Box Splines in Higher dimensions

Let V be a r-dimensional real vector space. lattice Λ.
Let $X = [a_1, a_2, \dots, a_N]$ be a sequence (a multiset) of N non zero vectors in Λ. Here $N > r$.
Zonotope $Z(X)$

$$Z(X) := \{\sum_{i=1}^{N} t_i a_i \,;\, t_i \in [0,1]\}.$$

Definition of the Box spline $B(X)(h)$:
$B(X)(h)$ is the volume of the slice of the hypercube

$$C_N := \{t_i \,;\, t_i \in [0,1]\}$$

with the affine space: $\sum_{i=1}^{N} t_i a_i = h$ (this gives r equations, and we obtain slice of dimension $N - r$)

Clearly $B(X)$ is a positive measure supported on the zonotope.
EXAMPLE $X = [1,0],[0,1],[1,1],[-1,1]$
$B_X([h_1,h_2])$ get two equations $t_1 + t_3 - t_4 = h_1$
$t_3 + t_3 + t_4 = h_2$

An hyperplane of V (dimension r) generated by a subsequence of $r-1$ elements of X is called admissible.

$V_{\mathrm{reg,aff}}$ is the complement of the union of all the translates by Λ of admissible hyperplanes.

A connected component τ of the set of regular elements will be called a (affine) alcove.

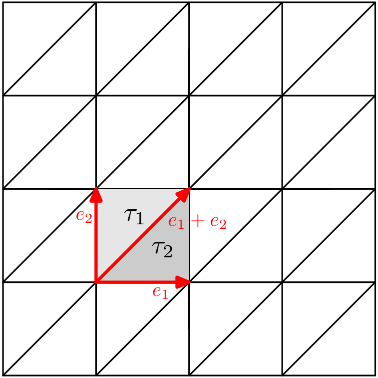

Figure: Affine alcoves for $X = [e_1, e_2, e_1 + e_2]$

The function $B_X(t)$ is given on each alcove by a polynomial function of degree $N - r$. Pictures in Procesi document.

Wonderful properties

$$\sum_{\lambda \in \Lambda} B(X)(t - \lambda) = 1.$$

Notations

We also associate to $a \in X$ three operators:

- the partial differential operator

$$(\partial_a f)(v) = \frac{d}{d\epsilon} f(v + \epsilon a),$$

- the difference operator

$$(\nabla_a f)(v) = f(v) - f(v - a),$$

- the integral operator

$$(I_a f)(v) = \int_0^1 f(v - ta) dt.$$

Series of differential operators

We define the IZ operator

$$IZ(X) = \prod_{i=1}^{N} \frac{(1 - exp(-\partial_{a_i}))}{\partial_{a_i}}$$

and its inverse

$$\textbf{Todd}(X) = \prod_{i=1}^{N} \frac{\partial_{a_i}}{(1 - exp(-\partial_{a_i}))}$$

$$= \prod_{i=1}^{N} (1 + \frac{1}{2}\partial_{a_i} - \frac{1}{12}(\partial_{a_i})^2 + \cdots)$$

We can apply $\textbf{Todd}(X)$ to the function $B(X)$ alcove by alcove. Then we take the limit "from the right", that is coming from small vectors in the cone generated by elements of X:

The Mother formula

X unimodular: Every σ basis of V extracted from X generates Λ.

$$(\mathbf{Todd}(X)B(X))_\Lambda = \delta_0$$

DEFINED BY LIMIT FROM THE $Cone(X)$ SIDE: (Unimodular case)
that is for any $\lambda \in \Lambda$ and any ϵ small regular in the cone generated by X:

- $\lambda \neq 0$

$$\lim_{t>0, t\to 0} (\mathbf{Todd}(X)B(X))(\lambda + t\epsilon) = 0$$

- $\lambda) = 0$

$$\lim_{t>0, t\to 0} (\mathbf{Todd}(X)B(X))(\lambda + t\epsilon) = 1$$

This formula implies Riemann-Roch theorem for smooth toric varities
Dahmen-Micchelli formula for Partition functions, unimodular case
Khovanskii-Pukhlikov Riemann-Roch theorem for polytopes.

Applications

Slightly more complicated formula when the system X is not unimodular.

Riemann-Roch theorem for any toric varieties

Brion-Vergne formula for number of integral points in rational convex polytopes.

and generalizations to Transversally elliptic operators: De Concini+Procesi+Vergne

The proof is based again on the relation between discrete convolution and continuous convolution.

Convolution against the Box Spline Let *test* be a smooth function of $t \in V$.

Definition

If *test* is a smooth function of $t \in V$, define

- The convolution

$$(Box(X) *_c test)(t) = \int_{u \in V} test(u) Box_N(t-u) du$$

- The semi-discrete convolution

$$(Box(X) *_d test)(t) = \sum_{u \in \Lambda} test(u) Box_N(t-u)$$

$*_c$ means usual convolution..

Then if *test* is a polynomial, we have

$$(B(X) *_c test)(t) = (\prod_{i=1}^{N} \frac{1 - e^{-\partial_{a_i}}}{\partial_{a_i}} test)(t).$$

Comparison between $B(X) *_c f$ and $B(X) *_d f$??
Analogue of the polynomials of degree strictly less than N.
Dahmen-Micchelli polynomials .

Dahmen-Micchelli polynomials

:

Definition: p is a Dahmen-Micchelli polynomial if: $\partial_Y p = 0$ for any cocircuit Y, that is Y is the complement in X of the elements in $a \in X \cap H$, where H is a hyperplane .

Terminology of Holtz-Ron: long subsets Y of X: the complement Y do not generate V.

Example $X = [1, 1, \dots, 1]$: $\frac{d}{dt}^N p = 0$: polynomials of degree $< N$.

Semi-discrete convolution

When is $B(X) *_d f$ equal to $B(X) *_c f$??: We need sufficiently many such polynomials f.

Theorem

If p is a polynomial, then

$$B(X) * -cf = \prod_i \frac{1 - e^{-\partial_{a_i}}}{\partial_{a_i}} f.$$

If p is a Dahmen-Micchelli polynomial then $B(X) *_d f$ equal to $B(X) *_c f$:

Problem very similar to Euler-MacLaurin formula.
???
I will define an EULER-MAC LAURIN formula for hyperplanes arrangement.

Euler MacLaurin formula in Dimension 1

Goal: Evaluate $\sum_{n\in\mathbb{Z}} f(n)$ for f smooth (rapidly decreasing), and compare it with the integral $\int_{\mathbb{R}} f(t)dt$

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{\mathbb{R}} f(t)dt + R_k$$

where R_k depends only of the k-th derivative of f

Bernoulli polynomials $\mathcal{B}_k(t)$

- $\mathcal{B}_0(t) = 1$
- $k > 0$ $\frac{d}{dt}\mathcal{B}_k(t) = \mathcal{B}_{k-1}(t)$
- $\int_0^1 \mathcal{B}_k(t) = 0$

I want them periodic: Take $\mathcal{B}_k$ between $[0, 1]$ and repeat it by translation

$$\mathcal{B}_k(t) = B_k(t - [t])$$

$\mathcal{B}_1(t)$ not continuous.

Bernoulli polynomials: One variable

Other definition by Fourier series:
$0 < t < 1$

$$\mathcal{B}_k(t) = -\sum_{n \neq 0} \frac{e^{2i\pi nt}}{(2i\pi n)^k}$$

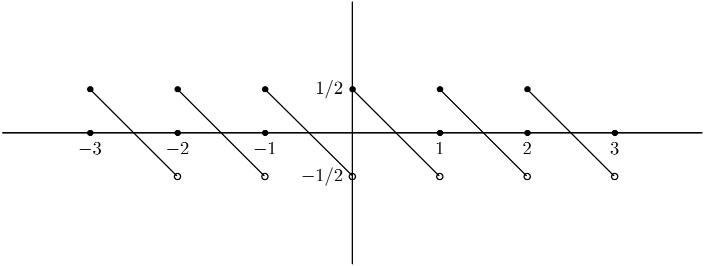

Figure: Graph of $-\mathcal{B}_1(t) = \frac{1}{2} - t + [t]$

Euler-Mac Laurin formula

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{\mathbb{R}} f(t)dtn + \int_{\mathbb{R}} \mathcal{B}_k(t)((\frac{d}{dt})^k f)(t).$$

Proof: Repeated use of the fundamental formula of Calculus $\int_a^b f' = f(b) - f(a)$.
Compute $\int_{\mathbb{R}} f'(t)\mathcal{B}_1(t)$ interval by interval. Other proof: Poisson formula.

Euler-Mac Laurin formula For Hyperplanes arrangements

Need the analog of Bernoulli periodic polynomials: some locally polynomial functions on alcoves that satisfy some equations. I will call them W not to mix them with the Box spline.. If $a \in X$, $X - a$ is another system in V, while X/a is a system in $V/\mathbb{R}a$: pull back of a function f on $V/\mathbb{R}a$ still denoted by f.

If X do not span V, define $W(X) = 0$.

If X is a basis σ , $W_X(\sum_i t_i a_i) = \det(\sigma) \prod_{i=1}^r \mathcal{B}_1(t_i)$.

Theorem

(Zagier, Szenes)

There exists unique piecewise polynomials functions satisfying:

$$\partial_a W(X) = W(X - a) - W(X/a)$$

$$\int_{V/\Lambda} W(X) = 0$$

Formula for $W(X)$

$$W(X)(v) = \sum_{\gamma \in \Gamma, <a_i, \gamma> \neq 0} \frac{e^{2i\pi\langle v, \gamma\rangle}}{\prod_{a \in X} 2i\pi\langle a, \gamma\rangle}.$$

Notation $\partial_Y = \prod_{a \in Y} \partial_a$.
Let $\mathcal{R}$ be the collection of subspaces in V spanned by some subsets of elements a_i. On $V/\mathfrak{s}$, we have the arrangement $X/\mathfrak{s}$, and the function $W(X/\mathfrak{s})$ a locally polynomial function on $V/\mathfrak{s}$. We lift it up to V by $V \to V/\mathfrak{s}$.
If $\mathfrak{s} \in \mathcal{R}$, we consider $X \setminus \mathfrak{s}$ the list of vectors in X not lying in the subspace $\mathfrak{s}$.

Theorem

(Boysal+V)

$$\sum_{\lambda \in \Lambda} f(\lambda) = \sum_{\mathfrak{s} \in \mathcal{R}} \int_V (\partial_{X-\mathfrak{s}} f)(t) W(X/\mathfrak{s})(t) dt$$

Example $X = [1, 1, ..., 1]$, then $\mathcal{R}$ consists of two elements: $\mathfrak{s} = V$ and $\mathfrak{s} = \{0\}$
Then:

$$\sum_{n \in \mathbb{Z}} f(n) = \int_V f(t) dt + \int_V f^{(k)}(t) \mathcal{B}_k(t) dt.$$

Proof: Use The most beautiful formula in mathematics: the Poisson formula

f smooth, rapidly decreasing:

Let

$$\hat{f}(y) = \int_V e^{2i\pi\langle y,x\rangle} f(x)dx.$$

THEN

$$\sum_{\lambda\in\Lambda} f(\lambda) = \sum_{\gamma\in\Gamma} \hat{f}(\gamma).$$

$$\Gamma = \Lambda^*.$$

And we group together the term of the Poisson formula.

Proof of Euler Mac Laurin formula for hyperplanes arrangement

That is instead of writing

$$\sum_{u \in \Gamma} (\hat{f})(u)$$

we group together the terms in $\Gamma = \Lambda^*$ in the same strata according to the hyperplane arrangement $a_i = 0$,
then we take the primitives that we can :
Example $\sum_{n \neq 0} e^{int} = \frac{d}{dt} \sum_{n \neq 0} \frac{e^{int}}{n}$.

A formula for semi-discrete convolution

- Recall the difference operator

$$(\nabla_a f)(v) = f(v) - f(v - a),$$

If I, J are subsequences of X, we define the operators $\partial_I = \prod_{a \in I} \partial_a$ and $\nabla_J = \prod_{b \in J} \nabla_b$.

Theorem

Let f be a smooth function on V. We have

$$B(X) *_d f - B(X) *_c f$$

$$= \sum_{\mathfrak{s} \in \mathcal{R}; \mathfrak{s} \neq V} \sum_{I \subset X \setminus \mathfrak{s}} (-1)^{|I|} B((X \cap \mathfrak{s}) \sqcup I) *_c (W(X/\mathfrak{s}) \partial_I \nabla_J f).$$

In this formula J is the complement of the sequence I in $X \setminus \mathfrak{s}$.

Proof: Use our Euler-Mac Laurin formula and
$\partial_Y B(X) = \nabla_Y B(X \setminus Y)$, if Y is a subsequence of X.

Dahmen-Micchelli polynomials

:
If p is a Dahmen-Micchelli polynomial if: For any $\mathfrak{s} \neq V$, $\partial_{X \setminus \mathfrak{s}} p = 0$.
In particular if I, J are subsets of X such that $I \cup J = X \setminus \mathfrak{s}$, then $\partial_I \nabla_J p = 0$.
Corollary:

Theorem

p Dahmen-Micchelli: Then $B(X) *_c p = B(X) *_d p$

The Mother Formula

THEOREM X unimodular

$$(Todd(X)B(X))_\Lambda = \delta_0.$$

Slightly more complicated for the general case. Proof: By induction, there exists a Dahmen Micchelli polynomial p such that $p(0) = 1$ and $p(z) = 0$ for all points in $(Z(X) - \epsilon) \cap \Lambda$. Then same proof.

Inversion formula

Let f be a function on Λ. Then

$$f(\lambda) = Todd(X)(B(X) *_d f)(\lambda)$$

Applications to representation theory

Next lecture