Some properties of a conjugation-free geometric presentation of fundamental groups of arrangements

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Importance and Applications of fundamental groups

- Used by Chisini, Kulikov and Kulikov-Teicher in order to distinguish between connected components of the moduli space of surfaces.

- The Zariski-Lefschetz hyperplane section theorem:

$$\pi_1(\mathbb{CP}^N \setminus S) \cong \pi_1(H \setminus (H \cap S)),$$

  where S is an hypersurface and H is a generic 2-plane. This invariant can be used also for computing the fundamental group of complements of hypersurfaces in $\mathbb{CP}^N$.

- Getting more examples of Zariski pairs: A pair of plane curves is called a *Zariski pair* if they have the same combinatorics, but their complements are not homeomorphic.

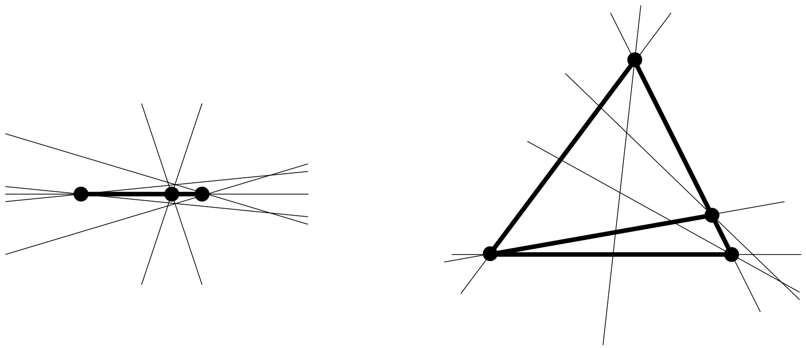
- Exploring new finite non-abelian groups which are serving as fundamental groups of complements of plane curves in general.

- Computing the fundamental group of the Galois cover of a surface: By the fundamental group of a complement of a branch curve of a surface, we can find the fundamental group of the Galois cover of the surface, with respect to a generic projection of the surface onto $\mathbb{CP}^2$.

Graph of multiple points

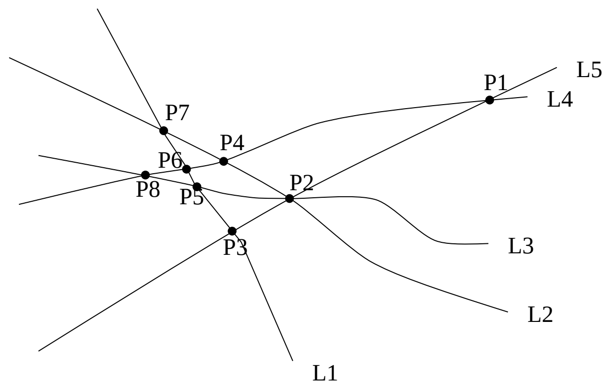
Line arrangement in $\mathbb{CP}^2$: An algebraic curve in $\mathbb{CP}^2$ which is a union of projective lines. An arrangement is called *real* if its defining equations can be written with real coefficients.

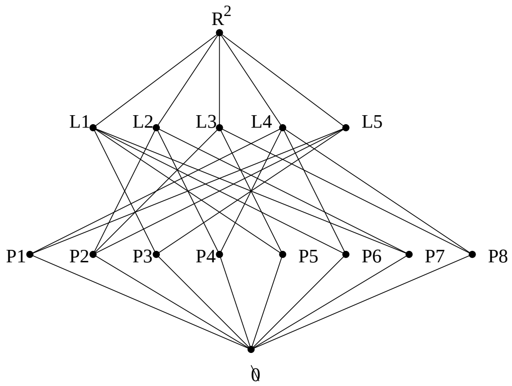
$G(\mathcal{L})$:
Vertices: Multiple points
Edges: Segments on lines with more than two multiple points.



(a) (b)

Lattice of an arrangement





A generic presentation of the fundamental group

(Arvola, Randell, Cohen-Suciu, ...)

Let $\mathcal{L}$ be an arrangement of n lines.

Then $\pi_1(\mathbb{C}^2 - \mathcal{L})$ is generated by $x_1, \ldots, x_n$ - the natural topological generators.

The relations: for each intersection point of multiplicity k:

$$x_{i_k}^{s_k} x_{i_{k-1}}^{s_{k-1}} \cdots x_{i_1}^{s_1} = x_{i_{k-1}}^{s_{k-1}} \cdots x_{i_1}^{s_1} x_{i_k}^{s_k} = \cdots = x_{i_1}^{s_1} x_{i_k}^{s_k} \cdots x_{i_2}^{s_2}$$

where $a^b = b^{-1}ab$ and s_i are words in $\langle x_1, \ldots, x_n \rangle$ $(1 \leq i \leq k)$.

A Conjugation-free geometric presentation of fundamental group

A *conjugation-free geometric presentation* of a fundamental group is a presentation with the natural topological generators $x_1, \dots, x_n$ and the cyclic relations:

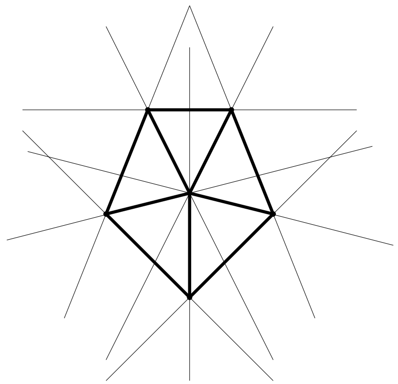
$$x_{i_k} x_{i_{k-1}} \cdots x_{i_1} = x_{i_{k-1}} \cdots x_{i_1} x_{i_k} = \cdots = x_{i_1} x_{i_k} \cdots x_{i_2}$$

with no conjugations on the generators.

Main importance: For this family, the lattice determines the fundamental group. Moreover, one can read this presentation directly from the arrangement.

Eliyahu-G-Teicher (2008): if $G(\mathcal{L})$ is a union of disjoint cycles, then $\pi_1(\mathbb{C}^2 - \mathcal{L})$ has a conjugation-free geometric presentation.

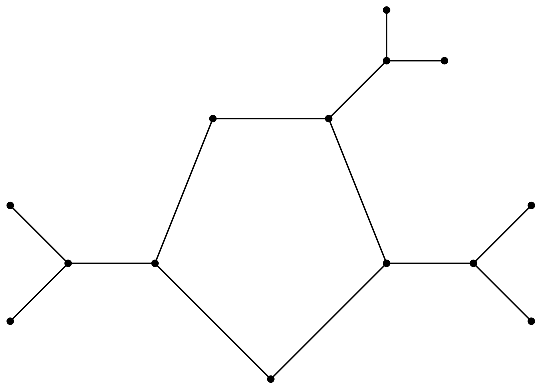
Family A_n:



Computationally proved: A_5, A_6 have a conjugation-free geometric presentation. A_3 (Ceva) and A_7 have no conjugation-free geometric presentation.

Expansion of the conjugation-free geometric presentation's family

A *cycle-tree* graph is a graph which consists of a cycle, where each vertex of the cycle can be a root of a tree.



Eliyahu-G-Teicher (2010): if $G(\mathcal{L})$ is a union of disjoint cycle-tree graphs, then $\pi_1(\mathbb{C}^2 - \mathcal{L})$ has a conjugation-free geometric presentation.

Idea of proof: Adding a line which does not close a new cycle, preserves the conjugation-free property.

Complemented presentations

Let $(\mathcal{S}, \mathcal{R})$ be a semigroup presentation: $\mathcal{S}$ is a nonempty set and $\mathcal{R}$ is a family of pairs of nonempty words in the alphabet $\mathcal{S}$ (relations).

$\langle \mathcal{S} | \mathcal{R} \rangle^{+} \cong \mathcal{S}^{*} / \equiv_{\mathcal{R}}^{+}$ is the monoid presented by $(\mathcal{S}, \mathcal{R})$.

Definition: A semigroup presentation $(\mathcal{S}, \mathcal{R})$ is called *complemented* if, for each $s \in \mathcal{S}$, there is no relation $s \ldots = s \ldots$ in $\mathcal{R}$ and, for $s, s' \in \mathcal{S}$, there is at most one relation $s \ldots = s' \ldots$ in $\mathcal{R}$.

Lemma: A conjugation-free geometric presentation is a complemented presentation.

Proof: Any pair of lines intersect exactly once, hence their corresponding generators appear as prefixes in exactly one relation. Since there are no conjugations, this is their unique appearance as a pair of prefixes.

Remarks:

1. It is not correct in general presentations of fundamental groups (due to the conjugations).
2. This property does not hold in the homogeneous minimal presentations introduced by Yoshinaga.

Reversing

Definition: For $(\mathcal{S}, \mathcal{R})$ a semigroup presentation and $w, w' \in \mathcal{S} \cup \mathcal{S}^{-1}$, *$w$ reverses to w' in one step*, denoted $w \curvearrowright^1_{\mathcal{R}} w'$, if there exist a relation $sv' = s'v$ of $\mathcal{R}$ and u, u' satisfying:

$$w = us^{-1}s'u' \text{ and } w' = uv'v^{-1}u'.$$

We say that *w reverses to w' in k steps*, denoted $w \curvearrowright^k_R w'$, if there exist words $w_0, \ldots, w_k$ satisfying $w_0 = w, w_k = w'$ and $w_i \curvearrowright^1_R w_{i+1}$ for each i. The sequence $(w_0, \ldots, w_k)$ is called an R-reversing sequence from w to w'.

We write $w \curvearrowright w'$, if $w \curvearrowright^k_{\mathcal{R}} w'$ holds for some k.

Example: if $ac = ca$, then:

$$abc^{-1}a \curvearrowright abac^{-1}.$$

Complete presentations (Dehornoy)

Definition (Dehornoy): A semigroup presentation $(\mathcal{S}, \mathcal{R})$ is called *complete* if, for all words $w, w' \in \mathcal{S}^*$:

$$w \equiv^+_{\mathcal{R}} w' \quad \Rightarrow \quad w^{-1}w' \curvearrowright_{\mathcal{R}} \varepsilon.$$

Advantages of complete presentations (Dehornoy, 2003):

• Every monoid that admits a complete complemented presentation is left-cancellative (i.e. $xy = xz \Rightarrow y = z$).

• Assume that $(\mathcal{S}, \mathcal{R})$ is a complete semigroup presentation. If $(\mathcal{S}, \mathcal{R})$ is complemented, then the monoid $\langle \mathcal{S} | \mathcal{R} \rangle^+$ admits least common multiples.

• Assume that $(\mathcal{S}, \mathcal{R})$ is a complete semigroup presentation and there exists $\widehat{\mathcal{S}} \subseteq \mathcal{S}^*$ that includes $\mathcal{S}$ and satisfies the conditions:
(1) $\forall u, u' \in \widehat{\mathcal{S}}\ \exists v, v' \in \widehat{\mathcal{S}}\ (u^{-1}u' \curvearrowright_{\mathcal{R}} v'v^{-1})$,
(2) $\forall u, u' \in \widehat{\mathcal{S}}\ \forall v, v' \in \mathcal{S}^*\ (u^{-1}u' \curvearrowright_{\mathcal{R}} v'v^{-1}\ \Rightarrow\ v, v' \in \widehat{\mathcal{S}})$.
Then every $\mathcal{R}$-reversing sequence leads in finitely many steps to a positive–negative word. If $\widehat{\mathcal{S}}$ is finite, then the word problem of the presented monoid $\langle \mathcal{S} | \mathcal{R} \rangle^+$ is solvable in exponential time, and in quadratic time if $(\mathcal{S}, \mathcal{R})$ is complemented.

If, in addition, the monoid $\langle \mathcal{S} | \mathcal{R} \rangle^+$ is right-cancellative, the word problem of the presented group $\langle \mathcal{S} | \mathcal{R} \rangle$ is solvable in exponential time, and in quadratic time if $(\mathcal{S}, \mathcal{R})$ is complemented.

- A monoid M is called *Garside* if:

(1) it is cancellative,

(2) it contains no invertible element except 1,

(3) any two elements admit a left and right least common multiples and greatest common divisors.

(4) there exist an element $\Delta \in M$ such that the left and right divisors of Δ coincide, generate M, and are finite in number.

In the case of complemented presentations:
completeness $\Rightarrow$ cancellativity, existence of least common multiples.
For having a Garside structure, we need the existence of Garside element, which can be achieved by the longest element in the smallest set of words that includes $\mathcal{S}$ and is closed under the complement and right-lcm operations (Dehornoy, 2002).

Completeness of conjugation-free geometric presentations

Eliyahu-G-Teicher (2010)

Proposition: Let $\mathcal{L}$ be a real arrangement whose fundamental group has a conjugation-free geometric presentation and $G(\mathcal{L})$ is a triangle-free graph. Then, the presentation of the corresponding monoid is complete (and complemented).

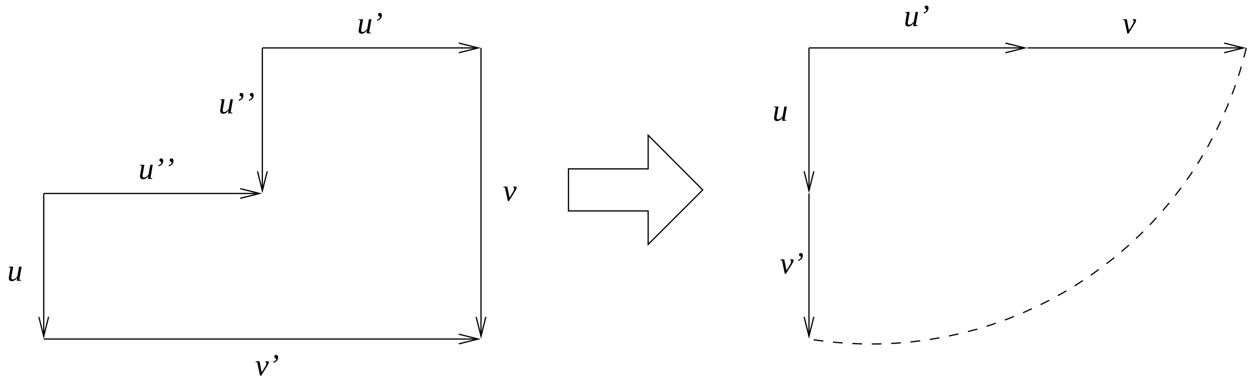
Consequences: The corresponding monoid is cancellative and has least common multiples. May help for verifying if the word problem is solvable and having a Garside structure.

Idea of proof

The cube condition: Assume that $(\mathcal{S}, \mathcal{R})$ is a semigroup presentation, and $u, u', u'' \in \mathcal{S}^*$. We say that $(\mathcal{S}, \mathcal{R})$ satisfies the *cube condition for (u, u', u'')* if:

$$u^{-1}u''u''^{-1}u' \curvearrowright_{\mathcal{R}} v'v^{-1} \qquad \Rightarrow \qquad (uv')^{-1}(vu') \curvearrowright_{\mathcal{R}} \varepsilon.$$

For $X \subseteq \mathcal{S}^*$, we say that $(\mathcal{S}, \mathcal{R})$ satisfies the cube condition on X if it satisfies the cube condition for every triple (u, u', u'') where $u, u', u'' \in X$.



Definition: A semigroup presentation $(\mathcal{S}, \mathcal{R})$ is said to be *homogeneous* if there exists an $\equiv^+_{\mathcal{R}}$-invariant mapping $\lambda : \mathcal{S}^* \to \mathbb{N}$ satisfying, for every letter $s \in \mathcal{S}$ and every word $w \in \mathcal{S}^*$,

$$\lambda(sw) > \lambda(w).$$

Proposition (Dehornoy, 2003): Assume that $(\mathcal{S}, \mathcal{R})$ is a homogeneous semigroup presentation. Then:

$(\mathcal{S}, \mathcal{R})$ is complete $\Leftrightarrow$ it satisfies the cube condition on $\mathcal{S}$.

Definition: For $(\mathcal{S}, \mathcal{R})$ a complemented semigroup presentation and $w, w' \in \mathcal{S}^*$, the *$\mathcal{R}$-complement of w' in w, denoted $w \backslash w'$*, ("w under w'"), is the unique word $v' \in \mathcal{S}^*$ such that $w^{-1}w'$ reverses to $v'v^{-1}$ for some $v \in \mathcal{S}^*$, if such a word exists.

Proposition (Dehornoy, 2003): Assume that $(\mathcal{S}, \mathcal{R})$ is a complemented semigroup presentation. Then, for all words $u, u', u'' \in \mathcal{S}^*$, the following are equivalent:
(1) $(\mathcal{S}, \mathcal{R})$ satisfies the cube condition on $\{u, u', u''\}$
(2) either $(u \backslash u') \backslash (u \backslash u'')$ and $(u' \backslash u) \backslash (u' \backslash u'')$ are $\mathcal{R}$-equivalent or they are not defined, and the same holds for all permutations of u, u', u''.

Idea of our proof: We have verified the equivalent version of the cube condition ($(u \backslash u') \backslash (u \backslash u'')$ and $(u' \backslash u) \backslash (u' \backslash u'')$ are $\mathcal{R}$-equivalent) for any triple of generators u, u', u''. In case of a triangle, it crashed.

THE END

Thank you!!!