Aim of the talk

Classical mechanics versus quantum mechanics
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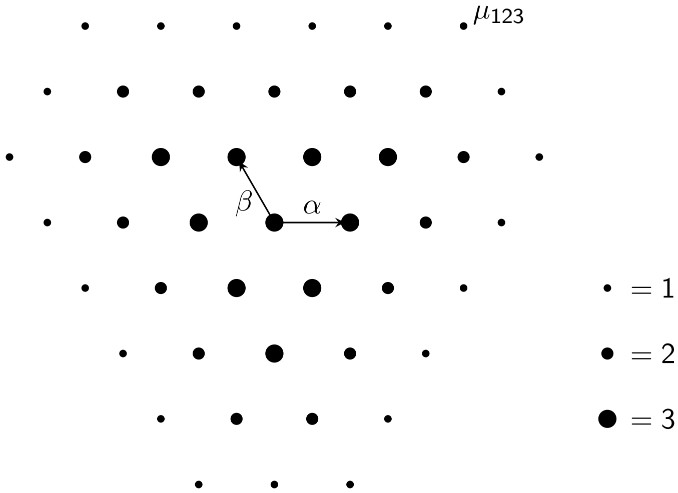
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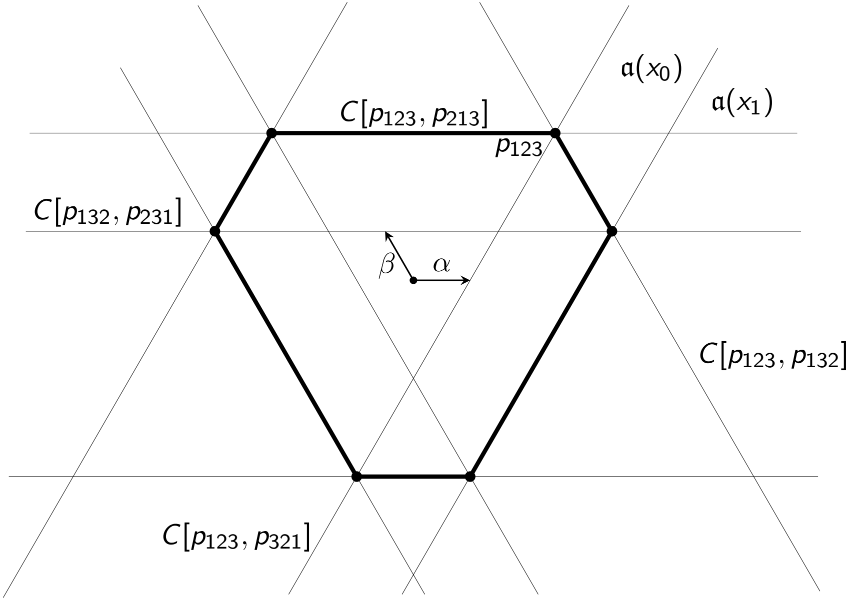

$\mathfrak{a}(x_0)$
$\mathfrak{a}(x_1)$
$C[p_{123}, p_{213}]$
p_{123}
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$C[p_{123}, p_{321}]$

Box Splines and representation theory

T is a torus with character lattice $\Lambda \subset \mathfrak{t}^*$. The torus T will be of dimension r. Its Lie algebra is $\mathfrak{t} = \mathbb{R}^r$ $H \in \mathfrak{t} := (\theta_1, \theta_2, \ldots, \theta_r)$

$$\exp H = (e^{i\theta_1}, \ldots, e^{i\theta_r}).$$

"Concrete torus" : Diagonal $r \times r$ matrices. Example $r = 3$

$$T := \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix}$$

If $\lambda \in \Lambda$, $t \in T$, use t^λ for the character of T determined by Λ.
If $H \in \mathfrak{t}$, and $t = \exp H$, I mean

$$t^\lambda = e^{i<\lambda,H>}$$

A finite representation of T is a function $m : \Lambda \to \mathbb{Z}$ with finite support

$$\Theta := \sum_\lambda m(\lambda) t^\lambda.$$

We will also consider formal characters: $m(\lambda)$ is any $\mathbb{Z}$-valued function on Λ

Example

$\mathbb{C}[z_1, z_2]$ polynomials in 2 variables, with action of $T = \{e^{i\theta_1}, e^{i\theta_2}\}$, two dimensional torus by $z_1 \to e^{i\theta_1} z_1, z_2 \to e^{i\theta_2} z_2$ then the corresponding representation of T in

$$\mathbb{C}[z_1, z_2] = \oplus_{n_1 \geq 0, n_2 \geq 0} \mathbb{C} z_1^{n_1} z_2^{n_2}$$

is the series

$$\Theta = \sum_{n_1 \geq 0} \sum_{n_2 \geq 0} e^{in_1\theta_1} e^{in_2\theta_2}.$$

It corresponds to the function identically 1 on the first quadrant.

Notations from last lectures

$X = [\alpha_1, \alpha_2, \ldots, \alpha_N]$ list of weights:

$$C_N = [0,1]^N \subset R_+^N$$

hypercube.

$$R_+^N \to \mathfrak{t}^*$$

$$(t_1, t_2, \ldots, t_N) \to \sum_i t_i \alpha_i$$

Then the Box spline $Box(X)(h)$ is the volume of the slice of C_N by $\sum_i t_i \alpha_i = h$

We need ALSO the spline $T(X)$: the volume of the slice of $\mathbb{R}_+^N$.

$T(X)(h)$ volume of $\sum_i t_i \alpha_i = h$, $t_i \geq 0$

If h is in a one dimension space, volume of the standard simplex $h^{N-1}/(N-1)!$.

Classical mechanic

M symplectic manifold of dimension $2N$: In local coordinates there is a two form $\Omega = \sum_{i=1}^{N} dp_i \wedge dq_i$.
Example $M = \mathbb{R}^2$: $M = T^*B$ cotangent bundle with coordinates $(q \in B, p \in T_q^*B)$.
Then there is a Lie algebra structure of the space of functions:
Impose $[p_i, q_j] = \delta_i^j$.
We have also the Liouville measure: β_M. In local coordinates $\beta_M = dp_1 dq_1 \cdots dp_n dq_n$.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra $f_1, f_2, \ldots f_r$ (closed under Poisson bracket) of functions on M. Here $\dim \mathfrak{g} = r$.
Then we get the moment map $\phi : M \to \mathfrak{g}^*$: evaluation at m.

$$< \phi(m), H >= H(m)$$

(H is a function on M)
A function f gives a vector field: take df and use the symplectic form:
Assume that the moment map ϕ is proper: Then we can pushforward the Liouville measure and obtain a measure on $\mathfrak{g}^*$ supported on the image of M.

The moment map ϕ for $P_1(\mathbb{C})$. Action of S^1 by rotation around the vertical axes. Correspond to the function z: (Darboux coordinates $dz \wedge d\phi$), so the vector field corresponding to z is ∂_ϕ.

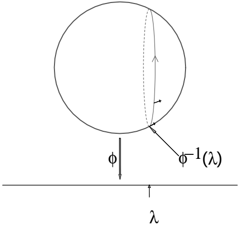


Image of the measure: the Box spline B_1 !!

We want that the Hamiltonians of the functions f_i integrate to an action of a Lie group G.

The Credo of quantum mechanics

there is a way to associate to this situation a representation of the Lie group G and that we can read everything on this representation from the moment map.

(Too simple)

Example $M = \mathbb{R}^{2N}$. Then all functions $(x_1^2 + y_1^2), \ldots, (x_N^2 + y_N^2)$. commute for the Poisson structure. Hamiltonians vector fields (dual under $dx_i \wedge dy_i$) $x_i \partial_{y_i} - y_i \partial_{x_i}$: rotations under the axes. Generates an action of the torus $T = S_1^N$.
Identify $\mathbb{R}^{2N}$ with $\mathbb{C}^N$. Take then for example $f := |z_1|^2 + |z_2|^2 + \cdots + |z_N|^2$. generate the homothety $z \to e^{i\theta} z$. Moment map $\phi(z) = \|z\|^2$ then $\phi(M)$ is supported on $t \geq 0$, and ...

$$\phi_*(\beta_M)(t) = \frac{t^{N-1}}{(N-1)!}$$

More generally, action of T a r dimensional torus on $M := \mathbb{C}^N = \mathbb{R}^{2N}$, with list of weights $X := [\alpha_1, \alpha_2, \dots, \alpha_N]$. Function corresponding to $J \in \mathfrak{t}$: The function

$$\sum_{i=1}^{N} \langle \alpha_i, J \rangle |z_i|^2$$

: will generate a weighted rotation: $\sum_i \langle \alpha_i, J \rangle (x_i \partial_{x_i} - y_i \partial_{x_i})$:

β_M Lebesgue measure.
Then $\phi : M \to \mathfrak{t}^*$ the moment map is defined by
$\phi(z) = \sum_{i=1}^{N} |z_i|^2 \alpha_i$.
The image $\phi_* \beta_M$: spline function $T(X)$ constructed on
$\alpha_1, \alpha_2, \ldots, \alpha_N$.
Indeed:
$T(X)(t)$ is the volume of the slice of the positive quadrant so again use Fubini.

M symplectic with $\mathfrak{g}$ a Lie algebra of functions on M.
EXAMPLE:

$$M = \mathbb{C}^N$$

: Diagonal action of a torus with weights all on one side:
$\mathbb{C}[z_1, z_2, \ldots, z_N]$ representation space of T with character

$$\sum_{\lambda} P_X(\lambda) t^{\lambda}$$

where $P_X(\lambda)$ is the dimension of the space of monomials $z_1^{p_1} \cdots z_N^{p_N}$ with $p_i \geq 0$, $\sum_i p_i \alpha_i = \lambda$.

$\phi : \mathbb{C}^N \to \mathfrak{t}^*$ given by

$$\phi(z) = \sum_{i=1}^{N} |z_i|^2 \alpha_i.$$

β Lebesgue measure on $M = \mathbb{C}^N = \mathbb{R}^{2N}$
Theorem

$$P_X *_d Box(X) = \phi_* \beta$$

Obvious: this means

$$\sum_{\lambda \in \Lambda} P_X(\lambda) Box(X)(t - \lambda) = T(X)(t).$$

Proof: Paving on $\mathbb{R}_+^N$ by hypercubes.

$T(X)$ is given by a polynomial function on each of the connected components of the regular values of the moment map (Cones)
Inversion formula (unimodular case)

$$P_X(\lambda) = \lim_{t \to 0, t>0} (Todd(X) T_X)(\lambda + t\epsilon)$$

ϵ in the cone generated by X:
This implies that P_X is given by a polynomial formula on the closure of these cones. (same regularity properties that the image of the measure)
P_X is obtained by applying the Todd operator to the Duistermaat-Heckman measure.

Centered Box Spline

Put the cube $C_N = [-1/2, 1/2]^N$ Then $Box_c(X)$ is supported on $\sum_i t_i\alpha_i \; -1/2 \leq t_i \leq 1/2$.

Kirillov formula

Symplectic manifold M with a Lie algebra of functions.
An example: $M = Gf \subset \mathfrak{g}^*$, a coadjoint orbit of G, is a symplectic manifold.
The restriction to M of the linear coordinates form a finite dimensional Lie algebra isomorphic to $\mathfrak{g}$.
Kirillov Orbit Method says that (under some conditions):
to M corresponds a unitary representation of G.
He proposed a "universal character formula" for the trace of this representation (1968)

Branching rules: study of the restriction of a representation of G to a subgroup H should be read from the geometry of the moment map $\phi : M \to \mathfrak{g}^*$ and on the pushforward of the Liouville measure.

Formula of Kirillov too simple: but work for all "generic cases" (With Berline, we proposed a more general formula in terms of equivariant cohomology: see European Congress of Mathematicians)

My talk here deals with the cases where Kirillov formula applies and for compact group actions (eventually on non compact symplectic manifolds with proper moment maps).

We remarked with Michel Duflo that indeed

in cases where we have Kirillov character formula, we can say something qualitative (and non trivial) on branching rules.

(More general theorems by Paradan, but difficult proofs, via Transversally elliptic operators)

"easy proof": Inversion formula for the semi-discrete convolution with the Box splines of numerical analysis !!

Here I stay in the torus case, and I give the spirit of the method.
Theorem of Duistermaat-Heckman: T a torus acting on a symplectic manifold and moment map proper, then the push forward of the Liouville measure is a locally polynomial function on $\mathfrak{t}^*$.
Meinrenken-Sjamaar, Paradan, Ma:
One can read the multiplicities of the index of the Dirac operator (version of Dolbeaut cohomology) (with boundary conditions if M is non compact) on the image of the moment map and on the Liouville measure.

Compact groups

Let K be a compact connected Lie group, T its maximal torus; $\mathfrak{k}$ the Lie algebra of K, $\mathfrak{t}$ the Lie algebra of T. $\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$ triangular decomposition. (For $U(n)$, decomposition of a matric in diagonal$+$ upper triangular$+$ lower triangular)
Δ_+ roots of T in $\mathfrak{n}^+$ EXAMPLE $(e_i - e_j), i < j$ for $U(n)$.
Then the centered Box spline $Box_c(\Delta_+)$ is a measure on $\mathfrak{t}^*$. The vertices of $Z_c(\Delta_+)$ is the convex hull of the orbit by the Weyl group W of ρ.
(SEE PAGE 19 PROCESI for B_2)

Let $O_\rho \subset \mathfrak{k}^*$ be the orbit of ρ by the group K.
This is a homogeneous space for K isomorphic to the flag variety.
Flags for $U(3)$: $L_1 \subset L_2 \subset \mathbb{C}^3$. Basic flag: coordinate subspaces

$$\mathbb{C}e_1 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3.$$

Then O_ρ has a canonical measure.

Project $O_\rho \to \mathfrak{t}^*$ by the natural projection.

Theorem

(Harish-Chandra) The image of O_ρ is the centered zonotope: $Z_c(\Delta_+)$ and the push forward of the measure is the Box Spline $Box_c(\Delta^+)$

Example $U(3)$. Identify $\mathfrak{g}^*$ with hermitian matrices by $< X, Y >= iTr(XY)$

$$\rho := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Orbit M: all hermitian matrices A
with eigenvalue $(1, 0, -1)$.
moment map: $A \to [a_{11}, a_{22}, a_{33}]$ (diagonal)

$Box_c(X)$ convolution of the measure of intervals

$$\int_{t=-1/2}^{t=1/2} e^{it<H,\alpha>} dt = \frac{e^{i<\alpha,H>/2} - e^{-i<\alpha,H>/2}}{i<\alpha,H>}$$

Thus

$$\hat{B}oc_c(X)(H) = \prod_{\alpha\in X} \frac{e^{i<\alpha,H>/2} - e^{-i<\alpha,H>/2}}{i<\alpha,H>}$$

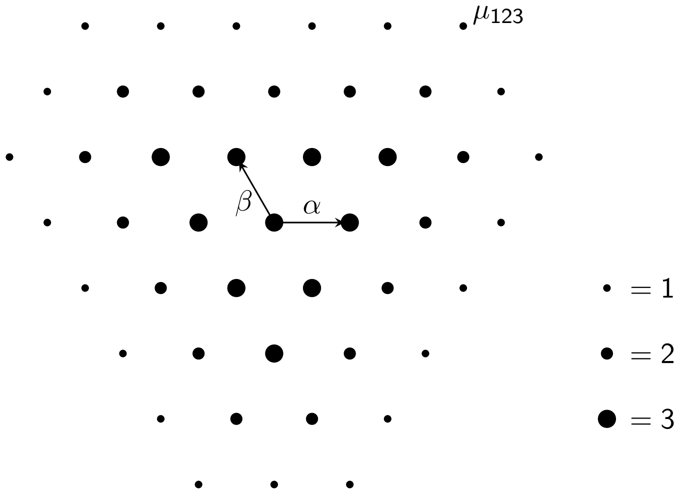
Then for X a system of positive roots,

$$\hat{B}oc_c(X)(H) = \prod_{\alpha\in X} \frac{\sum_w \epsilon(w) e^{i<w\rho,H>}}{i<\alpha,H>}$$

$M = K\lambda$ a coadjoint orbit of λ dominant regular such that $\lambda - \rho$ is in the lattice of weights.
It has a canonical Liouville measure $c_\lambda dk/dt$.
Then there exists a unique irreducible representation V_λ of K such that Hermann Weyl formula holds. $H \in \mathfrak{t}$

$$Tr_{V_\lambda}(expH) = \sum_{w\in W} \frac{\sum_w \epsilon(w) e^{i<w\lambda, H>}}{\prod_{\alpha>0} e^{i<\alpha,H>/2} - e^{-i<\alpha,H>/2}}$$

We consider χ_λ as a sum of characters.
$\chi_\lambda = \sum_\mu m^\lambda(\mu) e^{i\mu}$



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Kirillov Universal formula:

$$\chi_\lambda(\exp H)\frac{\prod_{\alpha>0} e^{i<\alpha,H>/2} - e^{-i<\alpha,H>/2}}{\prod_{\alpha>0} i < \alpha, H >} = c_\lambda \int_{K\lambda} e^{i<k\lambda,H>} dk.$$

(Universal: meaning for any Lie group G)

Let us write

$$\chi_\lambda(\exp H) = \sum_{\mu \in \hat{T}} m^\lambda(\mu) e^{i<\mu,H>}$$

We want to read the multiplicity function $\mu \to m^\lambda(\mu)$ on this formula.
We recognize all the terms

$$\sum_{\mu \in \hat{T}} m^\lambda(\mu) e^{i<\mu,H>} * \hat{B}ox_c(\Delta^+)(H)$$

$$= \int_{\mathfrak{t}^*} e^{i<f,H>} DH_\lambda(f)$$

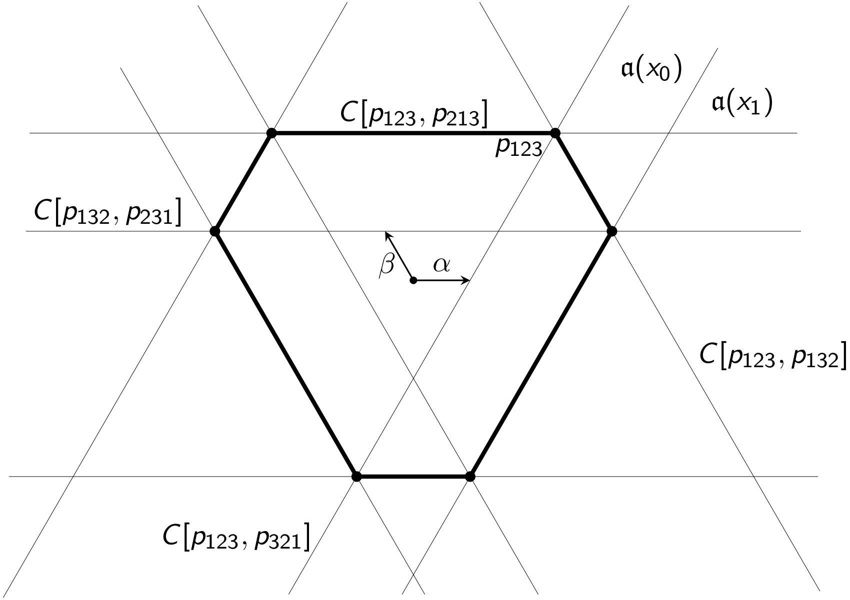
Here DH_λ (Duistermaat-Heckman) is a measure on $\mathfrak{t}^*$ supported on the convex hull of the points $w\lambda$.

Use Fourier transform.

$$\sum_{\mu} m^{\lambda}(\mu) Box(\Delta^{+})(f-\mu) = DH_{\lambda}$$

Use Inversion formula (case $U(n)$)

$$m_{\lambda}(\mu) = \prod_{\alpha>0} \left(\frac{\partial_{\alpha}}{e^{\partial_{\alpha}/2} - e^{-\partial\alpha/2}}\right) \cdot DH_{\lambda})(\mu)$$



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Philosophy

We obtain $m_\lambda(\mu)$ as a restriction to the lattice of a continuous at lattice points (centered case: no "Gibbs phenomenon" at lattice points) locally polynomial function on $\mathfrak{t}^*$ obtained by applyng the Todd operator to the Duistermaat-Heckman measure..

$$m_\lambda(\mu) = \prod_{\alpha>0} \Big(\frac{\partial_\alpha}{e^{\partial_\alpha/2} - e^{-\partial_\alpha/2}} DH_\lambda\Big)(\mu)$$

Furthermore: $m_\lambda(\mu)$ depends only of the geometry of the fiber $K\lambda \to \mathfrak{t}^*$ in μ

Same result for discrete series representations. (Duflo+Vergne) " Geometric Branching rule". Representations occuring in the restriction of a representation are associated to orbits in the image of the moment map: