Complex Matroids

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Combinatorics of linear dependencies

Let $M := \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$ be a $d \times n$ matrix.

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Maximal independent subsets of E:
Bases of $\operatorname{Im} M$.

Fact: E contains a basis.

Theorem: Given bases $B_1, B_2 \subseteq E$, $e_1 \in B_1 \setminus B_2$, there is $e_2 \in B_2 \setminus B_1$ s.t. $(B_1 \setminus e_1) \cup e_2$ and $(B_2 \setminus e_2) \cup e_1$ both are bases of V.

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Minimal supports of nonzero elements of $\ker M$: *Circuits.*

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Consider circuits $C_1 \neq C_2$ and $e \in C_1 \cap C_2$.
With $E = \{e, v_2, \ldots, v_n\}$, this means that there are coefficients $(\lambda_i)_{\geq 2}$ and $(\mu_i)_{i \geq 2}$ with

$$\begin{array}{rcr} e + \lambda_2 v_2 + \ldots + \lambda_n v_n & = 0 & \text{(minimal lin. dep.,} \\ & & \lambda_i = 0 \text{ if } v_i \notin C_1) \\ -e + \mu_2 v_2 + \ldots + \mu_n v_n & = 0 & \text{(minimal lin. dep.)} \\ & & \mu_i = 0 \text{ if } v_i \notin C_2) \\ \hline (\lambda_2 + \mu_2) v_2 + \ldots & = 0 & \text{(lin. dep. among} \\ & & \text{elements of } C_1 \cup C_2 \end{array}$$

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there is a circuit C_3 with $C_3 \subseteq (C_1 \cup C_2) \setminus e$.

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Minimal supports of nonzero elements of $\ker M$: *Circuits.*

$\emptyset$ is not a circuit.

Given minimal dependent subsets $C_1, C_2 \subset E$ and $e \in C_1 \cap C_2$, $(C_1 \cup C_2) \setminus e$ is dependent.
So, there is $C_3 \subseteq (C_1 \cup C_2) \setminus e$.

(Circuit elimination)

Combinatorics of linear dependencies

C is minimal dependent iff
- $C \not\subseteq B$ for all bases B;
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$(B \cup e)$ contains a circuit
if and only if $e \in E \setminus B$

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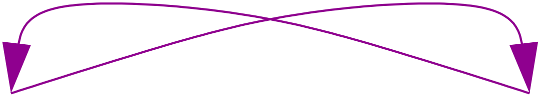

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a family $\mathscr{B}$ of subsets of E such that:

• $\mathscr{B} \neq \emptyset$

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COMBINATORICS OF LINEAR DEPENDENCIES

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Matroids

Definition. *A matroid on a finite ground set E is...*

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Two subsets $A, B \in \mathscr{F} \subseteq \mathscr{P}(E)$ are comodular if they are a modular pair of atoms in the lattice of unions of elements of $\mathscr{F}$

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Theorem [D.'09]. *A matroid on a finite ground set E is...*

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(Modular elimination axiom)

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Notice: If $\mathscr{B}$ is the set of bases of a matroid $\mathscr{M}$ on the ground set E, then

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Theorem.

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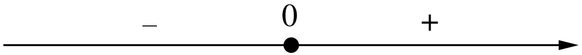
Linear dependencies over $\mathbb{R}$

There is a very natural stratification of $\mathbb{R}$ as:



LINEAR DEPENDENCIES OVER $\mathbb{R}$

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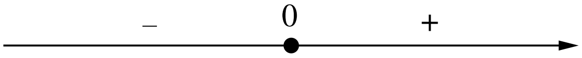
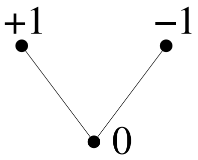


So we consider the following set of signs.

Definition. *The set* $\{-1, 0, +1\}$ *has a natural partial order coming from the stratification above*



COMBINATORICS OF LINEAR DEPENDENCIES OVER $\mathbb{R}$

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Every ordered element of $\mathscr{B}$ has a natural sign

Order $B \in \mathscr{B}$ as $\{v_1, \ldots, v_d\}$, then define

$$\chi(v_1, \ldots, v_d) := \operatorname{sgn} \det(v_1, \ldots, v_d)$$

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To every $C \in \mathscr{C}$ correspond $\lambda_i \in \mathbb{R}$ with $\lambda_1 v_1 + \ldots + \lambda_n v_n = 0$ where $\lambda_i \neq 0$ iff $v_i \in C$.

Given the λ_is, define $X : E \to \{0, \pm\}$ as

$$X(v_i) := \operatorname{sgn}(\lambda_i)$$

(Signed circuits)

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Given $\{x_0, \ldots, x_d, y_2, \ldots, y_d\} \subseteq E$,

$$\sum_{j=0}^{d} (-1)^j \det(x_0, \ldots, \widehat{x_j}, \ldots, x_d) \det(x_j, y_2, \ldots, y_d) = 0$$

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Let $P := \{(-1)^l \chi(x_0, \ldots, \widehat{x_l}, \ldots, x_d)\chi(x_l, y_2, \ldots, y_d) \mid 0 \leq l \leq d\}$.

If $P \neq \{0\}$, then $\{+1, -1\} \subseteq P$.

Signed circuits

Consider real coefficients $(\lambda_i)_{\geq 2}$ and $(\mu_i)_{i\geq 2}$ as above with

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By Carathéodory's theorem, there is

$$\nu_2 v_2 + \ldots + \nu_n v_n \quad = 0 \quad \text{(minimal lin. dep.)}$$

with $\operatorname{sgn} \nu_i \leq \operatorname{sgn}(\lambda_i + \mu_i)$.

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Given signed circuits X, Y and i, j
with $X(v_i) = -Y(v_i) \neq 0$ and $X(v_j) \neq -Y(v_j)$,
there is a signed circuit Z with $Z(v_i) = 0$, $Z(v_j) \neq 0$
and, for all i, $Z(v_i) \in \{0, X(v_i), Y(v_i)\}$.

Oriented matroids

Definition. *An oriented matroid on the finite ground set E is...*

an alternating function

$$\chi : E^d \to \{-,0,+\}$$

Such that:
For $x_0,\dots,x_d, y_2,\dots,y_d \in E$
and the set P given by
$\{(-1)^I \chi(x_0,..,\widehat{x_i},..,x_d)\chi(x_i,y_2..,y_d)\}$
either $P=\{0\}$ or $P \supseteq \{+,-\}$

(Chirotope axioms)

a subset $\mathcal{C} \subseteq \{-,0,+\}^E \setminus \underline{0}$
such that for $X,Y \in \mathcal{C}$:

- $\mathrm{supp}(X)=\mathrm{supp}(Y) \Rightarrow X = \pm Y$
- for $e,f \in \mathrm{supp}(X)\cap\mathrm{supp}(Y)$ with $X(e){=}{-}Y(e)$, $X(f){\neq}{-}Y(f)$, there is $Z\in\mathcal{C}$ with $f\in\mathrm{supp}(Z)\not\ni e$ $Z(g)\in\{0,X(g),Y(g)\}$ for all g.

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- for $e, f \in \mathrm{supp}(X) \cap \mathrm{supp}(Y)$ with $X(e) = -Y(e)$, $X(f) \neq -Y(f)$, there is $Z \in \mathcal{C}$ with $f \in \mathrm{supp}(Z) \not\ni e$ $Z(g) \in \{0, X(g), Y(g)\}$ for all g.

(Signed circuit axioms)

$\underline{\mathcal{B}} = \{\{x_1,.., x_d\} | (x_1,.., x_d) \in \mathrm{supp}(\chi)\}$ $\quad$ $\underline{\mathcal{C}} := \{\mathrm{supp}(X) \mid X \in \mathcal{C}\}$

are the set of bases, resp. circuits of the underlying matroid.

ORTHOGONALITY

Definition. *Two signed vectors* $X, Y : E \to \{-, 0, +\}$ *are orthogonal if for* $P := \{X(e)Y(e) \mid e \in E\}$*, either* $P = \{0\}$ *or* $P \supseteq \{+, -\}$.

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M^* is called dual to M, and if M represents $V \in G_{d,n}(\mathbb{R})$, then M^* represents the orthogonal complement $V^\perp \in G_{n-d,n}(\mathbb{R})$.

An oriented matroid is...

a pair of families $\mathcal{C}, \mathcal{D} \subset \{-, 0, +\}^E$ such that

- For $X, Y \in \mathcal{C}$ (or in $\mathcal{D}$), supp(X)=supp(Y)$\Rightarrow X = \pm Y$
- $\mathcal{C} = -\mathcal{C}$, $\mathcal{D} = -\mathcal{D}$
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(Axioms for dual pairs)

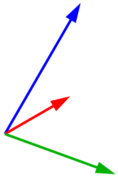
Chirotope axioms

Circuit axioms

Geometry and topology

Given a set $E = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$, consider $\Phi : \mathbb{R}^d \to \{-, 0, +\}^E$

$$\Phi : x \mapsto (\mathrm{sgn}(< x \mid v_1 >), \ldots, \mathrm{sgn}(< x \mid v_n >)).$$



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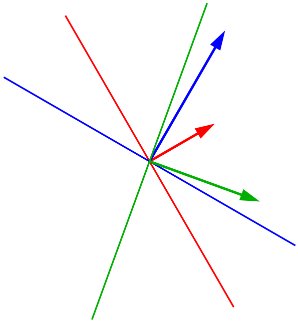

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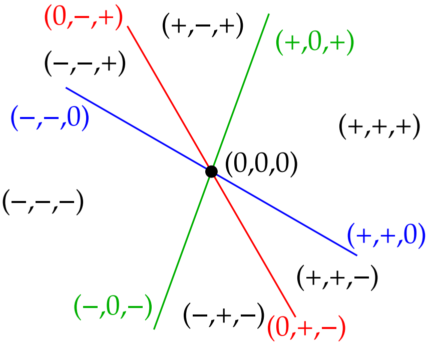

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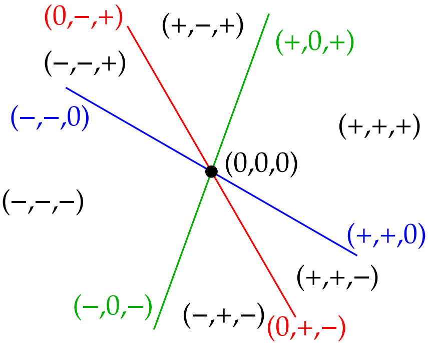

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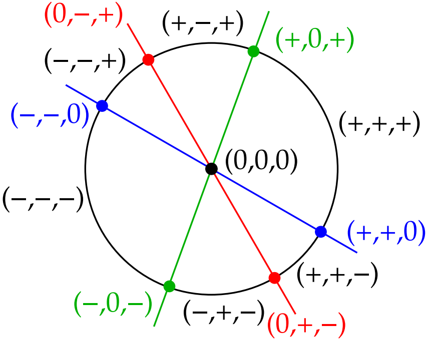

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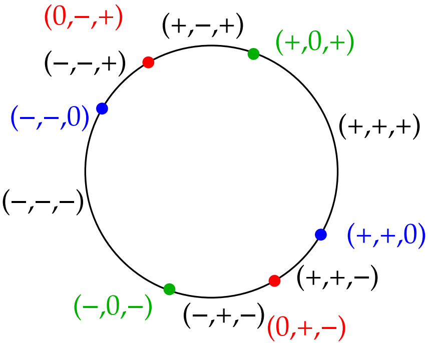


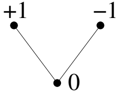
$\Phi^{-1}(\mathcal{C}^{\perp} \setminus \underline{0})$

Geometry and topology

Define a partial order on $\mathcal{C}^{\perp} \setminus \underline{0}$ by

$$(\sigma_1, \ldots, \sigma_n) \leq (\sigma'_1, \ldots, \sigma'_n) \Leftrightarrow \sigma_i \leq \sigma'_i \text{ for all } i,$$

where signs are ordered as in the poset



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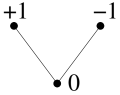

Theorem [Folkman, Lawrence, '78]. *If $\mathcal{C}$ is the set of signed circuits of an oriented matroid (of rank d), then*

$$\Delta(\mathcal{C}^\perp \setminus \underline{0}) \overset{hom}{\cong} S^{d-1}$$

(In fact, Oriented Matroids are cryptomorphic to arrangements of pseudospheres*...)*

Cryptomorphisms

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“Covector Axioms”

Combinatorics of linear dependencies over $\mathbb{C}$

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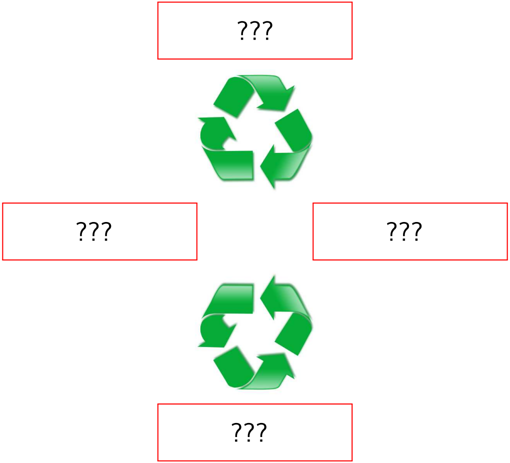
E. D.; *On generalizing complex matroids to a complex setting.*
Diploma thesis, ETH Zurich, 2003.

Focus: Orthogonality, Circuit duality, equivalence with Phirotopes.

The task

? – Cryptomorphisms?



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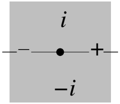

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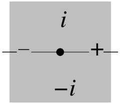

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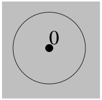

APPROACHING THE PROBLEM

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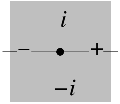

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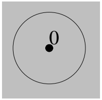

Our choice: Consider $S^1 \cup \{0\} \subset \mathbb{C}$ and let

$$\mathrm{ph} : \mathbb{C} \to S^1 \cup \{0\}$$

$$\mathrm{ph}(z) := \begin{cases} 0 & \text{if } z = 0 \\ e^{i\theta} & \text{if } z = re^{i\theta} \text{ for } r \in \mathbb{R}_{>0} \end{cases}$$

APPROACHING THE PROBLEM

2 – How to express orthogonality?

Two vectors $v, w \in \mathbb{C}^n$ are orthogonal if
$0 =< v \mid w >= \sum_{i=1}^{n} v_i \overline{w_i} = \sum_{i=1}^{n} \lambda_i \, \mathrm{ph}(v_i \overline{w_i})$
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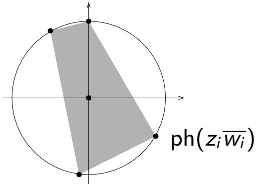

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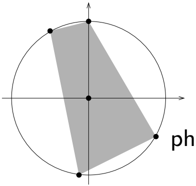

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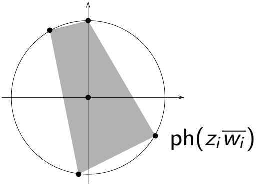

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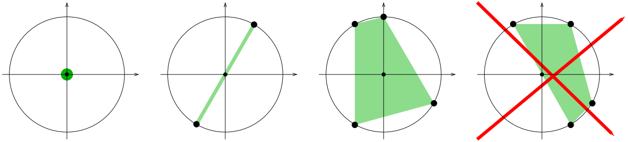

Bases

We start by mimicking the Grassmann-Plücker relations in $G_{d,n}(\mathbb{C})$.

Definition [B.,K.,R.-G.'03]. *A complex matroid of rank d on the ground set E is an alternating function*

$$\varphi : E^d \to S^1 \cup \{0\}$$

such that for all $x_0, x_1, \ldots, x_d, y_2, \ldots, y_d$,

$$0 \in \operatorname{relint} \operatorname{conv}\{(-1)^i \varphi(x_0, \ldots, \widehat{x_i}, \ldots, x_d)\varphi(x_i, y_2, \ldots, y_d) \mid 0 \leq i \leq d\}.$$

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- For χ a chirotope and $\iota : \{0, \pm 1\} \to S^1 \cup \{0\}$ the natural inclusion, $\iota \circ \chi$ is a (complexified) phirotope.

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Remarks:

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- Any phirotope φ uniquely defines the set $\mathcal{C}_\varphi$ of phased circuits X_C associated to φ

Dual pairs

Consider a phirotope φ. The associated $\mathcal{C}_\varphi$ satisfies

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Theorem [Anderson, D., '09]. *(Axioms for dual pairs)*
Given a finite set E, consider two families $\mathcal{C}, \mathcal{D} \subseteq (S^1 \cup \{0\})^E$. If both $\mathcal{C}$ and $\mathcal{D}$ satisfy (1),(2),(3), and if $\mathcal{C} \perp \mathcal{D}$, then there is a phirotope φ such that $\mathcal{C} = \mathcal{C}_\varphi$, $\mathcal{D} = \mathcal{C}_{\varphi^*}$.

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Remark. It follows that, as is the case for oriented matroids,

$$\mathcal{C}_{\varphi^*} = \min_{\text{supp}} \mathcal{C}^\perp$$

Phased circuits

Let us go back to the realizable case for inspiration.

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Consider complex numbers $(\zeta_i)_{\geq 2}$ and $(\xi_i)_{i\geq 2}$

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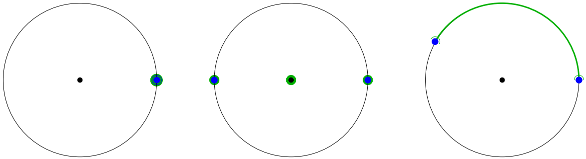

$\alpha = \beta$ $\qquad$ $\alpha = -\beta$ $\qquad$ $\alpha \neq \pm\beta$

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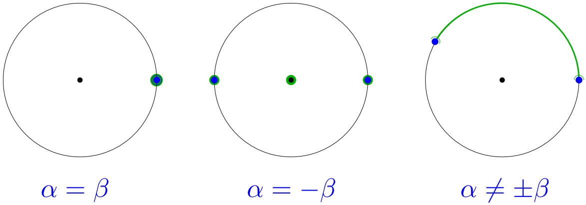

...so that $\boxed{\mathrm{ph}(z + w) \in [[\mathrm{ph}(z), \mathrm{ph}(w)]].}$

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Oriented Matroids also admit an axiomatization via Modular Elimination.

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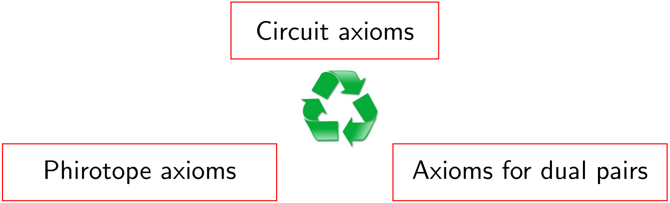

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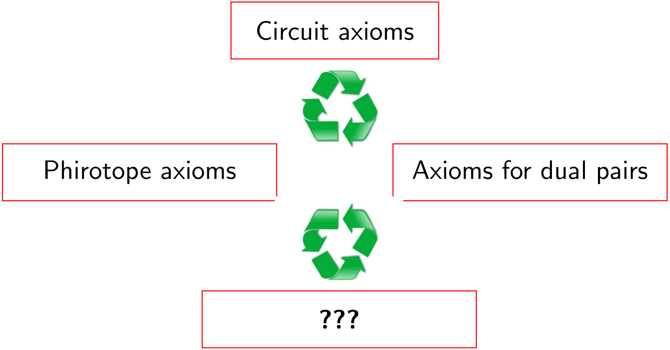

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In fact, there are two configurations V_2, V_2 (each of 4 vectors in $\mathbb{C}^2$) such that

- The two configuration have the same sets of signed circuits ($\mathcal{C}_{V_1} = \mathcal{C}_{V_2}$)
- For Φ as above, $\mathrm{Im}(\Phi_{V_1}) \neq \mathrm{Im}(\Phi_{V_2})$.

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