \text{colim } D(\sigma) = X_1 \times \cdots X_m.$$

Three basic points follow directly:

- (1) The generalized moment-angle complex

$$Z(K;(\underline{X},\underline{A}))$$

is a colimit (a union in this case).

- (2) In case $K=\Delta[m-1]$ the $(m-1)$-simplex, then

$$Z(K;(\underline{X},\underline{A}))=X_1\times\cdots\times X_m$$

which, by Theorem 1, admits a **natural homotopy equivalence**

$$H:\Sigma(X_1\times X_2\times\cdots\times X_m)\to\Sigma(\bigvee_{1\leq i_1<i_2\cdots<i_k\leq m}X_{i_1}\wedge\cdots\wedge X_{i_k}).$$

Three basic points follow directly continued:

- (3) Basic decompositions of the suspension of a moment-angle complex arise as wedge decomposition of smash moment-angle complexes.

  Thus the smash moment-angle complex is fundamental for the structure of the suspension as well as the structure of the homology.

Extensions to generalized moment-angle complexes to Theorem 2:

- In the special cases for which (X_i, A_i, x_i) are pointed triples of CW-complexes, Theorem 2 follows by naturality:

  Let K be an abstract simplicial complex with m vertices. Given $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^{m}$ where (X_i, A_i, x_i) are pointed triples of CW-complexes, the homotopy equivalence of Theorem 1 induces a

  natural pointed homotopy equivalence

$$H : \Sigma(Z(K; (\underline{X}, \underline{A}))) \to \Sigma(\bigvee_{I \subseteq [m]} \widehat{Z}(K_I; (\underline{X_I}, \underline{A_I}))).$$

Extensions to generalized moment-angle complexes to Theorem 2 continued:

- First observe that Theorem 2 is a restatement of Theorem 1 in the special case of

$$D(\sigma) = Y_1 \times \cdots \times Y_m.$$

- Next observe that the decompositions of Theorem 1 are natural. Thus these decompositions are compatible for all $D(\sigma) = Y_1 \times \cdots \times Y_m$ for all σ in K.

Extensions to generalized moment-angle complexes to Theorem 2 continued:

- Recall that the homotopy colimit of a diagram of spaces $D(P)$ is defined as

$$\text{hocolim}(D(P)) = \{\coprod_{p \in P} \Delta(P_{\leq p}) \times D(p)\}/\sim$$

  where $\sim$ is the natural equivalence relation obtained from the natural inclusions.
- The proof given below for the stable decompositions of moment-angle complexes relies on work of Welker-Ziegler-Živaljević on homotopy colimits of diagrams of spaces stated next.

Extensions to generalized moment-angle complexes to Theorem 2 continued:

- Let $D(P)$ be a diagram over P having the property that the map
$$\operatorname{colim}_{q>p} D(q) \quad \hookrightarrow \quad D(p)$$
is a closed cofibration. Then the natural projection map
$$\pi(D) : \operatorname{hocolim}(D(P)) \longrightarrow \operatorname{colim}(D(P))$$
induced by the projection
$$\Delta(P_{\leq p}) \times D(p) \quad \longrightarrow \quad D(p)$$
is a homotopy equivalence.

Extensions to generalized moment-angle complexes to Theorem 2 continued:

- The spaces $Z(K;(\underline{X},\underline{A})$ is the colimit of the $D(\sigma)$.
- In the special cases for which (X_i, A_i, x_i) are pointed triples of finite CW-complexes, the natural map

$$\pi(D) : \mathrm{hocolim}(D(\sigma)) \longrightarrow \mathrm{colim}(D(\sigma)) = Z(K;(\underline{X},\underline{A}))$$

  is a homotopy equivalence.
- The decomposition given in Theorem 2 follows from commutation with the decomposition maps H by naturality.

Extensions to generalized moment-angle complexes to Theorem 2 continued:

- The statement of Theorem 2 follows. Let K be an abstract simplicial complex with m vertices. Given $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ where (X_i, A_i, x_i) are pointed triples of CW-complexes, the homotopy equivalence of Theorem 1 induces a

natural pointed homotopy equivalence

$$H : \Sigma(Z(K; (\underline{X}, \underline{A}))) \to \Sigma(\bigvee_{I \subseteq [m]} \widehat{Z}(K_I; (\underline{X_I}, \underline{A_I}))).$$

Caution:

- Theorem 2 does not identify the homotopy types of the smash moment-angle complexes

$$\widehat{Z}(K_I; (\underline{X_I}, \underline{A_I})).$$

- More work is required.
- One example, Theorem 6, is developed next.

Identification of the smash moment-angle complex in a special case álaTheorem 6:

- Recall the Theorem 6:

  Let K be an abstract simplicial complex with m vertices and $(\underline{X}, \underline{A})$ have the property that all the A_i are contractible. Then there is a homotopy equivalence

  $$\widehat{Z}(K; (\underline{X}, \underline{A})) = \begin{cases} * & \text{if } K \text{ is not the simplex } \Delta[m-1], \text{ and} \\ \widehat{X}^{[m]} & \text{if } K \text{ is the simplex } \Delta[m-1]. \end{cases}$$

Identification of the smash moment-angle complex in a special case álaTheorem 6 continued:

- Theorem 2 and Theorem 6 have the following consequence.
- Let K be an abstract simplicial complex with m vertices and $(\underline{X}, \underline{A})$ have the property that all the A_i are contractible. Then there is a homotopy equivalence

$$\Sigma(Z(K; (\underline{X}, \underline{A})) \to \Sigma(\bigvee_{\sigma \in K} \widehat{X}^{\sigma})$$

where $\widehat{X}^{\sigma} = X_{i_1} \wedge \cdots \wedge X_{i_k}$ for $\sigma = (i_1, \ldots, i_k) \in K$.

Identification of the smash moment-angle complex in a special case álaTheorem 6 continued:

- This section gives a sketch of the proof of Theorem 6.
- Since each A_i is assumed to be contractible, the smash moment-angle complexes

$$\widehat{Z}(K;(\underline{X},\underline{A}))$$

are contractible for all K not equal to the $(m-1)$-simplex $\Delta[m-1]$.
- Furthermore,

$$\widehat{Z}(\Delta[m-1];(\underline{X},\underline{A}) = \widehat{X}^{[m]}.$$

- Theorem 6 follows.

A further application of Theorem 6:

- Theorem 6 will be used next to work out the cohomology ring of $Z(K;(\underline{X},\underline{A})$ with some natural hypotheses.
- Consider the natural inclusion

$$\iota : Z(K;(\underline{X},\underline{A}) \to X_1 \times \cdots \times X_m$$

together with the quotient space

$$W(K;(\underline{X},\underline{A})) = (X_1 \times \cdots \times X_m)/Z(K;(\underline{X},\underline{A}).$$

- The inclusion ι is one of closed CW-complexes. Thus, the map ι is a cofibration with cofibre

$$W(K;(\underline{X},\underline{A})).$$

A further application of Theorem 6:

- The suspension of the natural inclusion

$$\Sigma(\iota) : \Sigma(Z(K; (\underline{X}, \underline{A})) \to \Sigma(X_1 \times \cdots \times X_m)$$

  is split by Theorem 6.
- Since the inclusion ι is one of closed CW-complexes, the map ι is a cofibration with cofibre

$$W(K; (\underline{X}, \underline{A})).$$

- Furthermore, the cofibre of this inclusion is split after suspendng and is a wedge of smash products of the X_i.

Applications of Theorem 6 to cohomology:

- A finite set of path-connected spaces $X_1, \ldots, X_m$ is said to satisfy the **strong form** of the Künneth Theorem over R provided that the natural map

$$\otimes_{1 \leq j \leq k} H^*(X_{i_j}; R) \to H^*(X_{i_1} \times \ldots \times X_{i_k}; R)$$

  is an isomorphism for every sequence of integers $1 \leq i_1 < i_2 < \ldots, i_k \leq m$.
- Assume throughout this section that the cohomology ring $H^*(X; R)$ satisfies the natural strong form of the Künneth theorem for the cohomology of X. Thus the natural map

$$H^*(X; R)^{\otimes m} \to H^*(X^m; R)$$

  is an isomorphism.

Cohomological consequences:

- Assume that the cohomology ring $H^*(X; R)$ which satisfies natural strong form of the Künneth theorem for the cohomology of X so that the natural map

$$H^*(X; R)^{\otimes m} \to H^*(X^m; R)$$

is an isomorphism.

Cohomological consequences:

- With these assumptions, define the **generalized Stanley-Reisner ideal**

$$I(K) \subset H^*(X;R)^{\otimes m}$$

as the two-sided ideal generated by all elements

$$x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_r}$$

for which $x_{j_t} \in \bar{H}^*(X_{j_t};R)$ and the sequence $J = (j_1, \ldots, j_r)$ is not a simplex of K.

Cohomological consequences:

- Let K be an abstract simplicial complex with m vertices and let

$$(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^{m}$$

be m pointed, connected CW-pairs. If all of the A_i are contractible, and coefficients are taken in a ring R for which either

1. R is a field, or
2. the cohomology of X with coefficients in R satisfies the strong form of the Künneth Theorem.

There is an isomorphism of algebras

$$(\bigotimes_{i=1}^{m} H^*(X_i; R)/I(K) \to H^*(Z(K; (\underline{X}, \underline{A})); R).$$

Cohomological consequences:

- The cohomology rings of the Davis-Januskiewicz spaces as well as $Z(K;(S^1,*))$ had been computed earlier.
- The main feature to note is that the structure arising from the suspension of these spaces forces the structure of the associated cohomology rings for many of the spaces $Z(K;(X,A))$.

Some history: 1

- The spaces $Z(K;(D^2,S^1))$ are at the confluence of work of many people. A short introduction to a small sample of some of this work is given next.

Some history: 2

- Generalized moment-angle complexes, have been studied by topologists since the 1960's thesis of G. Porter. In the 1970's E. B. Vinberg and in the late 1980's S. Lopez de Medrano developed some of their features.

Some history: 3

- In seminal work during the early 1990's, M. Davis and J. Januszkiewicz introduced quasi-toric manifolds, a topological generalization of projective toric varieties which were being studied intensively by algebraic geometers. They observed that every quasi-toric manifold is the quotient of a moment-angle complex $Z(K;(D^2,S^1))$ by the free action of a real torus.
- Namely, a quasi-toric manifold M is given by the quotient

$$M = Z(K;(D^2,S^1))/T^d$$

where

$$T^d \subset T^m$$

is a sub-torus of T^m, and where T^d acts freely on $Z(K;(D^2,S^1))$.

Some history: 4

- Let R denote the ring given by $\mathbb{Z}$, $\mathbb{Q}$, or a finite field. Given K, there is an associated ring known as the Stanley-Reisner ring of K, defined below, and denoted $R[K]$ here. The ring $R[K]$ is a quotient of a finitely generated polynomial ring $P(K) = R[v_1, \ldots, v_m]$ with generators v_i for each vertex of K and relations given by

$$v_{i_1} \cdots v_{i_k} = 0$$

  for every simplex $\sigma = (i_1, \cdots, i_k)$ in K.
- M. Hochster, in purely algebraic work, calculated the Tor-modules $\mathrm{Tor}_{P(K)}(R[K], R)$ in terms of the full subcomplexes of K. In this work Hochster also produced an algebraic decomposition of these Tor-modules.

Some history: 5

- Subsequently, and independently, Goresky-MacPherson studied the cohomology of complements of subspace arrangements $U(\mathcal{A})$ and related decompositions of their cohomology. These spaces included complements of certain coordinate subspace arrangements. A more direct proof was subsequently given by Ziegler-Zivaljević.

Some history: 6

- Later, as well as independently, Davis-Januszkiewicz introduced manifolds now called quasi-toric varieties, a topological generalization of projective toric varieties. They proved that a certain choice of Borel construction for the space $Z(K; (D^2, S^1))$ which they define precisely had cohomology ring given by $R[K]$ for $R = \mathbb{Z}$, the Stanley-Reisner ring of K. These spaces are now known as the Davis-Januszkiewicz spaces.

Some history: 7

- Buchstaber-Panov synthesized these different developments by proving that the spaces $Z(K;(D^2,S^1))$ are strong deformation retracts of complements of certain coordinate subspace arrangements $U(\mathcal{A})$ appearing earlier in work of Goresky-MacPherson. They also proved that the cohomology algebra of $Z(K;(D^2,S^1))$ is isomorphic to $\mathrm{Tor}_{P(K)}(\mathbb{Z}[K],\mathbb{Z})$ which had been considered earlier by Hochster.

Some history: 8

- There has been further, extensive work on moment-angle complexes. A few samples are Notbohm-Ray, Grbic-Theriault, Strickland, Baskakov, Buchstaber-Panov, Panov, Baskakov-Buchstaber-Panov, Buchstaber-Panov-Ray, M. Franz, Panov-Ray-Vogt, and Kamiyama-Tsukuda.
- Further elegant closely related results are due to De Concini-Procesi, Danilov, Hu, Jewell, Jewell-Orlik-Shapiro. Extensions to generalized moment-angle complexes had been defined earlier in work of Anick.
- Applications to robotics are the focus of work by Cohen-Haynes-Koditschek.

Some history: 9

- The direction of this current joint work is guided by the development in elegant work of Denham-Suciu.

Next lecture on cup products and robotics:

Thank you very much.

- Please remember to hand in the homework !