A Krammer representation for complex braid groups

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1. Introduction

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$$\sigma_k . v_{ij} = \begin{cases} v_{ij} & \text{if} \quad k > i-1 \text{ or } j < k \\ v_{i-1,j} + (1-q)v_{ij} & \text{if} \quad k = i-1 \\ tq(q-1)v_{i,i+1} + qv_{i+1,j} & \text{if} \quad k = i < j-1 \\ tq^2 v_{ij} & \text{if} \quad k = i = j-1 \\ v_{ij} + tq^{k-i}(q-1)^2 v_{k,k+1} & \text{if} \quad i < k < j-1 \\ v_{i,j-1} + tq^{j-i}(q-1)v_{j-1,j} & \text{if} \quad k = j-1 \\ (1-q)v_{ij} + qv_{i,j+1} & \text{if} \quad k = j \end{cases}$$

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$$< v_{ij}, v_{kl} >= c \times \begin{cases} -q^2t^2(q-1) & \text{if} \quad i = k < j < l \text{ or } i < k < j = \\ (1-q) & \text{if} \quad k = i < l < j \text{ or } k < i < j = \\ t(q-1) & \text{if} \quad i < j = k < l \\ q^2t(q-1) & \text{if} \quad k < l = i < j \\ -t(q-1)^2(1+qt) & \text{if} \quad k < i < l < j \\ (1-qt)(1+q^2t) & \text{if} \quad k = i, j = l \end{cases}$$

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with

$$c = (t-1)(1+qt)(q-1)^2 t^{-2} q^{-3}$$

2. Preliminaries

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Fact : every reflection group is a direct product of irreducible reflection groups.

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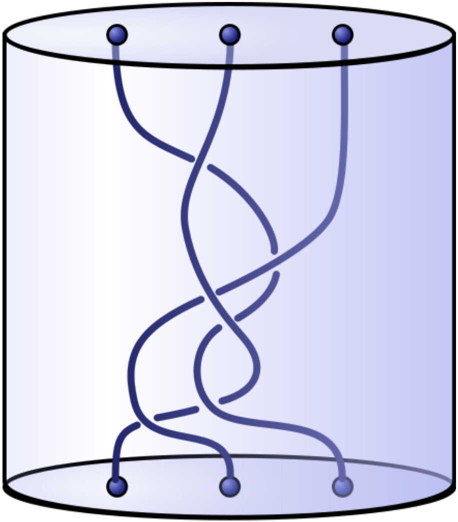
Archetype : $W = \mathfrak{S}_n$

B is the classical braid group on n strands.



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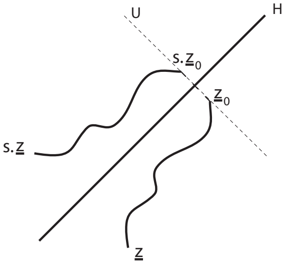

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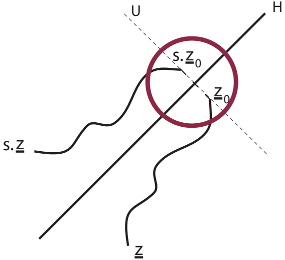

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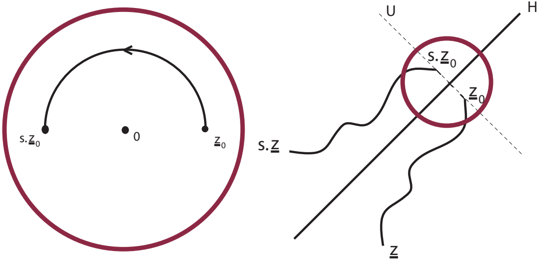

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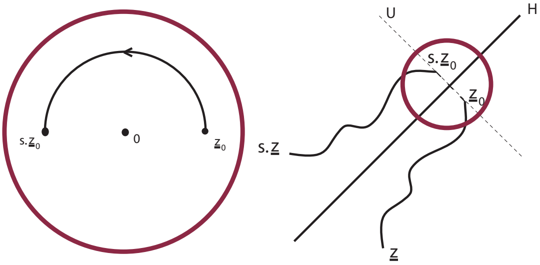

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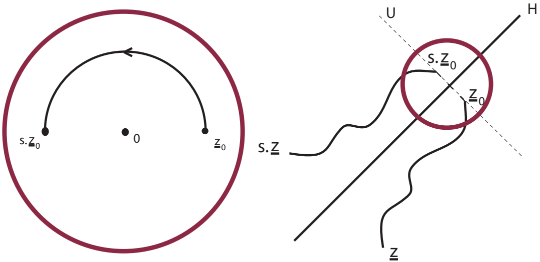

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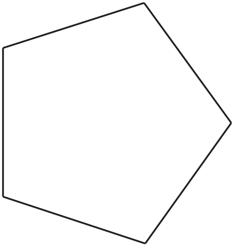

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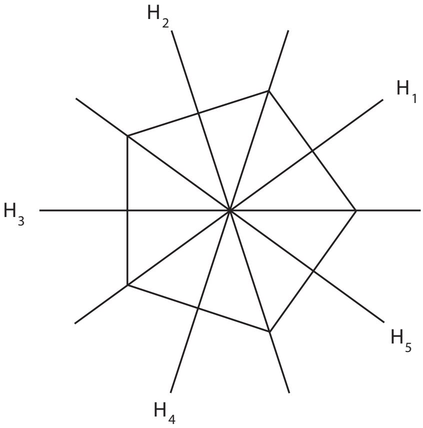

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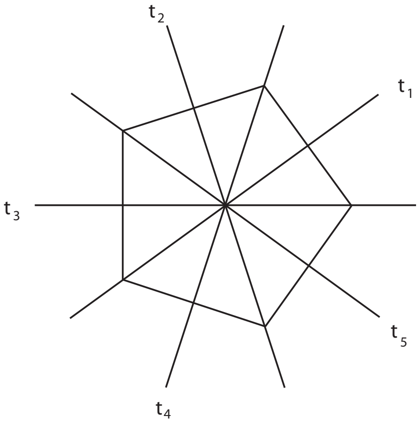

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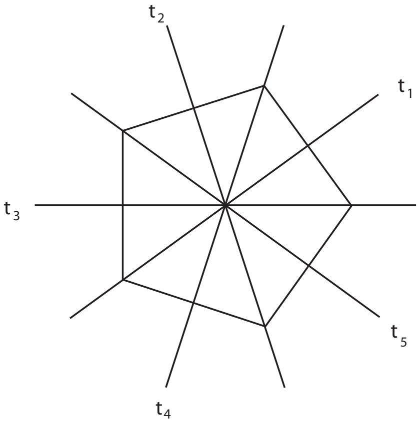

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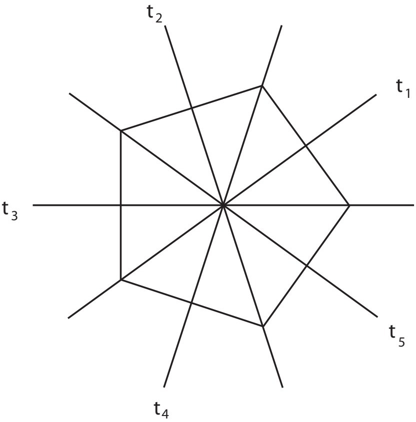

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Remark. When $W = \mathfrak{S}_n$, $\mathcal{T}$ is also known as the Lie algebra of (horizontal) chord diagrams.

3. Monodromy

Monodromy representations

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This was the only contruction known so far which worked for arbitrary complex reflection groups.

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$$U_N^{\varepsilon}(K) = \{x \in \mathrm{GL}_N(K) \mid {}^t\varepsilon(x) = x^{-1}\}$$

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Such morphisms exist.

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This conjecture is corroborated by : $X, X/W, X \to X/W$ are defined over $\mathbb{Q}$ (I.M., Jean Michel).

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If conjecture 1 (+ conjecture BMR) holds true, then the Hecke algebra representations are unitarizable when $|q| = 1$ *and* q *is close to 1.*

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Proposition

If $L \subset \mathbb{R}((h))$ is a finitely generated extension of $\mathbb{R}(h)$ such that $\varepsilon(L) = L$, then there exists $L^* \subset \mathbb{R}(\{h\})$ such that $\varepsilon(L^*) = L^*$ and $L^*/\mathbb{R}(h) \simeq L/\mathbb{R}(h)$ in a ε-equivariant way.

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Reflection representation case : Couwenberg, Heckman, Looijenga 2005.

Application 2 : Zariski closure

The monodromy construction of $H_W(q)$-representations is a consequence of the strange fact that

$$\varphi : \mathcal{T} \to \mathbb{C}W, t_H \mapsto s_H \in \mathcal{R}$$

is a Lie algebra morphism, $\mathbb{C}W$ being considered as a Lie algebra for $[a, b] = ab - ba$.

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$\mathcal{H}$ is reductive, with center of dimension the number of conjugacy classes of reflections in W.

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Type A : Freedman, Larsen, Wang 2002.

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4. Quest for a Krammer representation

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Let (P, P) be the commutator subgroup of P.

Theorem

R_h factors through $B/(P,P)$, and is faithful as a representation of $B/(P,P)$ if $h \notin \mathbb{Q}$. If $h \in \mathbb{Z}$, then R_h factors trough W. $h \mapsto R_h$ is $\kappa(W)$-periodic for some $\kappa(W) \in \mathbb{Z}_{\geq 2}$.

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(V. Beck) $\kappa(W)$ is the order of the extension

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The formulas

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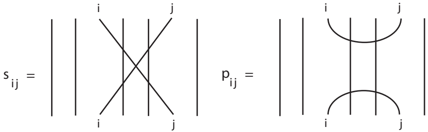

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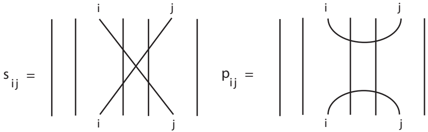

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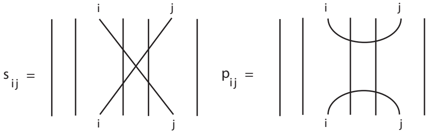

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- Among exceptional groups, only G_{13} has $\#\mathcal{R}/W > 1$, and its braid group is isomorphic to the one of Coxeter type $I_2(6)$.

5. Residual nilpotence

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Proposition

(I.M.) If W is a Coxeter group, or of type G_{25}, G_{26}, G_{32}, then P is residually torsion-free nilpotent.

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so under conjecture 2 this settles the case of $\#\mathcal{R}/W = 1$ for W a reflection group.

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This is the right one !

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If $\mathcal{A}$ is a pseudo-reflection arrangement, then $\pi_1(X)$ is residually torsion-free nilpotent.

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(Recall that residual torsion-free nilpotent groups are bi-orderable and residually p for all p.)