

Hodge numbers, rational points and discriminant varieties

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Let X be an algebraic variety over K, an algebraic number field.

X has an associated \mathbb{C} -variety $X(\mathbb{C})$, but may also be 'reduced mod \mathfrak{p} ' for primes \mathfrak{p} of K, obtaining $\overline{\mathbb{F}_q}$ -varieties $X_\mathfrak{p}$.

There are several cohomology theories associated with this data, in particular the de Rham cohomology $H^j_{dR}(X)$ and ℓ -adic cohomology $H^j(X_{\mathfrak{p}}, \mathbb{Q}_{\ell})$.



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My first purpose is to explain how the Hodge structure is determined by the Galois (and hence Frobenius) action on ℓ -adic cohomology. Most of this is joint work with Mark Kisin.

The second is to show how counting fixed points of twisted Frobenius maps on X_p is sometimes very effective in computing group actions on $H_{dR}(X)$.



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Let X be a variety over $K \subset \mathbb{C}$, a number field, and let G be a finite group of K-morphisms of X.

Problem: describe the (graded) action of G on the usual (Betti, or singular) cohomology $H^*(X,\mathbb{C})$.

Interpret as: compute for any $g \in G$

$$P_X(g,t) := \sum_i \operatorname{trace}(g,H^i(X,\mathbb{C}))t^i$$



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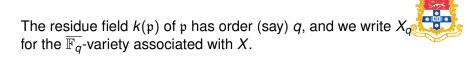
$$P_X(g,t) := \sum_i \operatorname{trace}(g,H^i(X,\mathbb{C}))t^i.$$

The residue field $k(\mathfrak{p})$ of \mathfrak{p} has order (say) q, and we write X_q for the $\overline{\mathbb{F}_q}$ -variety associated with X.

There are two elements to the method: first, given an isomorphism $\overline{\mathbb{Q}}_{\ell}\stackrel{\sim}{\to}\mathbb{C}$, we have isomorphisms of G-modules

$$H^i(X(\mathbb{C}),\mathbb{C})\stackrel{\sim}{\to} H^i(X_q,\overline{\mathbb{Q}_\ell}).$$

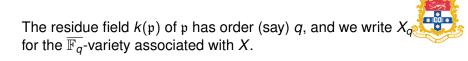
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$$|X_q^{g\mathcal{F}}| = \sum_i (-1)^i \operatorname{trace}(g\mathcal{F}, H_c^i(X_q, \overline{\mathbb{Q}_\ell}))$$

= $\sum_i (-1)^i q^{m_i} \operatorname{trace}(g, H_c^i(X(\mathbb{C}), \mathbb{C})).$

If we know the left side for almost all q, and $i \mapsto m_i$ is injective, then we have the compact supports version of $P_X(g, t)$.



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$$X = \mathbb{C}^*$$
, $G = \operatorname{Sym}_2$ acting via $r : z \mapsto z^{-1}$

Here Frob_q acts on $H_c^i(X)$ as q^{i-1} , i=1,2 because X is a hyperplane complement.

Now $X^{\operatorname{Frob}_q}$ has q-1 points, while $z\in X^{r\operatorname{Frob}_q}\iff z^{-q}=z$,

and there are q+1 such points.



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Conclusion:
$$H_c^2(X) = 1_{\mathrm{Sym}_2}$$
, and $H_c^1(X) = \varepsilon_{\mathrm{Sym}_2}$. So $P_X(t) = 1_{\mathrm{Sym}_2} + t\varepsilon_{\mathrm{Sym}_2}$ by Poincaré duality.



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K: an algebraic number field, \overline{K} : its algebraic closure.

S: a finite set of primes of K.

 $K_S \subset \overline{K}$: the maximal subfield of \overline{K} , unramified outside S.

$$G := \operatorname{Gal}(\overline{K}/K) \xrightarrow{\operatorname{onto}} G_{K,S} := \operatorname{Gal}(K_S/K).$$

These are both profinite topological groups; subgroups of finite index are open.

 ℓ : a rational prime, all of whose prime factors in K lie in S.

If $\mathfrak{p} \notin S$ is a prime of K, there is an element $\operatorname{Frob}_{\mathfrak{p}} \in G_{K,S}$ well defined up to conjugation.





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Cohomology and Filtrations. The setup.



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If $q_{\mathfrak{p}} := |\kappa(\mathfrak{p})|$ ($\kappa(\mathfrak{p})$ is the residue field of \mathfrak{p}) then $\mathrm{Frob}_{\mathfrak{p}}$ induces the $q_{\mathfrak{p}}$ -power map on the extension of $\kappa(\mathfrak{p})$ arising from $K_{\mathcal{S}}$.



Let X be an algebraic variety (i.e. a reduced scheme of finite type) over the number field K.

There are 3 cohomology theories naturally associated with X. The interrelationships among them are the key to this work.

1. de Rham Cohomology. This is a sequence $H^j_{dR}(X)$ $j=0,1,2,\ldots$ of K-vector spaces, which come naturally with a (Hodge) filtration $\mathbf{F}^{\bullet}H^j_{dR}(X)$:

$$\mathbf{F}^k H^j_{\mathrm{dR}}(X) \supseteq \mathbf{F}^{k+1} H^j_{\mathrm{dR}}(X).$$

2. Betti (usual) Cohomology. For any embedding $\sigma: K \hookrightarrow \mathbb{C}$, $X_{\sigma} := X \otimes_K \mathbb{C}$ has \mathbb{C} -points which may be identified with a complex analytic (algebraic) variety $X_{\sigma}(\mathbb{C})$.



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Its complex cohomology $H^j(X_\sigma(\mathbb C),\mathbb C)$ is a sequence of $\mathbb C$ -vector spaces.

Betti cohomology comes with 2 natural filtrations: the first, \mathbf{F}^{\bullet} ("de Rham filtration"), arises from that of H^{j}_{dR} via the extension of scalars isomorphism:

$$H^{j}_{dR}(X) \otimes_{K} \mathbb{C} \stackrel{\sim}{\longrightarrow} H^{j}(X_{\sigma}(\mathbb{C}), \mathbb{C}) \stackrel{\sim}{\longrightarrow} H^{j}(X_{\sigma}(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

$$\mathbf{F}^p H^j(X_{\sigma}(\mathbb{C}), \mathbb{C}) \cap \bar{\mathbf{F}}^q H^j(X_{\sigma}(\mathbb{C}), \mathbb{C})$$





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3. ℓ -adic Étale Cohomology. With ℓ a rational prime as above, we have a sequence of \mathbb{Q}_{ℓ} -vector spaces $H^{j}(X_{\overline{K}}, \mathbb{Q}_{\ell})$, the ℓ -adic cohomology of $X_{\overline{K}} := X \otimes_{K} \overline{K}$.

Important: $G=\operatorname{Gal}(\overline{K}/K)$ acts on $X_{\overline{K}}$, and hence on $H^j(X_{\overline{K}'},\mathbb{Q}_\ell)$; in particular, so does $\operatorname{Frob}_{\mathfrak{p}}$ for any prime $\mathfrak{p} \not\in S$.



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Given $\overline{\sigma}:\overline{K}\to\mathbb{C}$ which extends σ , and an embedding $\mathbb{Q}_\ell\to\mathbb{C}$, we have canonical isomorphisms

$$(^*) H^j(X_{\overline{K}}, \mathbb{Q}_{\ell}) \otimes_{\mathbb{Q}_{\ell}} \mathbb{C} \xrightarrow{\sim} H^j(X_{\sigma}(\mathbb{C}), \mathbb{C}) \xrightarrow{\sim} H^j_{\mathsf{dR}}(X) \otimes_{K} \mathbb{C}.$$

These permit the transfer of information from each setting to the others.

Weights

Each of the 3 cohomology theories (independently) carries an increasing weight filtration W_{\bullet} (due to Deligne).





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 B_{dR} has a decreasing filtration $\mathit{Fil}^{ullet}B_{\mathit{dR}}$, whose associated graded components are $G_{\mathfrak{l}}$ -modules of the form $\mathbb{C}_{\ell}\otimes_{\mathbb{Q}_{\ell}}\mathbb{Q}_{\ell}(d)$ (Tate twist).

Fundamental result: (Fontaine-Messing, Faltings, Kisin): there is an isomorphism of **filtered** $K_I G_I$ modules:

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$$h^{d,m-d}(j) = \dim_{K_{\mathbb{I}}} \left(\operatorname{Gr}_m^{\mathbb{W}} H^j(X, \mathbb{Q}_{\ell}) \otimes_{\mathbb{Q}_{\ell}} \mathbb{C}_{\ell}(d) \right)^{G_{\mathbb{I}}}$$

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Important: on the left the filtration is Hodge $\otimes Fil^{\bullet}$, while on the right it is purely number theoretic. (from Fil^{\bullet}).

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The first such application is:

Then $\mathrm{Gr}_{\mathsf{F}}^p\mathrm{Gr}_{\mathsf{F}}^qH^j(X_\sigma(\mathbb{C}),\mathbb{C})$ has dimension r_i if p=q=i, and is 0 otherwise.

NB The hypothesis is about eigenvalues of Frobenius, while the conclusion is about the Hodge filtration, which does not exist in ℓ -adic cohomology.

Say that *X* is *mixed Tate* (mt) if it satisfies the conditions of the theorem.

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Prop: $p: X \to Y$ a smooth morphism of smooth K-varieties such that each fibre $p^{-1}(y)$ is K-isomorphic to a fixed Z. Assume that the local systems $R^j p_* \mathbb{C}$ induced by $p: X_{\sigma}(\mathbb{C}) \to Y_{\sigma}(\mathbb{C})$ are constant for each j. If any 2 of X, Y, Z are mt then so is the third.



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Prop Suppose X is such that for almost all q, Frob_q has eigenvalues of absolute value $q^{\frac{i}{2}}$ on $H^i_c(X_{\bar{K}}, \bar{\mathbb{Q}}_\ell)$. Then the following are equivalent:

- (1) X is mt.
- (2) $|X(\mathbb{F}_{q^m})| = P_X(q^m)$ for all q,m >> 0, some $P_X(t) \in \mathbb{Z}[t]$.

They imply that

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Let $h^{p,q} = \sum_{j} (-1)^{j} h^{p,q}(j)$ (taken for cohomology with compact supports). These are the Euler-Hodge numbers of X.

Define $H_X(x, y) = \sum_{p,q} h^{p,q} x^p y^q$. Katz has recently proved:

Theorem: (N. Katz, 2009)Suppose there is a polynomial $P(t) \in \mathbb{C}[t]$ such that for almost all q, $|X(\mathfrak{F}_q)| = P_X(q)$. Then H(x,y) = P(xy).

This follows quite easily from Theorem 1 above. The following equivariant version follows from our methods.

Theorem 3. Assume the hypotheses of Katz's theorem. If G is a group of automorphisms of X, then the virtual G modules $\sum_j (-1)^j \operatorname{Gr}_{2m}^W H_c^j(X(\mathbb{C}))$ and $\sum_j (-1)^j \operatorname{Gr}_{\mathbf{F}}^m \operatorname{Gr}_{\mathbf{F}}^m H_c^j(X(\mathbb{C}))$ are equal in the Grothendieck group of G.



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Deligne has shown that $h^{p,q}(j) = \dim_{\mathbb{C}} \operatorname{Gr}_{\mathbf{F}}^{p} \operatorname{Gr}_{\mathbf{F}}^{q} H_{c}^{j}(X_{\sigma}(\mathbb{C}), \mathbb{C}) = 0$ if p < j - n or q < j - n, where $n = \dim X$.

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 \mathcal{A}_G is the set of its reflecting hyperplanes, M_G the corresponding complement, and $X_G:=M_G/G$ the corresponding discriminant variety.

Examples: when $G = \operatorname{Sym}_n$, X is the usual discriminant variety $\mathcal{D}_n := \{f(t) = t^n + a_1 t^{n-1} + \dots + a_n \in \mathbb{C}[t] \mid f \text{ has distict roots}\};$ when G = G(r, 1, n) (the monomial group) then $X_G = \mathcal{D}_n^0 := \{f(t) \in \mathcal{D}_n \mid f(0) \neq 0\}$

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Then (for non-trivial N/G) there is just the 'classical case' $\widetilde{G} = G(r,1,n) \supset G = G(r,r,n)$ and 6 exceptional cases. The latter include $D_4 \subset F_4 \subset Aut(F_4)$ and 4 complex cases.

I shall outline the computation in the classical case. Here $\Gamma=\mu_r$. So we will calculate the cohomology of discriminant varieties with coefficients in certain line bundles.

The reflecting hyperplanes of G are $z_i - \zeta z_j = 0$ ($\zeta \in \mu_r$). So the map $(z_1, \ldots, z_n) \mapsto (\prod_i (t + z_i^r), \prod_i z_i)$ identifies $X_n := X_G$ with $\{(f(t), \xi) \in \mathcal{D}_n \times \mathbb{C} \mid \xi^n = a_n\}$.

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Proposition: If M is any hyperplane complement (defined over a number field) of dimension n, then $\operatorname{Gr}_m^{\mathbf{W}} H_c^j(X) = 0$ unless m = 2(j-n)

Thus M is minimally pure (mp) in the sense of Dimca-L (1997). It means that $P_c^G(t)$ may be calculated by computing rational points as in the example at the start of the talk.

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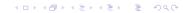
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Suppose X is a K-variety with Γ action, such that $X = \coprod_i X^{(i)}$ is a decomposition into locally closed pieces which are permuted by Γ .

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Theorem (L, 2003).

$$P^{\mu_r}(X_n,t) = \begin{cases} (1+t)1_{\mu_r} \text{ if } r \text{ or } n \text{ is odd} \\ (1+t)\mu_r + (t^{n-1}+t^n)\varepsilon_{\mu_r} \text{ if } r \text{ and } n \text{ are even} \end{cases}$$



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It follows after some calculation, that $W_c^{\mu_r}(X_n,t)=W_c^{\mu_r}(\widetilde{Y}_n,t)+W_c(\widetilde{X}_{n-1}(t)\mathbf{1}_{\mu_r},$

where \widetilde{X}_n is the discriminant variety of $\widetilde{G} := G(r, 1, n)$,

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The proof is a combinatorial exercise in counting rational points.

It depends on a power series identity, as follows.

Frobenius orbits on $\bar{\mathbb{F}}_q^{\times}$ correspond to irreducible polynomials over \mathbb{F}_q . For such a polynomial a(t), define $\tau_r(a) = a(0)^{\frac{q-1}{r}}$.

Assume r|q-1 and identify μ_r as a subgroup of \mathbb{F}_q . For $\alpha \in \mu_r$, let $m_d^r(\alpha,q)$ be the number of such polynomials a(t) of degree d such that $\tau_r(a) = \alpha$

For $\lambda \in \hat{\mu}_r$ define the power series $F_{\lambda}^{(r)}(t) := 1 + \sum_{d \geq 1} \sum_{\alpha \in \mu_r} m_d^r(\alpha, q) \lambda(\alpha) t^d$



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We then have the following power series identity (generalisation of the 'cyclotomic identity')

$$F_{\lambda}(t) = \begin{cases} 1 \text{ if } |\lambda| \neq 1 \text{ or } 2\\ \frac{1 - qt^2}{1 - t^2} \text{ if } |\lambda| = 2\\ \frac{1 - t^2q}{(1 - tq)(1 + t)} \text{ if } \lambda = 1. \end{cases}$$

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Remarks about the exceptional cases



There are 6 exceptional cases: 3 are two-dimensional; then $G(3,3,3) < G_{26}$, $W(D_4) < W(F_4)$, and $W(F_4) < \widetilde{W}(F_4)$ (extension by graph automorphism)

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The problem of comoplete description of the cohomology of the spaces X_G with local coefficients is far from solved.

There are close connections with the equivariant cohomology of the Milnor fibre, which Alex Dimca spoke about here in May.

Rational point methods may have limited applicability, but there are still plenty of opportunities to exploit them. In general, even when the cohomology is not mixed Tate, it has a mt part, which conjecturally (the Tate conjectures) is spanned by a subvariety.



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