The University of Sydney

Hodge numbers, rational points and discriminant varieties

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NSW 2006
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Centro Ennio de Giorgi,
Pisa, 25th June, 2010

Introduction

Let X be an algebraic variety over K, an algebraic number field.

X has an associated $\mathbb{C}$-variety $X(\mathbb{C})$, but may also be 'reduced mod $\mathfrak{p}$' for primes $\mathfrak{p}$ of K, obtaining $\overline{\mathbb{F}_q}$-varieties $X_\mathfrak{p}$.

There are several cohomology theories associated with this data, in particular the de Rham cohomology $H^j_{dR}(X)$ and ℓ-adic cohomology $H^j(X_\mathfrak{p}, \mathbb{Q}_\ell)$.

The former has associated Hodge numbers $h^{p,q}(j)$, while the latter has an action of Frob_q, the q-Frobenius endomorphism.

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These cohomology theories are related in several ways by comparison theorems.

My first purpose is to explain how the Hodge structure is determined by the Galois (and hence Frobenius) action on ℓ-adic cohomology. Most of this is joint work with Mark Kisin.

The second is to show how counting fixed points of twisted Frobenius maps on $X_{\mathfrak{p}}$ is sometimes very effective in computing group actions on $H_{dR}(X)$.

The third is to apply these methods to the case of discriminant varieties, which are defined as $X := M_G/G$, where G is a unitary reflection group, and M_G is its corresponding configuration space (or hyperplane complement). This amounts to computing the cohomology of X with some local coefficients.

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Example

Let X be a variety over $K \subset \mathbb{C}$, a number field, and let G be a finite group of K-morphisms of X.

Problem: describe the (graded) action of G on the usual (Betti, or singular) cohomology $H^*(X, \mathbb{C})$.

Interpret as: compute for any $g \in G$

$P_X(g, t) := \sum_i \text{trace}(g, H^i(X, \mathbb{C}))t^i$.

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Let $\mathfrak{p}$ be a prime ideal in K.

The residue field $k(\mathfrak{p})$ of $\mathfrak{p}$ has order (say) q, and we write X_q for the $\overline{\mathbb{F}_q}$-variety associated with X.

There are two elements to the method: first, given an isomorphism $\overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$, we have isomorphisms of G-modules

$$H^i(X(\mathbb{C}), \mathbb{C}) \xrightarrow{\sim} H^i(X_q, \overline{\mathbb{Q}_\ell}).$$

In practice, one has such results for "almost all q", and this suffices.

Second, assume that we knew that the Frobenius morphism $\mathcal{F}$ acts on $H^i_c(X_q, \overline{\mathbb{Q}_\ell})$ with just a single eigenvalue q^{m_i}.

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For any $g \in G$, compute $|X_q^{g\mathcal{F}}|$ using Grothendieck's fixed point theorem:

$|X_q^{g\mathcal{F}}| = \sum_i (-1)^i \text{trace}(g\mathcal{F}, H_c^i(X_q, \overline{\mathbb{Q}_\ell}))$
$= \sum_i (-1)^i q^{m_i} \text{trace}(g, H_c^i(X(\mathbb{C}), \mathbb{C}))$.

If we know the left side for almost all q, and $i \mapsto m_i$ is injective, then we have the compact supports version of $P_X(g, t)$.

Remark: The assumptions hold for all hyperplane complements, the moduli space of n points on a genus 0 curve, and smooth toric varieties; but there are more complicated cases, like the Milnor fibre of an arrangement.

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Special case–baby example

Take $X = \mathbb{C}^*$, $G = \mathrm{Sym}_2$ acting via $r : z \mapsto z^{-1}$

Here Frob_q acts on $H^i_c(X)$ as q^{i-1}, $i = 1, 2$ because X is a hyperplane complement.

Now X^{Frob_q} has $q - 1$ points, while $z \in X^{r\mathrm{Frob}_q} \iff z^{-q} = z$,

and there are $q + 1$ such points.

Conclusion: $H^2_c(X) = 1_{\mathrm{Sym}_2}$, and $H^1_c(X) = \varepsilon_{\mathrm{Sym}_2}$. So $P_X(t) = 1_{\mathrm{Sym}_2} + t\varepsilon_{\mathrm{Sym}_2}$ by Poincaré duality.

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Cohomology and Filtrations.
The setup.

K: an algebraic number field, $\overline{K}$: its algebraic closure.

S : a finite set of primes of K.

$K_S \subset \overline{K}$: the maximal subfield of $\overline{K}$, unramified outside S.

$G := \mathrm{Gal}(\overline{K}/K) \xrightarrow{\text{onto}} G_{K,S} := \mathrm{Gal}(K_S/K)$.

These are both profinite topological groups; subgroups of finite index are open.

ℓ : a rational prime, all of whose prime factors in K lie in S.

If $\mathfrak{p} \notin S$ is a prime of K, there is an element $\mathrm{Frob}_{\mathfrak{p}} \in G_{K,S}$ well defined up to conjugation.

If $q_{\mathfrak{p}} := |\kappa(\mathfrak{p})|$ ($\kappa(\mathfrak{p})$ is the residue field of $\mathfrak{p}$) then $\mathrm{Frob}_{\mathfrak{p}}$ induces the $q_{\mathfrak{p}}$-power map on the extension of $\kappa(\mathfrak{p})$ arising from K_S.

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Cohomology theories

Let X be an algebraic variety (i.e. a reduced scheme of finite type) over the number field K.

There are 3 cohomology theories naturally associated with X. The interrelationships among them are the key to this work.

1. de Rham Cohomology. This is a sequence $H^j_{dR}(X)$ $j = 0, 1, 2, \ldots$ of K-vector spaces, which come naturally with a (Hodge) filtration $\mathbf{F}^\bullet H^j_{dR}(X)$:

$$\mathbf{F}^k H^j_{\mathrm{dR}}(X) \supseteq \mathbf{F}^{k+1} H^j_{\mathrm{dR}}(X).$$

2. Betti (usual) Cohomology. For any embedding $\sigma : K \hookrightarrow \mathbb{C}$, $X_\sigma := X \otimes_K \mathbb{C}$ has $\mathbb{C}$-points which may be identified with a complex analytic (algebraic) variety $X_\sigma(\mathbb{C})$.

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1. de Rham Cohomology. This is a sequence $H^j_{dR}(X)$ $j = 0, 1, 2, \ldots$ of K-vector spaces, which come naturally with a (Hodge) filtration $\mathbf{F}^\bullet H^j_{dR}(X)$:

$$\mathbf{F}^k H^j_{\mathrm{dR}}(X) \supseteq \mathbf{F}^{k+1} H^j_{\mathrm{dR}}(X).$$

2. Betti (usual) Cohomology. For any embedding $\sigma : K \hookrightarrow \mathbb{C}$, $X_\sigma := X \otimes_K \mathbb{C}$ has $\mathbb{C}$-points which may be identified with a complex analytic (algebraic) variety $X_\sigma(\mathbb{C})$.

Its complex cohomology $H^j(X_\sigma(\mathbb{C}),\mathbb{C})$ is a sequence of $\mathbb{C}$-vector spaces.

Betti cohomology comes with 2 natural filtrations: the first, $\mathbf{F}^\bullet$ ("de Rham filtration"), arises from that of H^j_{dR} via the extension of scalars isomorphism:

$$H^j_{\mathrm{dR}}(X)\otimes_K\mathbb{C}\xrightarrow{\sim}H^j(X_\sigma(\mathbb{C}),\mathbb{C})\xrightarrow{\sim}H^j(X_\sigma(\mathbb{C}),\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}.$$

The second filtration $\overline{\mathbf{F}}^\bullet$ comes from the first via complex conjugation. Together, they provide the *Hodge filtration*:

$$\mathbf{F}^pH^j(X_\sigma(\mathbb{C}),\mathbb{C})\cap\bar{\mathbf{F}}^qH^j(X_\sigma(\mathbb{C}),\mathbb{C})$$

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Finally, we have

3. ℓ-adic Étale Cohomology. With ℓ a rational prime as above, we have a sequence of $\mathbb{Q}_\ell$-vector spaces $H^j(X_{\overline{K}}, \mathbb{Q}_\ell)$, the ℓ-adic cohomology of $X_{\overline{K}} := X \otimes_K \overline{K}$.

Important: $G = \mathrm{Gal}(\overline{K}/K)$ acts on $X_{\overline{K}}$, and hence on $H^j(X_{\overline{K}}, \mathbb{Q}_\ell)$; in particular, so does $\mathrm{Frob}_\mathfrak{p}$ for any prime $\mathfrak{p} \notin S$.

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Interrelationships

Given $\overline{\sigma} : \overline{K} \to \mathbb{C}$ which extends σ, and an embedding $\mathbb{Q}_\ell \to \mathbb{C}$, we have canonical isomorphisms

$$(^*)\ H^j(X_{\bar{K}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C} \xrightarrow{\sim} H^j(X_\sigma(\mathbb{C}), \mathbb{C}) \xrightarrow{\sim} H^j_{\mathrm{dR}}(X) \otimes_K \mathbb{C}.$$

These permit the transfer of information from each setting to the others.

Weights

Each of the 3 cohomology theories (independently) carries an increasing *weight filtration* $W_\bullet$ (due to Deligne).

The isomorphisms (*) above respect the weight filtrations.

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The Hodge numbers are defined by $h^{p,q}(j) = \dim_{\mathbb{C}} \mathrm{Gr}_{\mathbf{F}}^p \mathrm{Gr}_{\overline{\mathbf{F}}}^q H^j(X_\sigma(\mathbb{C}), \mathbb{C})$.

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Let $\mathfrak{l}$ be a prime of K which divides ℓ, and write $K_{\mathfrak{l}}$ for the $\mathfrak{l}$-adic completion of K. The decomposition group $G_{\mathfrak{l}}$ of ℓ maps into $G_{K,S}$, and we may restrict Galois representations to $G_{\mathfrak{l}}$

The 'Fontaine ring' B_{dR} is a discretely valued field with residue field denoted $\mathbb{C}_\ell$; it contains $K_{\mathfrak{l}}$ (hence also $\mathbb{Q}_\ell$ and K).

B_{dR} has a decreasing filtration $Fil^\bullet B_{dR}$, whose associated graded components are $G_{\mathfrak{l}}$-modules of the form $\mathbb{C}_\ell \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(d)$ (Tate twist).

Fundamental result: (Fontaine-Messing, Faltings, Kisin): there is an isomorphism of **filtered** $K_{\mathfrak{l}}G_{\mathfrak{l}}$ modules:

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Important: on the left the filtration is Hodge $\otimes Fil^\bullet$, while on the right it is purely number theoretic. (from $Fil^\bullet$).

Theorem 1. (Kisin-L) We have

$$h^{d,m-d}(j) = \dim_{K_\iota}\left(\mathrm{Gr}^{\mathbf{W}}_m H^j(X,\mathbb{Q}_\ell)\otimes_{\mathbb{Q}_\ell}\mathbb{C}_\ell(d)\right)^{G_\iota}$$

Note: the left side involves geometry; the right side is number-theoretic, involving only the Galois action.

This isomorphism is the key to applying knowledge of rational points, or Frobenius action, to the complex manifold $X_{\mathbb{C}}$.

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Theorem 2.(Kisin-L P&AMQ (Coates issue) 2006) Let K, S etc. be as above, and let X be a variety over K. Assume that for each prime $\mathfrak{p} \notin S$, the eigenvalues of $\mathrm{Frob}_{\mathfrak{p}}$ on $H^j(X, \mathbb{Q}_\ell)$ are all of the form $\zeta q_{\mathfrak{p}}^i$ ($i \in \mathbb{N}$, ζ a root of unity.) and that for any $i \in \mathbb{N}$, there are r_i of these.

Then $\mathrm{Gr}^p_{\mathbf{F}} \mathrm{Gr}^q_{\bar{\mathbf{F}}} H^j(X_\sigma(\mathbb{C}), \mathbb{C})$ has dimension r_i if $p = q = i$, and is 0 otherwise.

NB The hypothesis is about eigenvalues of Frobenius, while the conclusion is about the Hodge filtration, which does not exist in ℓ-adic cohomology.

Say that X is *mixed Tate* (mt) if it satisfies the conditions of the theorem.

Examples: $\mathbb{A}^n, \mathbb{P}^n$, hyperplane complements, reductive group schemes, homogeneous spaces, hyperplane complements, toral analogues,...

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Prop: $p : X \to Y$ a smooth morphism of smooth K-varieties such that each fibre $p^{-1}(y)$ is K-isomorphic to a fixed Z. Assume that the local systems $R^j p_* \mathbb{C}$ induced by $p : X_\sigma(\mathbb{C}) \to Y_\sigma(\mathbb{C})$ are constant for each j. If any 2 of X, Y, Z are mt then so is the third.

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The following consequence is relevant to the Springer fibres $\mathcal{B}_u$ in the flag variety.

Prop Suppose X is such that for almost all q, Frob_q has eigenvalues of absolute value $q^{\frac{i}{2}}$ on $H_c^i(X_{\bar{K}}, \bar{\mathbb{Q}}_\ell)$. Then the following are equivalent:

(1) X is mt.

(2) $|X(\mathbb{F}_{q^m})| = P_X(q^m)$ for all $q, m >> 0$, some $P_X(t) \in \mathbb{Z}[t]$.

They imply that

(3) $H_c^j(X, \mathbb{Q}_\ell) = 0$ for j odd.

Example: this implies that the Springer fibre $\mathcal{B}_u$ has vanishing odd cohomology if and only if $\mathrm{Ind}_{B^F}^{G^F}(u)$ is a polynomial in q for an infinite number of characteristics.

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Katz's theorem

Let $h^{p,q} = \sum_j (-1)^j h^{p,q}(j)$ (taken for cohomology with compact supports). These are the Euler-Hodge numbers of X.

Define $H_X(x,y) = \sum_{p,q} h^{p,q} x^p y^q$. Katz has recently proved:

Theorem: (N. Katz, 2009)Suppose there is a polynomial $P(t) \in \mathbb{C}[t]$ such that for almost all q, $|X(\mathfrak{F}_q)| = P_X(q)$. Then $H(x,y) = P(xy)$.

This follows quite easily from Theorem 1 above. The following equivariant version follows from our methods.

Theorem 3. Assume the hypotheses of Katz's theorem. If G is a group of automorphisms of X, then the virtual G modules $\sum_j (-1)^j \mathrm{Gr}^{\mathbf{W}}_{2m} H^j_c(X(\mathbb{C}))$ and $\sum_j (-1)^j \mathrm{Gr}^m_{\mathbf{F}} \mathrm{Gr}^m_{\mathbf{F}} H^j_c(X(\mathbb{C}))$ are equal in the Grothendieck group of G.

Write $E^G_{c,m}(X,t)$ for the module above. The equivariant weight polynomial $W^G_c(X) = \sum_m E^G_{c,m}(X) t^m \in R(G)[t]$ (the Grothendieck ring).

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Minimally pure varieties

Deligne has shown that $h^{p,q}(j) = \dim_{\mathbb{C}} \mathrm{Gr}^p_{\mathbf{F}} \mathrm{Gr}^q_{\bar{\mathbf{F}}} H^j_c(X_\sigma(\mathbb{C}), \mathbb{C}) = 0$ if $p < j - n$ or $q < j - n$, where $n = \dim X$.

We therefore say X is minimally pure (mp) if $\mathrm{Gr}^{j-n}_{\mathbf{F}} \mathrm{Gr}^{j-n}_{\bar{\mathbf{F}}} H^j_c(X_\sigma(\mathbb{C}), \mathbb{C}) = H^j_c(X_\sigma(\mathbb{C}), \mathbb{C})$.

Proposition: X is mp if and only if Frob_q acts on $H^j_c(X, \mathbb{Q}_\ell)$ with eigenvalues q^{j-n} for each j.

Examples: hyperplane complements (L, 1990); toral arrangment complements; ...

New from old: if $\mathcal{A}$ is an arrangement of mp varieties in X (mp), the complement in X of $\mathcal{A}$ is mp.

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Unitary reflection groups

Let G be a finite unitary reflection group acting on $V := \mathbb{C}^n$

$\mathcal{A}_G$ is the set of its reflecting hyperplanes, M_G the corresponding complement, and $X_G := M_G/G$ the correponding discriminant variety.

Examples: when $G = \mathrm{Sym}_n$, X is the usual discriminant variety $\mathcal{D}_n := \{f(t) = t^n + a_1 t^{n-1} + \cdots + a_n \in \mathbb{C}[t] \mid f \text{ has distict roots}\}$; when $G = G(r, 1, n)$ (the monomial group) then $X_G = \mathcal{D}_n^0 := \{f(t) \in \mathcal{D}_n \mid f(0) \neq 0\}$

If $\widetilde{G} \subseteq N_{GL(V)}(G)$ then $\Gamma := \widetilde{G}/G$ acts on X_G. Problem: compute $W_c^\Gamma(X_G, t)$.

Equivalent formulation: for each representation ρ of Γ, compute $H_c^j(X_G, \mathcal{L}_\rho)$, $\mathcal{L}_\rho =$ the corresponding local system on $X_{\widetilde{G}}$; ($M_{\widetilde{G}}/G \to X_{\widetilde{G}}$ is a Galois Γ-covering.

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Then (for non-trivial N/G) there is just the 'classical case' $\widetilde{G} = G(r,1,n) \supset G = G(r,r,n)$ and 6 exceptional cases. The latter include $D_4 \subset F_4 \subset Aut(F_4)$ and 4 complex cases.

I shall outline the computation in the classical case. Here $\Gamma = \mu_r$. So we will calculate the cohomology of discriminant varieties with coefficients in certain line bundles.

The reflecting hyperplanes of G are $z_i - \zeta z_j = 0 \quad (\zeta \in \mu_r)$. So the map $(z_1, \dots, z_n) \mapsto (\prod_i (t + z_i^r), \prod_i z_i)$ identifies $X_n := X_G$ with $\{(f(t), \xi) \in \mathcal{D}_n \times \mathbb{C} \mid \xi^n = a_n\}$.

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Reductions of the problem

A key fact which permits the application of rational point methods is:

Proposition: If M is any hyperplane complement (defined over a number field) of dimension n, then $\mathrm{Gr}^{\mathbf{W}}_m H^j_c(X) = 0$ unless $m = 2(j-n)$

Thus M is minimally pure (mp) in the sense of Dimca-L (1997). It means that $P_c^G(t)$ may be calculated by computing rational points as in the example at the start of the talk.

This implies, in particular, for $g \in \Gamma$, that

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Further reduction

Suppose X is a K-variety with Γ action, such that $X = \amalg_i X^{(i)}$ is a decomposition into locally closed pieces which are permuted by Γ.

Then $W_c^\Gamma(X,t) = \sum_{\mathcal{O}_j \in \mathcal{O}} \mathrm{Ind}_{\Gamma_j}^{\Gamma}(W_c^{\Gamma_j}(X^{(j)},t))$,

where $\mathcal{O}$ is the set of orbits of Γ on the pieces, and Γ_j is the isotropy group in Γ of a point of $\mathcal{O}_j$.

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NB: $\widetilde{Y}_n \longrightarrow \widetilde{X}_n \simeq \mathcal{D}_n^0$ is an unramified μ_r-covering

Theorem (L, 2003).

$$P^{\mu_r}(X_n, t) = \begin{cases} (1+t)1_{\mu_r} & \text{if } r \text{ or } n \text{ is odd} \\ (1+t)\mu_r + (t^{n-1} + t^n)\varepsilon_{\mu_r} & \text{if } r \text{ and } n \text{ are even} \end{cases}$$

Equivalent formulation: $\pi_1(\mathcal{D}_n^0)$ is the Artin braid group of type B_n. For $\omega \in \hat{\mu}_r$, let ζ_ω be the representation of $\pi_1(\mathcal{D}_n^0)$ which takes the long root generators to 1, and the other generator to $\omega(\sigma)$, where $\mu_r = \langle \sigma \rangle$. Let $\mathcal{L}_\omega$ be the corresponding local system on $\mathcal{D}_n^0$.

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Remarks about the proof.

The proof is a combinatorial exercise in counting rational points.

It depends on a power series identity, as follows.

Frobenius orbits on $\bar{\mathbb{F}}_q^\times$ correspond to irreducible polynomials over $\mathbb{F}_q$. For such a polynomial $a(t)$, define $\tau_r(a) = a(0)^{\frac{q-1}{r}}$.

Assume $r | q-1$ and identify μ_r as a subgroup of $\mathbb{F}_q$. For $\alpha \in \mu_r$, let $m_d^r(\alpha, q)$ be the number of such polynomials $a(t)$ of degree d such that $\tau_r(a) = \alpha$

For $\lambda \in \hat{\mu}_r$ define the power series $F_\lambda^{(r)}(t) := 1 + \sum_{d \geq 1} \sum_{\alpha \in \mu_r} m_d^r(\alpha, q)\lambda(\alpha)t^d$.

Remarks about the proof.

The proof is a combinatorial exercise in counting rational points.

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We then have the following power series identity (generalisation of the ‘cyclotomic identity’)

$$F_\lambda(t) = \begin{cases} 1 \text{ if } |\lambda| \neq 1 \text{ or } 2 \\ \frac{1-qt^2}{1-t^2} \text{ if } |\lambda| = 2 \\ \frac{1-t^2q}{(1-tq)(1+t)} \text{ if } \lambda = 1. \end{cases}$$

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Remarks about the exceptional cases

There are 6 exceptional cases: 3 are two-dimensional; then $G(3,3,3) < G_{26}$, $W(D_4) < W(F_4)$, and $W(F_4) < \widetilde{W}(F_4)$ (extension by graph automorphism)

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Concluding remarks

The problem of comoplete description of the cohomology of the spaces X_G with local coefficients is far from solved.

There are close connections with the equivariant cohomology of the Milnor fibre, which Alex Dimca spoke about here in May.

Rational point methods may have limited applicability, but there are still plenty of opportunities to exploit them. In general, even when the cohomology is not mixed Tate, it has a mt part, which conjecturally (the Tate conjectures) is spanned by a subvariety.

There are examples to show that Hodge structure is connected with symmetries of the ambient variety. This is a line we are pursuing.

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References

A Dimca and G I Lehrer, “Hodge structure of the Milnor fibre of an arrangement” *arXiv, 2010*

M Kisin and G I Lehrer, “Eigenvalues of Frobenius and Hodge numbers”, *Pure and Appl. Math. Quart.* 2 (2006), 497-518.

G I Lehrer, “Coxeter group actions on the cohomology of toric varieties” *Ann. Fourier 58, 2008*, 671–688

Lehrer, G. I. “Rational points and cohomology of discriminant varieties”, *Adv. Math.* 186 (2004), no. 1, 229–250.

References

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References

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References

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Lehrer, G. I. “Rational points and cohomology of discriminant varieties”, *Adv. Math.* **186 (2004), no. 1, 229–250.**