On the representation of the symmetric group on the cohomology of the toric variety associated with the type A Coxeter complex

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Joint work with John Shareshian

COMMUTATIVE ALGEBRA - Rees construction

$\Updownarrow$ Björner and Welker

POSET TOPOLOGY - Rees product of posets

$\Updownarrow$ Shareshian and MW

ENUMERATIVE COMBINATORICS - q-analog of Euler's formula

$\Updownarrow$ Shareshian and MW

SYMMETRIC FUNCTIONS - Eulerian quasisymmetric functions

$\Updownarrow$???

TORIC VARIETIES - Decomposition and lifting of cohomology

The toric variety

X_n = the toric variety associated with the type A_{n-1} Coxeter complex

$H^{2j}(X_n)$ = the $2j$ th cohomology of X_n for $j \in \{0, \ldots, n-1\}$.

Symmetric group $\mathfrak{S}_n$ acts naturally on X_n and this induces a linear representation of $\mathfrak{S}_n$ on each $H^{2j}(X_n)$.

Procesi gave a recurrence relation for this representation in 1985 and Stanley used this recurrence relation to give a generating function formula.

In terms of symmetric functions

Let $\mathrm{ch}\, V$ denote the Frobenius characteristic of a representation V of $\mathfrak{S}_n$.
Let $h_n = h_n(x_1, x_2, \dots)$ be the complete homogeneous symmetric function of degree n and let

$$H(z) = \sum_{n \geq 0} h_n z^n$$

Theorem (Procesi (1985) → Stanley (1989))

$$\sum_{n \geq 0} \sum_{j=0}^{n-1} \mathrm{ch} H^{2j}(X_n)\, t^j z^n = \frac{(1-t)H(z)}{H(zt) - tH(z)},$$

Proof uses geometric methods.

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Proof uses geometric methods.

Why do combinatorialists care about toric varieties?

$$\sum_{j=0}^{n-1} \dim H^{2j}(X_n) t^j = A_n(t),$$

where $A_n(t)$ is the Eulerian polynomial, which is defined by

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)}.$$

By Hard Lefschetz Theorem, $A_n(t)$ is symmetric and unimodal in t.

$$A_4(t) = 1 + 11t + 11t^2 + t^3$$

$$A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$$

Why do combinatorialists care about toric varieties?

Stanley (1980): If X is the toric variety associated with a simplicial d-dimensional polytope then

$$(\dim H^0(X), \dim H^2(X), \ldots, \dim H^{2d}(X))$$

equals the h-vector of the boundary complex of the polytope, where the h-vector is related to the f-vector by

$$h_i = \sum_{j=0}^{i} \binom{d-j}{d-i} (-1)^{i-j} f_{j-1} \qquad f_i = \sum_{j=0}^{d} \binom{d-j}{d-i-1} h_j$$

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Stanley used toric varieties in 1980 to prove one direction of McMullen's g-conjecture, which characterizes the h-vector of the boundary complex of a simplicial polytope. Symmetry and unimodality are part of this characterization.

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The h-vector of the type A_{n-1} Coxeter complex is $(a_{n,0}, a_{n,1}, \ldots, a_{n,n-1})$ where $a_{n,j}$ is an Eulerian number, i.e,.

$$A_n(t) = a_0 + a_1 t + \ldots a_{n-1} t^{n-1}.$$

Eulerian Permutation Statistics - MacMahon (1913)

For $\sigma \in \mathfrak{S}_n$,

Descent set: $\mathrm{DES}(\sigma) := \{i \in [n-1] : \sigma(i) > \sigma(i+1)\}$

$$\sigma = 3.25.4.1 \qquad \mathrm{DES}(\sigma) = \{1, 3, 4\}$$

Define $\mathrm{des}(\sigma) := |\mathrm{DES}(\sigma)|$. So

$$\mathrm{des}(32541) = 3$$

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Excedance set: $\mathrm{EXC}(\sigma) := \{i \in [n-1] : \sigma(i) > i)\}$

$$\sigma = 32541 \qquad \mathrm{EXC}(\sigma) = \{1, 3\}$$

Define $\mathrm{exc}(\sigma) := |\mathrm{EXC}(\sigma)|$. So

$$\mathrm{exc}(32541) = 2$$

Eulerian Permutation Statistics - MacMahon (1913)

$\mathfrak{S}_3$	des	exc
123	0	0
132	1	1
213	1	1
231	1	2
312	1	1
321	2	1

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Eulerian polynomial

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)} = \sum_{j=0}^{n-1} a_{n,j} t^j$$

$$A_3(t) = 1 + 4t + t^2$$

Euler's Formula

Euler's exponential generating function formula:

$$\sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{1-t}{e^{z(t-1)}-t}$$

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Procesi-Stanley formula:

$$\sum_{n\geq 0}\sum_{j=0}^{n-1} \mathrm{ch} H^{2j}(X_n)\, t^j \; z^n = \frac{(1-t)H(z)}{H(zt)-tH(z)},$$

Expansions of $\mathrm{ch} H^{2j}(X_n)$

Stembridge (1992): Expansion of symmetric function $\mathrm{ch} H^{2j}(X_n)$ in

- the basis of Schur functions - Schur positive
- the basis of power symmetric functions

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Stembridge proves

$$\sum_{j=0}^{n-1} \mathrm{ch} H^{2j}(X_n) t^j = \sum_{\nu \vdash n} \left(A_{l(\nu)}(t) \prod_{i=1}^{l(\nu)} [\nu_i]_t\right) \; z_\nu^{-1} p_\nu,$$

where $[m]_t := 1 + t + \cdots + t^{m-1}$.

So $\mathrm{ch} H^{2j}(X_n)$ is p-positive.

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Shareshian and MW: Expansion in the basis of fundamental quasisymmetric functions

Gessel's Theory of quasisymmetric functions

A quasisymmetric function is a formal power series $f(x_1, x_2, \ldots)$ of finite degree such that for all $a_1, \ldots, a_k \in \mathbb{P}$,

$$\mathrm{coeff}\; x_{i_1}^{a_1} \ldots x_{i_k}^{a_k} = \mathrm{coeff}\; x_{j_1}^{a_1} \ldots x_{j_k}^{a_k}$$

whenever $i_i < \cdots < i_k$ and $j_1 < \cdots < j_k$.

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Every symmetric function is a quasisymmetric function

Fix n and for $T \subseteq [n-1] := \{1, 2, \dots, n-1\}$, define the fundamental quasisymmetric function

$$F_T(x_1, x_2, \dots) := \sum_{\substack{s_1 \geq \cdots \geq s_n \\ i \in T \Rightarrow s_i > s_{i+1}}} x_{s_1} \dots x_{s_n}$$

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$F_\emptyset = h_n$ and $F_{[n-1]} = e_n$

$\{F_T : T \subseteq [n-1]\}$ forms a basis for the $\mathbb{Z}$-module of quasisymmetric functions of degree n.

Expansion in the basis of fundamental quasisymmetric functions

For $\sigma \in \mathfrak{S}_n$, let $\bar{\sigma}$ be obtained by placing bars above each excedance.

$$\bar{5}\bar{3}14\bar{6}2$$

View $\bar{\sigma}$ as a word over ordered alphabet

$$\{\bar{1} < \bar{2} < \cdots < \bar{n} < 1 < 2 < \cdots < n\}.$$

Define

$$\mathrm{DEX}(\sigma) := \mathrm{DES}(\bar{\sigma})$$

$$\mathrm{DEX}(531462) = \mathrm{DES}(\bar{5}.\bar{3}14.\bar{6}2) = \{1, 4\}$$

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We prove $\sum \mathrm{DEX}(\sigma) = \sum \mathrm{DES}(\sigma) - \mathrm{exc}(\sigma) =: \mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)$

Expansion in the basis of fundamental quasisymmetric functions

For all $j \in \{0, 1, \ldots, n-1\}$, define the Eulerian quasisymmetric function

$$Q_{n,j} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma) = j}} F_{\mathrm{DEX}(\sigma)}.$$

Theorem (Shareshian and MW)

$$\sum_{n \geq 0} \sum_{j=0}^{n-1} Q_{n,j}\, t^j z^n = \frac{(1-t)H(z)}{H(zt) - tH(z)},$$

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Corollary

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By the Hard Lefschetz Theorem and Schur's Lemma $Q_{n,j} - Q_{n,j-1}$ is Schur positive for all $j \leq n/2$.

Cycle-type Eulerian quasisymmetric functions

For $\sigma \in \mathfrak{S}_n$, let $\lambda(\sigma)$ denote the cycle type of σ. For $\lambda \vdash n$, define

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For all $j \in \{0, 1, \ldots, n-1\}$ and $\lambda \vdash n$, $Q_{\lambda,j}$ is a symmetric function and $Q_{\lambda,j} = Q_{\lambda,n-k-j}$, where $k = \#$ 1's in λ.

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Conjecture (Shareshian and MW)

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Conjecture (Shareshian and MW)

$Q_{\lambda,j}$ is Schur positive. Moreover $Q_{\lambda,j} - Q_{\lambda,j-1}$ is Schur positive for all $j \leq (n-k)/2$.

Computer verification up to $n = 8$.

Lifting the representation to $\mathfrak{S}_{n+1}$

Let $V_{\lambda,j}$ be the virtual representation whose Frobenius characteristic is $Q_{\lambda,j}$.
Then

$$H^{2j}(X_n) = \bigoplus_{\lambda \vdash n} V_{\lambda,j}$$

Theorem (Shareshian and MW)

For all $j = 0, \ldots, n-1$, the $\mathfrak{S}_n$-module $H^{2j}(X_n)$ is isomorphic to the restriction of $V_{(n+1),j+1}$ from $\mathfrak{S}_{n+1}$ to $\mathfrak{S}_n$

Is there a geometric explanation which shows that $V_{\lambda,j}$ is an actual representation?

Restricting the representation $V_{\lambda,j}$

Let $V_{\lambda,j}$ be the virtual representation whose Frobenius characteristic is $Q_{\lambda,j}$.
The subgroup C_n of $\mathfrak{S}_n$ generated by the cycle $(1, 2, \ldots, n)$ acts on $\mathfrak{S}_n$ by conjugation. Since the action preserves cycle type and number of excedances, C_n acts on

$$\mathfrak{S}_{\lambda,j} := \{\sigma \in \mathfrak{S}_n : \lambda(\sigma) = \lambda, \mathrm{exc}(\sigma) = j\}.$$

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Theorem (Sagan, Shareshian and MW)

The restriction of the virtual $\mathfrak{S}_n$-module $V_{\lambda,j}$ to the subgroup C_n is isomorphic to the permutation representation of C_n acting on $\mathfrak{S}_{\lambda,j}$ by conjugation.

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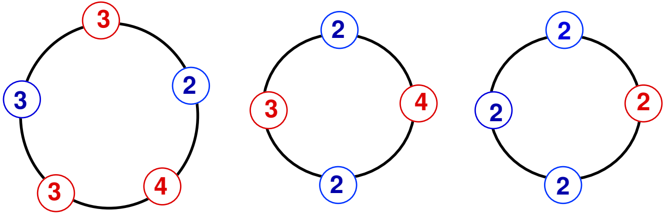
Alternative description of $Q_{\lambda,j}$

An ornament of type λ is a multiset of bicolored necklaces whose necklace sizes form partition λ



type $= (5, 4, 4)$ weight $= x_2^7 x_3^4 x_4^2$

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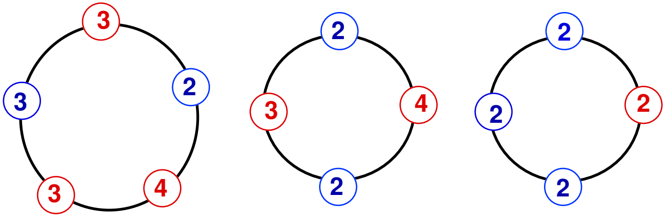

type $= (5,4,4)$ weight $= x_2^7 x_3^4 x_4^2$

Theorem (Shareshian and MW (2006))

Let $\mathcal{R}_{\lambda,j}$ = set of ornaments of type λ with j red letters. Then

$$Q_{\lambda,j} = \sum_{R \in \mathcal{R}_{\lambda,j}} wt(R)$$

Analogous to a result of Gessel and Reutenauer (1993).

Plethystic identity

For $\lambda = 1^{m_1} 2^{m_2} \cdots k^{m_k}$,

$$\sum_{j=0}^{n-1} Q_{\lambda,j} t^j = \prod_{i=1}^{k} h_{m_i(\lambda)} \left[\sum_{j=0}^{i-1} Q_{(i),j} t^j \right].$$

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Summing over all partitions λ yields,

$$\sum_{n,j \geq 0} Q_{n,j} t^j = \sum_{m \geq 0} h_m \left[\sum_{i,j \geq 0} Q_{(i),j} t^j \right].$$

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The plethystic inverse of $\sum_{m \geq 0} h_m$ is,

$$L := \sum_{n \geq 0} (-1)^n \, \mathrm{ch} \, lie_n,$$

where lie_n is the lie representation. Hence

$$\sum_{n,j \geq 0} Q_{(n),j} t^j = L \left[\sum_{i,j \geq 0} Q_{i,j} t^j \right].$$

Expansion in the power sum basis

$$\sum_{n,j\geq 0} Q_{(n),j} t^j = L\left[\sum_{i,j\geq 0} Q_{i,j} t^j\right].$$

It is well-known that

$$L = \sum_{d\geq 1} \frac{\mu(d)}{d} \sum_{i\geq 1} \frac{(-1)^{i-1}}{i} p_d^i,$$

where μ is the classical Möbius function and $p_d = \sum_{n\geq 1} x_n^d$.

Recall the result of Stembridge:

$$\sum_{n,j\geq 0} Q_{n,j} t^j = \sum_{\nu} (A_{l(\nu)}(t) \prod_{i=1}^{l(\nu)} [\nu_i]_t) \;\; z_\nu^{-1} p_\nu$$

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Plug in, do a tricky computation, and get ...

Expansion in the power sum basis

$$\sum_{j=0}^{n-1} Q_{(n),j} t^j = \sum_{\nu \vdash n} \left(t A_{k-1}(t) \prod_{i=1}^{k} [\nu_i]_t \right)_{\gcd(\nu_1,\ldots,\nu_k)} z_\nu^{-1} p_\nu,$$

where

$$\left(\sum_{j \geq 0} a_j t^j\right)_k = \sum_{j : \gcd(j,k)=1} a_j t^j.$$

For example, $(t + 3t^2 + 5t^3 + 7t^4)_2 = t + 5t^3$.

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$$\left(\sum_{j\geq 0} a_j t^j\right)_k = \sum_{j:\gcd(j,k)=1} a_j t^j.$$

For example, $(t + 3t^2 + 5t^3 + 7t^4)_2 = t + 5t^3$.

So $Q_{(n),j}$ is p-positive.

Expansion in the power sum basis

$$\sum_{j=0}^{n-1} Q_{(n),j} t^j = \sum_{\nu \vdash n} \left(t A_{k-1}(t) \prod_{i=1}^{k} [\nu_i]_t \right)_{\gcd(\nu_1,\ldots,\nu_k)} z_\nu^{-1} p_\nu,$$

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The above formula was a key step in our proof that $V_{\lambda,j} \downarrow_{C_n}^{\mathfrak{S}_n}$ is the conjugation representation of C_n on $\mathfrak{S}_{\lambda,j}$

q-Analogs

q-analogs arise in combinatorics, representation theory, algebraic geometry, algebraic topology, etc.,

Classical Example:

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = [n]_q!$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$ and

$$[n]_q! = [n]_q[n-1]_q \ldots [1]_q$$

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Recall: the major index of a permutation σ is defined to be

$$\mathrm{maj}(\sigma) = \sum_{i \in \mathrm{DES}(\sigma)} i$$

$$\mathrm{maj}(32541) = \mathrm{maj}(3.25.4.1) = 1 + 3 + 4 = 8$$

q-Analogs

Theorem (MacMahon (1913))

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = [n]_q!$$

$\mathfrak{S}_3$	inv	maj
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

$$1 + 2q + 2q^2 + q^3 = (1 + q + q^2)(1 + q)$$

q-Eulerian polynomials

Eulerian quasisymmetric function formula:

$$\sum_{n\geq 0}\sum_{j=0}^{n-1} Q_{n,j}\, t^j z^n = \frac{(1-t)H(z)}{H(zt)-tH(z)},$$

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Specialization: $x_i \to q^{i-1}$, $z \to z(1-q)$:

Theorem (Shareshian and MW)

$$\sum_{n\geq 0}\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)} \frac{z^n}{[n]_q!} = \frac{(1-t)\exp_q(z)}{\exp_q(zt)-t\exp_q(z)}$$

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Set $q=1$

$$\sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{(1-t)e^z}{e^{zt}-te^z}$$

q-Eulerian polynomials

To obtain the specialization we use Gessel's theory of quasisymmetric functions:

$$F_T(1, q, q^2, \dots) = \frac{q^{\sum T}}{(1-q)(1-q^2)\dots(1-q^n)}$$

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Recall $\sum \mathrm{DEX}(\sigma) = \mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)$.
So

$$F_{\mathrm{DEX}(\sigma)}(1, q, q^2, \dots) = \frac{q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}}{(1-q)(1-q^2)\dots(1-q^n)}$$

which implies

$$Q_{\lambda,j}(1, q, q^2, \dots) = \frac{\sum_{\sigma \in \mathfrak{S}_{\lambda,j}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}}{(1-q)(1-q^2)\dots(1-q^n)},$$

where $\mathfrak{S}_{\lambda,j} := \{\sigma \in \mathfrak{S}_n : \lambda(\sigma) = \lambda, \mathrm{exc}(\sigma) = j\}$.

q-Eulerian polynomials

Let

$$A_n(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

and

$$A_\lambda(q,t) := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \lambda(\sigma)=\lambda}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

Theorem (Shareshian and MW)

$A_n(q,t)$ is a symmetric and unimodal polynomial in t.

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Conjecture (Shareshian and MW)

$A_\lambda(q,t)$ is a unimodal polynomial in t. (symmetry is easy)

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Conjecture (Shareshian and MW)

$A_\lambda(q,t)$ *is a unimodal polynomial in* t. *(symmetry is easy)*

Consequence of the Schur-positivity of $Q_{\lambda,j+1} - Q_{\lambda,j}$ conjecture.

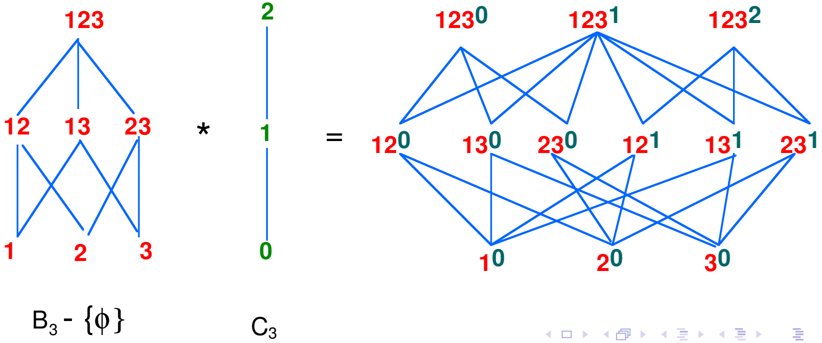
Rees Product-Björner & Welker, 2003

The Rees product of ranked posets P and Q is defined by

$$P * Q := \{(p, q) \in P \times Q : r(p) \geq r(q)\}$$

$(p_1, q_1) \leq (p_2, q_2)$ if the following holds

- $p_1 \leq_P p_2$
- $q_1 \leq_Q q_2$
- $r(p_2) - r(p_1) \geq r(q_2) - r(q_1)$



Rees Product - Björner & Welker, 2003

Theorem (Björner & Welker)

The Rees product of any Cohen-Macaulay poset with any acyclic Cohen-Macaualy poset is Cohen-Macaulay (CM means that homology of each interval vanishes below its top dimension.)

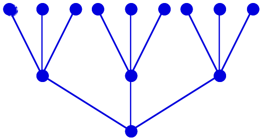
Theorem (Jonsson (conjectured by Björner & Welker))

$\dim \tilde{H}_{n-1}((B_n \setminus \{\emptyset\}) * C_n) = \#$ *derangements in* $\mathfrak{S}_n$.

Rees product of a Boolean Algebra and a Tree

Let $T_{t,n}$ be the poset whose Hasse diagram is the complete t-ary tree of height n with the root at the bottom.

$T_{3,2} =$



Theorem (Shareshian and MW)

The order complex of $(B_n * T_{t,n}) \setminus \{\hat{0}\}$ has the homotopy type of a wedge of $A_n(t)$ spheres of dimension $n-1$. Moreover

$$\mathrm{ch}\tilde{H}_{n-1}((B_n * T_{t,n}) \setminus \{\hat{0}\}) = \frac{(1-t)E(z)}{E(zt) - tE(z)},$$

where $E(z) = \sum_{i \geq 0} e_i z^i$.

Rees product of a Boolean Algebra and a Tree

Corollary

As $\mathfrak{S}_n$-modules,

$$\tilde{H}_{n-1}((B_n * T_{t,n}) \setminus \{\hat{0}\}) = \mathrm{sgn} \otimes \bigoplus_{j=0}^{n-1} H^{2j}(X_n) t^j$$

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