A Tutte polynomial for toric arrangements

Luca Moci

Pisa, June, 21 2010

Real hyperplane arrangements

Take $V = \mathbb{R}^n$ and $\mathcal{H}$ a collection of affine hyperplanes.

Problem

In how many regions V is divided by the hyperplanes?

Take $H \in \mathcal{H}$ and set: $\mathcal{H}_1 \doteq \mathcal{H} \setminus \{H\}$, $\mathcal{H}_2 \doteq \{H \cap K, K \in \mathcal{H}_1\}$.
Clearly $reg(\mathcal{H})$ is obtained from $reg(\mathcal{H}_1)$ by adding the number of regions of $\mathcal{H}_1$ which are cut in two parts by H. But this number equals $reg(\mathcal{H}_2)$.
Thus we have the recursive formula

$$reg(\mathcal{H}) = reg(\mathcal{H}_1) + reg(\mathcal{H}_2).$$

This method is known as deletion-restriction.

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Complex hyperplane arrangements

If $V = \mathbb{C}^n$, removing hyperplanes does not disconnect V.
In this way we get an object $\mathcal{M}$ with a rich topology and geometry.
Then one wants to compute invariants of the complement $\mathcal{M}$.
These are related with the combinatorics of the intersection poset $\mathcal{L}$.

Problem

Compute the Poincaré polynomial $\mathcal{M}$ and the characteristic polynomial $\mathcal{L}$.

Also these polynomials can be computed by deletion-restriction.
Tutte's idea: find the most general deletion-restriction invariant. This is a polynomial $T(x, y)$. (It was originally defined for graphs).
In this talk we will introduce another kind of arrangements, and provide an analogue of the Tutte polynomial.

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An example

Take $V = \mathbb{C}^2$ with coordinates (x, y), $T = \mathbb{C}^{*2}$ with coordinates (t, s), and

$$X = \{(2, 0), (0, 3), (1, -1)\} \subset \Lambda = \mathbb{Z}^2.$$

We associate to X three objects:

1. a finite hyperplane arrangement $\mathcal{H}$ given in V by the equations

$$2x = 0, 3y = 0, x - y = 0;$$

2. a periodic hyperplane arrangement $\mathcal{A}$ given in in V by the conditions

$$2x \in \mathbb{Z}, 3y \in \mathbb{Z}, x - y \in \mathbb{Z};$$

3. a toric arrangement $\mathcal{T}$ given in T by the equations:

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Hyperplane and toric arrangements

Let X be a finite list of vectors in a lattice Λ. Assume X to span the vector space $U = \Lambda \otimes \mathbb{C}$.

A hyperplane arrangement in the complex vector space $V = U^*$ is a family of hyperplanes $\mathcal{H}_X = \{U_\lambda\}_{\lambda \in X}$, where $U_\lambda \doteq \{v \in V | \lambda(v) = 0\}$.
A toric arrangement in the complex torus $T = Hom(\Lambda, \mathbb{C}^*)$ is a family of hypersurfaces $\mathcal{T}_X = \{T_\lambda\}_{\lambda \in X}$, where $T_\lambda \doteq \{t \in T | \lambda(t) = 1\}$.

Remark: if in the previous example (i.e. $X = \{(2,0), (0,3), (1,-1)\}$) we replace $(2,0)$ by $(1,0)$ or $(5,0)$, we get the same $\mathcal{H}_X$, but different $\mathcal{T}_X$. So $\mathcal{H}_X$ depends only on the linear algebra of X, whereas $\mathcal{T}_X$ also depends on its arithmetics.
In fact $\mathcal{H}_X$ is related to a number of differentiable problems and objects, $\mathcal{T}_X$ with their discrete counterparts.

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The partition function

Problem

In how many ways an amount of k euro can be paid in 20 euro and 50 euro banknotes?

We call this number $\mathcal{P}(k)$, and we study the partition function $k \mapsto \mathcal{P}(k)$.
On every equivalence class *mod* 100, $\mathcal{P}$ is a (linear) polynomial in k.

In general, given $\lambda \in \Lambda$, we define $\mathcal{P}(\lambda)$ as the number of solutions of the equation

$$\lambda = \sum_{\lambda_i \in X} x_i \lambda_i \text{ , with } x_i \in \mathbb{N}.$$

We say that a function $\mathcal{Q} : \Lambda \to \mathbb{C}$ is quasipolynomial if there is a sublattice of Λ such that the restriction of $\mathcal{Q}$ to every coset is polynomial.
$\mathcal{P}$ is piecewise quasipolynomial.

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Differential and difference operators

For every $\lambda \in X$, let ∂_λ be the usual directional derivative

$$\partial_\lambda f(x) \doteq \frac{\partial f}{\partial \lambda}(x)$$

and let ∇_λ be the difference operator

$$\nabla_\lambda f(x) \doteq f(x) - f(x - \lambda).$$

Then for every $A \subset X$ we define the differential operator

$$\partial_A \doteq \prod_{\lambda \in A} \partial_\lambda$$

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Dahmen-Micchelli spaces

We can now define the differentiable Dahmen-Micchelli space

$$D(X) \doteq \{f : U \to \mathbb{C} \mid \partial_A f = 0 \; \forall A \text{ such that } r(X \setminus A) < n\}$$

and the discrete Dahmen-Micchelli space

$$DM(X) \doteq \{f : \Lambda \to \mathbb{C} \mid \nabla_A f = 0 \; \forall A \text{ such that } r(X \setminus A) < n\}.$$

$D(X)$ is a space of polynomials, introduced to study the *box spline*. This space is naturally graded.
$DM(X)$ is a space of quasipolynomials, arising from the *partition function*.

$D(X)$ and $DM(X)$ are deeply related respectively with the $\mathcal{H}_X$ and $\mathcal{T}_X$.

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$DM(X)$ is a space of quasipolynomials, arising from the *partition function*.

$D(X)$ and $DM(X)$ are deeply related respectively with the $\mathcal{H}_X$ and $\mathcal{T}_X$.

Dahmen-Micchelli spaces

We can now define the differentiable Dahmen-Micchelli space

$$D(X) \doteq \{f : U \to \mathbb{C} \mid \partial_A f = 0 \; \forall A \text{ such that } r(X \setminus A) < n\}$$

and the discrete Dahmen-Micchelli space

$$DM(X) \doteq \{f : \Lambda \to \mathbb{C} \mid \nabla_A f = 0 \; \forall A \text{ such that } r(X \setminus A) < n\}\,.$$

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Tutte polynomial

We recall that the Tutte polynomial associated to a list of vectors X is

$$T_X(x, y) \doteq \sum_{A \subseteq X} (x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)}.$$

This polynomial embodies a lot of information on $\mathcal{H}_X$ and $D(X)$:

1. The number of regions of the complement in $\mathbb{R}^n$ is $T_X(2, 0)$;
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Moreover $T_X(x, y)$ can be computed by deletion-restriction:

$$T_X(x, y) = T_{X_1}(x, y) + T_{X_2}(x, y)$$

where X_1 is obtained from X by removing a linearly dependent vector λ, and X_2 is the quotient of X_1 by λ.

The Tutte polynomial is the most general deletion-restriction invariant.

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Multiplicity Tutte polynomial

Problem

Define a "Tutte polynomial" for $\mathcal{T}_X$ and $DM(X)$.

Let be $X \subset \Lambda$. For every $A \subseteq X$ let us define

$$m(A) \doteq [\Lambda \cap \langle A \rangle_{\mathbb{Q}} : \langle A \rangle_{\mathbb{Z}}].$$

Then we define a multiplicity Tutte polynomial $M_X(x, y)$:

$$M(x, y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)}.$$

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Relations with $\mathcal{T}_X$ and $DM(X)$

Theorem

1. *The number of regions of the complement in $\mathbb{S}_1{}^n$ is $M_X(1,0)$;*
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$DM(X)$ is isomorphic to a direct sum of spaces $D(X_p)$, one for every "point" p of $\mathcal{T}_X$.Thus also $DM(X)$ is a graded space.

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Deletion-restriction and positivity for $M_X(x, y)$

Theorem

1. $M_X(x, y) = M_{X_1}(x, y) + M_{X_2}(x, y)$;
2. $M_X(x, y)$ *is a polynomial with positive coefficients.*

Then the coefficients "count something". What?
Still open, but we can answer for the coefficients of $M_X(1, y)$, and also of $M_X(x, 1)$...

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The zonotope

Let $U_{\mathbb{R}}$ be the real vector space spanned by the elements of X.
Then we define a convex polytope in $U_{\mathbb{R}}$

$$\mathcal{Z}(X) \doteq \left\{ \sum_{\lambda \in X} t_\lambda \lambda, 0 \leq t_\lambda \leq 1 \right\}.$$

Theorem

1. $M_X(1,1)$ *equals the volume of the zonotope* $\mathcal{Z}(X)$*;*
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