Analytic continuation of polytopes and wall crossing

Work in progress with Nicole Berline.

REFERENCES: Varchenko: Combinatorics and topology of the disposition of affine hyperplanes in real space: Functional Analysis and its Applications:1987.

Paradan: Wall crossing formulas in Hamiltonian Geometry *arXiv:math/0411306*

De Concini+Procesi+Vergne: Vector partition functions and generalized Dahmen-Micchelli spaces, arXiv 0805.2907.

Karshon-Tolman: The moment map and line bundles over presymplectic toric manifolds. J. diff geometry (1993)

We define a set theoretic "analytic" continuation of a simple polytope.
For the regular values of the parameter, our construction coincides with the "combinatorial connection" introduced by Varchenko on "mirages".
We study what happens to the polytope when reaching a wall, and crossing it.

Let V be a real vector space, and let $[\alpha_1, \alpha_2, \ldots, \alpha_N]$ be a sequence of linear forms on V. Consider on V an arrangement of affine hyperplanes given by the equations $\langle \alpha_j, x \rangle - z_j = 0$. Assume that for a particular value z^0, the set

$$\mathfrak{p}(z^0) = \{v; \langle \alpha_j, v \rangle \leq z_j^0, 1 \leq j \leq N\}$$

is non empty, and that $\mathfrak{p}(z^0)$ is a simple polytope.

If z varies in a neighborhood U of z^0, it is intuitively clear that the nearby polytope

$$\mathfrak{p}(z) = \{v; \langle \alpha_j, v \rangle \leq z_j, 1 \leq j \leq N\}.$$

varies "analytically" with z.
For example, the integral over $\mathfrak{p}(z)$ of a polynomial function $f(v)$ depends polynomially on z.

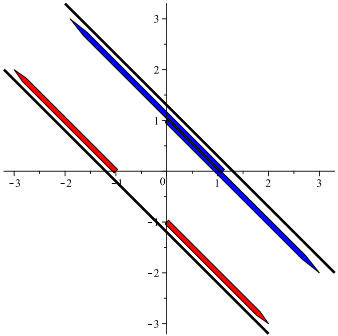
That is
there exists a polynomial function P of $(z_1, z_2, \dots, z_N)$, such that for z in a neighborhood of z^0

$$\int_{\mathfrak{p}(z)} f(v) = P(z).$$

Varchenko showed that it is possible to interpret the polynomial function $z \to P(z)$ as the integral of f on an "analytic continuation" $X(\mathfrak{p}, z)$ of the polytope $\mathfrak{p}(z)$.
Similar result for number of integral points, if the arrangement is rational.
(We will not make any such assumption however)

Here, I will define $X(\mathfrak{p}, z)$, for any $z \in \mathbb{R}^N$, prove the "continuity" of $z \to X(\mathfrak{p}, z)$, when z reaches a singular value, and describe the jump of $X(\mathfrak{p}, z)$ when z crosses a wall.

The idea beyond analytic continuation is quite simple: it follows for example from Brion's formula that the integral of a polynomial over a polytope can be computed from the geometry of the tangent cones: indeed the characteristic function of the polytope $\mathfrak{p}(z)$ is a signed combination of the indicator functions of the tangent cones to the faces of $\mathfrak{p}(z)$. So we move z, follow invidually each vertex, move the tangent cones accordingly, and look at the result.



Let $z_1 < z_2$. Consider the closed interval

$$C := [z_1, z_2] = [z_1, +\infty] + [-\infty, z_2] - [-\infty, \infty].$$

If (z_1, z_2) moves and cross each other so that now $z_2 \leq z_1$, we see that C becomes a point $\{z_1 = z_2\}$
THEN
$]z_2, z_1[$ with a minus sign.

If z_1, z_2 are integers, and we count the number of points in the closed interval $[z_1, z_2]$ for $z_1 \leq z_2$, this is the function $(z_2 - z_1) + 1$, and, if $z_2 < z_1$, the function $(z_2 - z_1) + 1$ is indeed minus the number of points in the interior of the interval (z_2, z_1).

Let us state more precisely our result.
It is easier to express it when the vector space V varies, but equations are fixed. Thus we consider a r dimensional vector space F, and $\Phi := [\phi_1, \phi_2, \ldots \phi_N]$ a sequence of elements of F. We assume that Φ span F and span a pointed cone.

Consider $V(\lambda)$ the affine space

$$\sum_{i=1}^{N} x_i \phi_i = \lambda.$$

All $V(\lambda)$ are "the same" vector space V: affine translations of the vector space $V := \{\sum_i x_i \phi_i = 0\}$ and we consider the "mirage" defined by the fixed equations $x_i = 0$ on the moving space $V(\lambda) \sim V$.

Definition

If $\lambda \in F$, define

$$\mathfrak{p}(\Phi, \lambda) := \{(x_1, x_2, \ldots, x_N); x_i \geq 0; \sum_i x_i \phi_i = \lambda\}.$$

That is this is the intersection of $V(\lambda)$ with the positive quadrant. It is a bounded polytope.

We will need intersections with other quadrants; so let $A \cup B = [1, 2, \ldots, N]$ and let

$$Q(A, B) := \{x_a \geq 0, a \in A, \qquad x_b < 0, b \in B.\}$$

Define

$$\mathfrak{q}(A, B, \lambda) = V(\lambda) \cap Q(A, B).$$

Bounded if and only if $\{\phi_a; a \in A\} \cup \{(-\phi_b), b \in B\}$ generates a pointed cone.

"walls": hyperplanes in F generated by $(r-1)$ independent vectors.

An element f which is not on one of these walls be called regular.

A connected component C_O of the open set of regular elements will be called a chamber.

Let λ_0 be regular and in the cone generated by the ϕ_i.
Then $\mathfrak{p}(\Phi, \lambda_0)$ is a simple polytope with non empty interior relatively to $V(\lambda_0)$.

Let C_0 be the chamber which contains λ_0.
We denote by $\mathcal{G}(\Phi, C_0)$ the set of subsets $I \subseteq \{1, \dots N\}$ such that C_0 is contained in the cone generated by the $\phi_i, i \in I$.
For any $I \in \mathcal{G}(\Phi, C_0)$, define

$$\mathfrak{t}(\Phi, I) = \{x \in \mathbb{R}^N, x_k \geq 0 \text{ for } k \in I^c\}. \quad (1)$$

In this equation, the set I^c is the complementary set of indices to I in $\{1, 2, \dots, N\}$.
(No condition on x_i for $i \in I$)

Let $\lambda \in C_0$. When I varies in the set $\mathcal{G}(\Phi, C_0)$, the affine cones

$$\mathfrak{t}(\Phi, I, \lambda_0) = \mathfrak{t}(\Phi, I) \cap V(\lambda_0) \quad (2)$$

describe all tangent cones to the faces of $\mathfrak{p}(\Phi, \lambda_0)$. The corresponding face is of dimension $|I| - \dim F$. Vertex if I corresponds to a feasible basis ϕ_i: $\lambda = \sum_i \lambda_i \phi_i$ with $\lambda_i \geq 0$.

Denote by $[S]$ the indicator function of a set $S \subset V(\lambda)$.
By Brianchon-Gram, the polytope is the signed sum of its tangent cones:
for $\lambda \in C_0$:

$$[\mathfrak{p}(\Phi, \lambda_0)] = \sum_{I \in \mathcal{G}(\Phi, C_0)} (-1)^{|I| - \dim F} [\mathfrak{t}(\Phi, I) \cap V(\lambda_0)].$$

Define the "analytic continuation" of $\mathfrak{p}(\Phi, \lambda_0)$ to be:

$$X(\Phi, C_0, \lambda) = \sum_{I \in \mathcal{G}(\Phi, C_0)} (-1)^{|I| - \dim F} [\mathfrak{t}(\Phi, I) \cap V(\lambda)].$$

Recall:

$$\mathfrak{t}(\Phi, I) = \{x \in \mathbb{R}^N, x_k \geq 0 \text{ for } k \in I^c\}. \tag{3}$$

and $\mathfrak{t}(\Phi, I) \cap V(\lambda)$ is always "the same cone".

Theorem

For any $\lambda \in F$, $X(\Phi, C_O, \lambda)$ is a combination of characteristic functions of bounded polytopes.
It is "continuous" on the closure: when λ is in $\overline{C}_0$, $X(\Phi, C_O, \lambda) = [\mathfrak{p}(\Phi, \lambda)]$.
(for the moment, only weaker statement: $X(\Phi, C_O, \lambda)$, for λ singular in the closure, is equal to $[\mathfrak{p}(\Phi, \lambda)]$ (not simple) modulo indicator functions of cones with lines.)

First assertion due to Varchenko for regular values, and $X(\Phi, C_O, \lambda)$ coincides with Varchenko construction on regular values.
We can think of this theorem, as a set theoretic analogue of the finiteness of the Euler characteristic for a line bundle over a complete toric manifold;
and of the continuity theorem of Boutot.

More precise wall crossing theorem later.

$\lambda - > X(\Phi, \lambda, C_0)$ is "the" analytic continuation of $\mathfrak{p}(\Phi, \lambda)$ (intially defined "analytically " for λ in the open set $\in C_0$). Indeed the cones $[\mathfrak{t}(\Phi, I) \cap V(\lambda)]$ are just affine "analytic translations" of the tangent cones to $\mathfrak{p}(\Phi, \lambda_0)$.

Let us see how $X(\Phi, C_0, \lambda)$ varies in the following example:

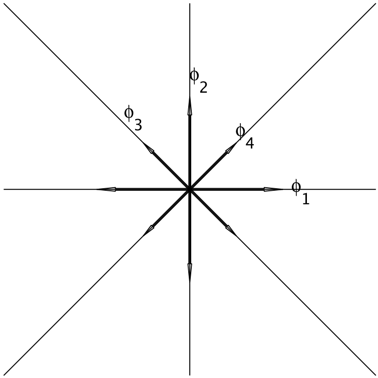


Figure: Chambers

Assume that λ_0 lies in the open conic chamber C_0 generated by ϕ_2 and ϕ_4.
To draw $\mathfrak{p}(\lambda)$, we parametrize $V(\lambda) = \{\sum_i x_i \phi_i = \lambda\} \subset \mathbb{R}^4$ by $\mathbb{R}^2$:

$$(x_1, x_2) \to [\lambda_1 + x_1, \lambda_2 + x_2, \frac{(x_1 - x_2)}{2}, -\frac{(x_1 + x_2)}{2}].$$

We start by λ in the chamber ϕ_2, ϕ_4.
$\mathfrak{p}(\lambda)$ is the quadrilateral

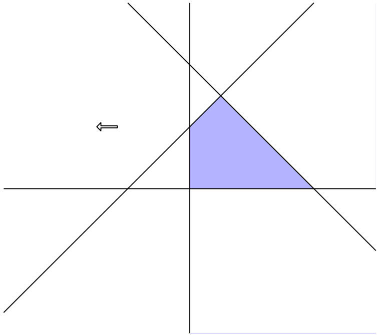


Figure: OUR START

described by the inequations $x_1 \geq -\lambda_1$, $x_2 \geq -\lambda_2$ $x_1 \geq x_2$, $x_1 + x_2 \leq 0$

Let us describe its analytic continuation, as the parameter λ visits all the chambers. When λ_1 increases (the vertical line moves to the right), we watch the jump appear. It is -with a minus sign- the semi-open triangle.

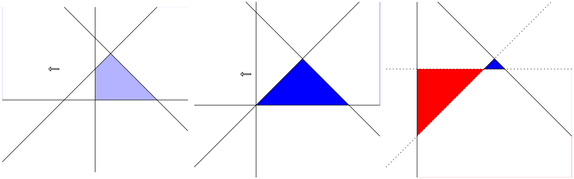


Figure: JUMP OVER THE FIRST WALL ϕ_4

We picture how vary the polytope turning in clockwise direction (there are some semi open pieces, and minus signs in red):

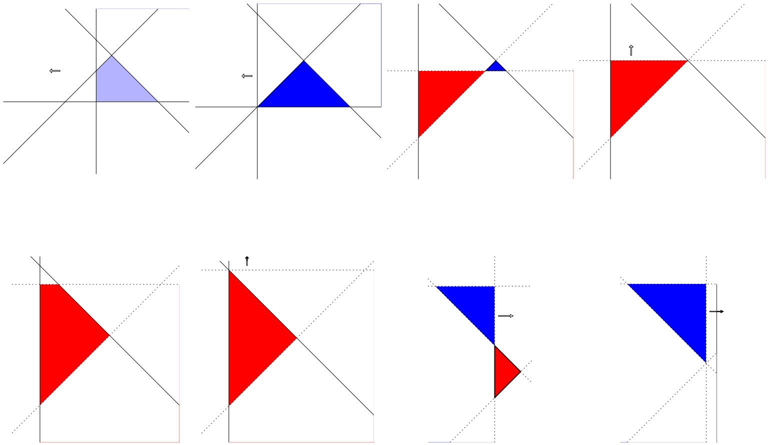
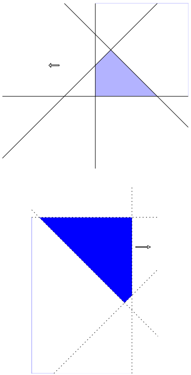


Figure: Turning around

Thus after one half turn, we go to $-\lambda$, and we picture the start and the end.



which is just the interior of the original polytope (turned by 180 degree rotation).

We now give a description of how $X(\Phi, C_O, \lambda)$ varies when crossing a wall W.
Let C_0, C_1 be adjacent chambers. We assume C_0 contained in the cone generated by the ϕ_i.

Define $B \subset \{1, 2, \ldots, N\}$ to be the set of ϕ_b such that ϕ_b is (strictly) on the side of C_1 with respect to the wall W. Let A be the complement of B in $[1, 2, \ldots, N]$. So $A \cup B = \{1, 2, \ldots N\}$
Remark that

$$\{\phi_a, a \in A\} \cup \{-\phi_b, b \in B\}$$

span a pointed cone.

Theorem

For $\lambda \in C_1$, an adjacent chamber, the analytic continuation is given by the formula:

$$[X(\Phi, C_O)(\lambda)] = [\mathfrak{p}(\Phi, \lambda)] + (-1)^{|B|}[\mathfrak{q}(A, B, \lambda)].$$

Or

$$\mathfrak{q}([1, 2, \ldots, N], \emptyset, \lambda) + (-1)^{|B|}[\mathfrak{q}(A, B, \lambda)].$$

Here $[\mathfrak{p}(\Phi, \lambda)]$ is the polytope associated to λ for the chamber C_1: while $[\mathfrak{q}(A, B, \lambda)]$ is also a polytope associated to the chamber C_1 but for a system $\Psi \subset \Phi \cup (-\Phi)$, where some of the Φ are reversed. (In the toric context where Φ are integral vectors, would correspond to a change of complex structure on $\mathbb{C}^N$)

In other-words , for $\lambda \in C_1$,
$[X(\Phi, C_O)(\lambda)]$ is the "sum" of two bounded sets: the intersection of $V(\lambda)$ with the positive quadrant
and (with some sign) the (bounded) intersection of $V(\lambda)$ with an explicit other semi-open quadrant.
(The first set $\mathfrak{p}(\Phi, \lambda)$ maybe empty, if C_1 is not contained in the cone generated Φ)

This theorem is a rephrasing in a set theoretic manner the jump formula of Paradan for number of points in polytopes. It implies it. Maybe we can get more information on $X(\Phi, C_O)(\lambda)$ for any λ, where the "zonotope" will appear: recall that the number of integral points in $X(\Phi, C_O)(\lambda)$ is given by a quasi-polynomial formula if $\lambda \in C_O - Zontope(\Phi)$ (Szenes-Vergne).
Here, we dont (yet) see this result.

Case of jumps of volumes (or Riemann Roch numbers) reductions of a vector space by a torus actions: implied by the set theoretic jump.

Question: is there a set theoretic version of the variation of the fibers of the moment map, that will explain Paradan jump formulae in the more general Hamiltonian setting.

Paradan: Wall crossing formulas in Hamiltonian Geometry *arXiv:math/0411306*

See also

De Concini+Procesi+Vergne: Vector partition functions and generalized Dahmen-Micchelli spaces, arXiv 0805.2907.

Also Examples in Boysal-Vergne (work in progress) for multiple Bernoulli polynomials of very similar jump formulae where I dont see set theoretic analogues, nor Hamiltonian analogues.