Generalized Moment-Angle Complexes, Lecture 3

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joint work with Tony Bahri, Martin Bendersky, and Sam Gitler

Outline of the lecture:

This lecture addresses the following points.

- Applications of the main decomposition theorems to derive cup-products in cohomology for generalized moment-angle complexes.
- Connections to a robotics problem described earlier.

Ingredients:

- Let $(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)_{i=1}^m\}$ denote a set of triples of CW–complexes with base-point x_i in A_i.
- Let K denote an abstract simplicial complex with m vertices labeled by the set

$$[m] = \{1, 2, \ldots, m\}.$$

- The main subject of this lecture is the relationship between the structure of the cup products in the cohomology ring of the generalized moment-angle complexes

$$Z(K; (\underline{X}, \underline{A})),$$

and the decompositions of the suspensions of the moment-angle complexes..

Methods:

- Given any space, the diagonal map

$$\Delta : Y \to Y \times Y$$

induces a map in cohomology

$$\Delta^* : H^*(Y \times Y) \to H^*(Y).$$

- The Eilenberg-Zilber map

$$H^*(Y) \otimes H^*(Y) \to H^*(Y \times Y)$$

composed with Δ^* gives a bilinear map

$$\cup : H^i(Y) \otimes H^j(Y) \to H^{i+j}(Y),$$

the cup-product in cohomology.

Two earlier examples:

- This product endows $H^*(Y)$ with the structure of a graded commutative, associative ring with identity. This ring structure has many applications in several subjects, and is basic in the world of moment-angle complexes.
- Two basic examples arose earlier as

$$H^*(Z(K;(X,A))$$

for $(X,A) = (\mathbb{CP}^\infty, *)$ or $(X,A) = (S^1, *)$ in terms of the Stanley-Reisner ring of the simplicial complex K, and variations due to Davis-Januszkiewicz, Charney-Davis, Denham-Suciu as well as others.

Stanley-Reisner rings, and their analogues:

- Let
  $$V_K$$
  denote the graded, free abelian group with generators $v_1, \cdots, v_m$ one for each vertex of K, and of **gradation two**.
- Let
  $$W_K$$
  denote the graded, free abelian group with generators (denoted ambiguously) $v_1, \cdots, v_m$ one for each vertex of K, and of **gradation one**.
- Caution: The same notation v_i is used for two different objects here.

Stanley-Reisner rings, and their analogues:

- Let
$$\mathbb{Z}[V_K]$$
denote the polynomial ring generated by the v_i (which are assumed to have gradation two).
- Let
$$I_V(K)$$
denote the Stanley-Reisner ideal, the ideal generated
$$v_{i_1} \cdots v_{i_r}$$
for every simplex $\sigma = (i_1, \cdots, i_r)$ not in K.

Stanley-Reisner rings, and their analogues:

- Given K, recall that the Stanley-Reisner ring of K, denoted

$$\mathbb{Z}[K]$$

  here is

$$\mathbb{Z}[V_K]/I_V(K),$$

  the quotient of the polynomial ring $\mathbb{Z}[V_K]$ by the relations given by

$$v_{i_1} \cdots v_{i_t} = 0$$

  for every simplex $\sigma = (i_1, \cdots, i_t)$ not in K.
- The cohomology ring of $Z(K; (\mathbb{CP}^\infty, *))$ is isomorphic to the Stanley-Reisner ring $\mathbb{Z}[K]$.

Stanley-Reisner rings, and their analogues:

- Let

  $$\Lambda[W_K]$$

  denote the exterior algebra generated by the v_i (which are assumed to have gradation one).

- Let

  $$I_W(K)$$

  denote the Stanley-Reisner ideal, the ideal generated

  $$v_{i_1} \cdots v_{i_k}$$

  for every simplex $\sigma = (i_1, \cdots, i_k)$ not in K.

Stanley-Reisner rings, and their analogues:

- Let
$$\Lambda^{\mathsf{SR}}[K] = \Lambda[W_K]/I_W(K).$$

- The cohomology ring of
$$Z(K;(S^1,*))$$
is isomorphic to
$$\Lambda^{\mathsf{SR}}[K]/ = \Lambda[W_K]/I_W(K).$$

Extensions:

- In case A is contractible, the cohomology ring of

$$Z(K; (X, A))$$

is analogous (with the assumptions that the strong form of the Künneth theorem hold).

- The purpose of the next few sections is to develop methods which extend this product structure to a broader setting.

Methods returned:

- The main goal is to understand the diagonal map

$$\Delta : Z(K;(X,A)) \to Z(K;(X,A)) \times Z(K;(X,A))$$

 on the level of cohomology.
- It suffices to give the homological properties of the suspension of the diagonal map

$$\Sigma(\Delta) : \Sigma(Z(K;(X,A))) \to \Sigma(Z(K;(X,A)) \times Z(K;(X,A)))$$

 as the induced maps determine one another (at least on the level of cohomology).

Methods returned:

- The utility of considering the map

  $$\Sigma(\Delta) : \Sigma(Z(K;(X,A))) \to \Sigma(Z(K;(X,A)) \times Z(K;(X,A)))$$

  is that there is additional structure arising from two further decompositions.
- One such arises from the earlier decompositions $\Sigma(Z(K;(X,A)))$ as a bouquet of smash moment-angle complexes.
- The second arises from the decomposition of the suspension of a product space given in Lecture 1.
- Throughout all of this, there is an assumption that the strong form of the Künneth theorem holds. There are also natural extensions to other cohomology theories as noted below.

Starting point:

- The main work for the structure of the products is to analyze the combinatorics between the stable decompositions and how these 'fit' with the diagonal map.
- This method is partially successful and gives some new results.
- However, there is still some mystery behind these combinatorics (at least to one person here).

Extensions to generalized moment-angle complexes to Theorem 2:

- In the special cases for which (X_i, A_i, x_i) are pointed triples of CW-complexes, Theorem 2 is as follows:

  Let K be an abstract simplicial complex with m vertices. Given $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ where (X_i, A_i, x_i) are pointed triples of CW-complexes, the homotopy equivalence of Theorem 1 induces a

natural pointed homotopy equivalence

$$H : \Sigma(Z(K; (\underline{X}, \underline{A}))) \to \Sigma(\bigvee_{I \subseteq [m]} \widehat{Z}(K_I; (\underline{X_I}, \underline{A_I}))).$$

Isomorphisms:

- Let K be an abstract simplicial complex with m vertices. det $(\underline{X},\underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ be pointed triples of CW-complexes.
- By Theorem 2, there is a natural isomorphism of

graded abelian groups

$$H^*(Z(K;(\underline{X},\underline{A})) \to H^*(\bigvee_{I\subseteq[m]} \widehat{Z}(K_I;(\underline{X_I},\underline{A_I})).$$

- Similarly, if $i > 0$, there is a natural isomorphism of

graded abelian groups

$$H^i(Z(K;(\underline{X},\underline{A}))) \to \oplus_{I\subseteq[m]} H^i(\widehat{Z}(K_I;(\underline{X_I},\underline{A_I})).$$

Notation:

- Let K be an abstract simplicial complex with m vertices. det $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ be pointed triples of CW-complexes.
- For brevity, write

$$Z(K_I) = Z(K_I; (\underline{X_I}, \underline{A_I})),$$

and

$$\widehat{Z}(K_I) = \widehat{Z}(K_I; (\underline{X_I}, \underline{A_I})).$$

Partial diagonal maps:

- Let K be an abstract simplicial complex with m vertices. Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ be pointed triples of CW-complexes.
- Consider the natural diagonal maps

$$\Delta : Y \to Y \times Y$$

where

$$Y = X_1 \times \cdots \times X_m.$$

Partial diagonal maps continued:

- If
$$I = J \cup L,$$
there are induced maps
$$Z(K_I) \to Z(K_J) \times Z(K_L),$$
and
$$\widehat{\Delta}_I^{J,L} : \widehat{Z}(K_I) \to \widehat{Z}(K_J) \wedge \widehat{Z}(K_L)$$

$$\widehat{Z}(K_I) \to \widehat{Z}(K_J) \wedge \widehat{Z}(K_L)$$
induced by the natural restriction of the diagonal
$$\Delta : X_1 \times \cdots \times X_m \to (X_1 \times \cdots \times X_m) \times (X_1 \times \cdots \times X_m).$$

An example of the partial diagonal maps:

- Let $I = (1, 2, 3)$, $J = (1, 2)$, and $L = (2, 3)$ with the point $(a, b, c) \in X_1 \times X_1 \times X_3$. Then the induced partial diagonal map
$$\widehat{\Delta}_I^{J,L} : \widehat{Z}(K_I) \to \widehat{Z}(K_J) \wedge \widehat{Z}(K_L)$$
is given by
$$\widehat{\Delta}_I^{J,L}(a, b, c) = ((a, b), (b, c)) \in \widehat{Z}(K_J) \wedge \widehat{Z}(K_L).$$

Partial diagonal maps continued:

- The **partial diagonal maps**

$$\widehat{\Delta}_I^{J,L} : \widehat{Z}(K_I) \to \widehat{Z}(K_J) \wedge \widehat{Z}(K_L)$$

  induce a graded commutative, associative product product on

$$\bigoplus_{I \subseteq [m]} H^i(\widehat{Z}(K_I; (\underline{X_I}, \underline{A_I})).$$

- This product is denoted $*$. Namely, if

$$u \in H^s(\widehat{Z}(K_J)),$$

  and

$$v \in H^t(\widehat{Z}(K_L)),$$

  then define

$$u * v = (\widehat{\Delta}_I^{J,L})^*(u \otimes v) \in H^{s+t}(\widehat{Z}(K_I)).$$

The cup-product:

Theorem 7

- Let K be an abstract simplicial complex with m vertices. Assume that $(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^m$ is a family of based CW-pairs. Then the product induced by the pairing

$$u * v = (\widehat{\Delta}_I^{J,L})^*(u \otimes v) \in H^{s+t}(\widehat{Z}(K_I))$$

is the cup-product.

Structure theorems for the cup-product:

Theorem 8

- Let K be an abstract simplicial complex with m vertices. Assume that $(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^m$ is a family of based CW-pairs for which

$$X_i = \Sigma(Y_i), \text{ and } A_i = \Sigma(B_i)$$

with $B_i \subset Y_i$ for all i.

- If

$$I = J \cup L$$

with $J \cap L \neq \emptyset$ with $u \in H^s(\widehat{Z}(K_J))$, and $v \in H^t(\widehat{Z}(K_L))$ for $s, t > 0$, then

$$u * v = (\widehat{\Delta}_I^{J,L})^*(u \otimes v) = 0 \in H^{s+t}(\widehat{Z}(K_I)).$$

Remark:

- The previous theorem gives conditions which guarantee the vanishing of certain cup-products.
- The theorem does not state any conclusion about non-vanishing of cup-products.

A third structure theorems for the cup-product:

Theorem 9

- Let K be an abstract simplicial complex with m vertices. Assume that $(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^m$ is a family of based CW-pairs for which

$$X_i = \Sigma(Y_i), \text{ and } A_i = \Sigma(B_i)$$

with $B_i \subset Y_i$ for all i.

- The 'ungraded rings'

$$H^*(Z(K; (\underline{X}, \underline{A})))$$

and

$$H^*(Z(K; (\underline{\Sigma^{2q} X}, \underline{\Sigma^{2q} A})))$$

are isomorphic for all integers $q > 0$.

Remark:

- The previous theorem concerning ungraded structures of the cohomology ring is the precise analogue of the case with configuration spaces of points either Euculidean space

$$H^*Conf(\mathbb{R}^n, k), \text{ and } H^*Conf(\mathbb{R}^{n+2q}, k)$$

as well as

$$H^*Conf(\mathbb{R}^n \times M, k), \text{and } H^*Conf(\mathbb{R}^{n+2q} \times M, k)$$

where M is any manifold.

Remark:

- Several questions about torsion arose during this workshop where the analogue for the cohomology of

  $$H^*Conf(\mathbb{R}^n, k)/\Sigma_k$$

  and

  $$H^*Conf(\mathbb{R}^{n+2q}, k)/\Sigma_k)$$

  are quite different in case $q > 0$ as seen in Springer Lecture Notes in Math. v. 533. For example, the p-torsion in $H^*Conf(\mathbb{R}^2, k)/\Sigma_k$ is all of order p. On the other hand, there is arbitrarily large p-torsion in the cohomology of

  $$H^*Conf(\mathbb{R}^n, k)/\Sigma_k$$

  for all $n > 2$, and k sufficiently large.

A third structure theorems for the cup-product:

- Let K be an abstract simplicial complex with m vertices. Assume that $(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^m$ is a family of pointed, based CW-pairs.
- Let K be an abstract simplicial complex with m vertices. Assume that $(\underline{CX}, \underline{X}) = \{(CX_i, X_i, x_i)\}_{i=1}^m$ is a family of based CW-pairs such that the finite product

  $$(X_1 \times \cdots \times X_m) \times (Z(K_{I_1}; (D^1, S^0)) \times \cdots \times Z(K_{I_t}; (D^1, S^0)))$$

  for all $I_j \subseteq [m]$ satisfies the strong form of the Künneth theorem. Then the cup-product structure for the cohomology algebra $H^*(Z(K; (\underline{CX}, \underline{X})))$ is a functor of the cohomology algebras of X_i for all i, and $Z(K_I; (D^1, S^0))$ for all I.

A language and context for legged robotic motion

- This section is based on joint work with Clark Haynes, and Dan Koditschek.

- The problem is to devise a practical, useful language for describing legged motion of certain robots.

Setting

- The topological ingredients are a space of positions again, the so-called moment-angle complexes.
- The interiors of cells in a cell decomposition gives 'gait states' for the legs of a legged motion.
- The purpose here is to describe the possible gait states in terms of Young diagrams, then to construct vector fields on these interiors.
- Further applications are intended.

Definition of moment-angle complex

- The moment-angle complex determined by (X, A) and K denoted $Z(K; (X, A))$ is defined as follows:
- For every σ in K, let

$$D(\sigma) = \prod_{i=1}^{m} Y_i, \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if} \quad i \in \sigma \\ A_i & \text{if} \quad i \in [m] - \sigma. \end{cases}$$

  with $D(\emptyset) = A_1 \times \cdots \times A_m$.
- The generalized moment-angle complex is

$$Z(K; (X, A)) = \bigcup_{\sigma \in K} D(\sigma) = \text{colim } D(\sigma).$$

Spaces of legs

- Let
$$Legs(m, q)$$
denote the space of ordered m-tuples in a circle S^1 with at most q "off of the ground". That means at most q of the coordinates are in the open upper hemisphere U^+ of the circle, the complement of the closed lower hemisphere E_-.
- A mathematical starting point is as follows: If $q = 2$,
$$Legs(m, 2) = Z(K; (S^1, E_-))$$
where K is the complete graph with m vertices. If $q \geq 2$ and $K = \Delta[m-1]_{q-1}$, the $(q-1)$-skeleton of the $(m-1)$-simplex, then
$$Legs(m, q) = Z(K; (S^1, E_-)).$$

Enumerating 'gait states' in spaces of legs

- This section provides a language which describes 'gait states'.
- A convenient form of this language is in terms of **Young diagrams** a construction originally invented to study the representation theory of the symmetric groups.

Young diagrams

- A **Young diagram or Ferrers diagram** is an array of $n+1$ boxes in $k+1$ rows. Filling in these boxes with all of the integers $1, 2, \ldots, n, n+1$ gives all of the **Young tableaux**.
- These Young tableaux index cells in T^n (where $n+1$ boxes is not a misprint) in a way that is intuitively meaningful as well as a setting to compute.

Young diagrams, more formally

- The set of 'filled in Young diagrams', 'Young tableaux' $Y(n+1, k+1)$ is the set of arrays

$$[a_{i,j}] = \begin{array}{ccc} a_{1,1} & \cdots & a_{1,j_1} \\ a_{2,1} & \cdots & a_{2,j_2} \\ \ldots & \ldots & \ldots \\ a_{k+1,1} & \cdots & a_{k+1,j_{k+1}} \end{array}$$

  with $n+1$ entries given by the set of all of the integers between 1, and $n+1$.

Young diagrams, more formally

- The diagrams

  $$[a_{i,j}]$$

  are also specified by their rows

  $$(R_i)$$

  with notation

  $$[a_{i,j}] = (R_i)$$

  where

  $$1 \leq i \leq k + 1.$$

Face operations in Young diagrams

- Define

$$d_i([a_{i,j}]) = d_i(R_t) = (S_q) \in Y(n+1,k) \text{ with } 1 \le i \le k$$

where

$$S_q = \begin{cases} R_q & \text{if } q < i, \\ [R_q | R_{q+1}] & \text{if } q = i < k+1, \\ R_{q+1} & \text{if } q > i. \end{cases}$$

Face operations continued

- Define

$$d_{k+1}([a_{i,j}]) = d_i(R_t) = (S_q) \in Y(n+1,k)$$

where

$$S_q = \begin{cases} [R_1|R_{k+1}] & \text{if } q = 1, \\ R_q & \text{if } 1 < q \leq k. \end{cases}$$

Cyclic permutations and face operations in Young diagrams

- Recall that C_{k+1} denotes the cyclic group generated by the $(k+1)$-cycle $t_{k+1} = (1, 2, \cdots, k+1)$.
- The operations

$$d_i : Y(n+1, k+1) \to Y(n+1, k),$$

and

$$t_{k+1} : Y(n+1, k+1) \to Y(n+1, k+1)$$

satisfy the identities

$$d_i t_{k+1} = t_k d_{i-1} \text{ if } 1 \leq i \leq k,$$

and

$$d_0 t_{k+1} = d_k.$$

‘Local flows’:

- Each gait state corresponds to the interior of a cell in the moment-angle complex.
- Flows are defined on each cell to prescribe motion.

Conclusion

- The 'gait states' in a product of circles is enumerated by the 'cyclic-Delta' set structure given above.
- The open cells correspond to all possible 'gait states'.
- The 'motions' of the boxes in the Young diagrams via the natural action of the cyclic group corresponds to motions of legs.
- One application with 'before' and 'after' slides is given next.
- The above mathematical structures are naive as well as essentially classical. The application to a problem in engineering appears to be new, and gives an efficient solution.
- The precise answers and connection with cyclic homology are still unclear.

Thank you very much.

- Where is your homework ?