Diffusion processes on nilpotent Lie groups
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We consider a group $S = NA$, where the group multiplication in S is given by

$$(x, a)(x', a') = (x\Phi_a(x'), aa'),$$

Φ_a $a \in A$ being a one-parameter group of shrinking automorphisms of a nilpotent Lie group N. By "shrinking" we mean

$$\lim_{a\to 0} \Phi_a x = e \quad \forall x \in N.$$

For a measure μ on S we say that a Radon measure m on N is *μ-invariant*, if for continuous functions with compact support we have

$$\int\int f(s.x)d\mu(s)dm(x) = \int f(x)dm(x).$$

On $C_c^\infty(S)$ we consider a second order left-invariant operator

$$\mathcal{L} = \sum_{j=0}^{m} Y_j^2 + Y$$

such that $Y_0, ..., Y_m$ generate $\mathcal{S}$ as a Lie algebra, i.e. $\mathcal{L}$ satisfies the strong Hörmander condition. In appropriate coordinates $\mathcal{L}$ can be written as

$$\mathcal{L} = \mathcal{L}_\alpha = (a\partial_a)^2 - \alpha(a\partial_a) + \sum_{j=1}^{m} \Phi_a(X_j)^2 + \Phi_a(X), \quad \alpha \in \mathbf{Z},$$

where Φ_a are the automorphisms of N indexed by $a \in A$ and $X_1, ..., X_m$ generate $\mathcal{N}$. Let μ_t be the semi-group of probability measures on S whose infinitesimal generator is $\mathcal{L}$. Then there exists a $\check{\mu}_t$-invariant measure m on N, if and only, if $\alpha \geq 0$. Then we have

Main Theorem (Ewa Damek and A.H.)

Let Σ be the unit sphere in N w.r. to an Euclidean inner product in $\mathcal{N}$. For every $\sigma \in \Sigma$ the limit

$$\lim_{r\to\infty} r^{Q+\alpha} m_\alpha(\Phi_r(\sigma)) = c(\sigma),$$

where Q is the homogeneous dimension of $\mathcal{N}$, is finite and positive. Moreover, the function $\Sigma \ni \sigma \to c(\sigma)$ is continuous.