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The Link between the KdV Hierarchy and Pseudodifferential Calculus

The lectures give a short review of some of the landmark discoveries of I. M. Gel'fand, I. M., and L. A. Dickey, linking the algebra of classical pseudodifferential operators (in a single variable x), or equivalently the algebra **Symb** of their symbols, to the hierarchy based on the KdV equation $u_t = u_{xxx} - 6u_x$ (the coefficient -6 being replaceable by any other nonzero number).

The composition of two symbols $\sigma_i(x, \xi)$ ($i = 1, 2$; $x, \xi \in \mathbb{R}$) is the standard one (except that we omit the factors $\sqrt{-1}$):

$$\sigma_1 \# \sigma_2 = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\partial_\xi^n \sigma_1\right) \partial_x^n \sigma_2. \tag{1}$$

One looks at the odd roots, in the sense of (1), of the symbol $\xi^2 - u(x)$ of the Sturm-Liouville operator $\frac{d^2}{dx^2} - u(x)$; these roots are Laurent series in ξ^{-1}. Denote by $\operatorname{Res}(\xi - u(x))^{\#\left(m+\frac{1}{2}\right)}$ the coefficient of ξ^{-1} in the symbol expansion of $(\xi - u(x))^{\#\left(m+\frac{1}{2}\right)}$. Direct calculation shows that

$$\operatorname{Res}(\xi - u(x))^{\#\left(\frac{1}{2}\right)} = -\frac{1}{2} u(x); \tag{2}$$

and the following theorem is easily proved:

Theorem 0.1. *Let $L = \partial_x^3 - 4u(x)\partial_x - 2\partial_x u(x)$ be the Lenard operator. For every positive integer m,*

$$\partial_x \operatorname{Res}\left(z^2 - u(x)\right)^{\#\left(\frac{1}{2}m+1\right)} = \frac{1}{4} L \operatorname{Res}\left(z^2 - u(x)\right)^{\#\frac{1}{2}m}. \tag{3}$$

Since (3) is precisely the recurrence relation between KdV polynomials R_m, (2) allows us to conclude that

$$R_m\left(u, \partial_x u, ..., \partial_x^{2m+1} u\right) = -2^{\kappa_m} \partial_x \operatorname{Res}\left(z^2 - u(x)\right)^{\#\left(\frac{1}{2}m+1\right)}$$

for some $\kappa_m \in \mathbb{Z}$.