Calculus of Variations

≡ *Ennio De Giorgi*

Novelties in the Variational Approach to Mappings of Smallest Energy

Tadeusz Iwaniec (Helsinki & Syracuse)

Lectures in Pisa

(Centro De Giorgi, June 11 - 15, 2012)

On the isoperimetric property of the hypersphere in the class of sets whose oriented boundary has finite measure

by Ennio De Giorgi

Atti Accad. Naz. Lincei Mem. (**8**) **5** (1958), pp. 33-44

Report by Renato Caccioppoli (referee)

"***Dr. Ennio De Giorgi, in some previous papers, has made deep studies concerning a very general concept, introduced by Caccioppoli, of measure for the oriented boundary of a set in the Euclidean space; measure named* perimeter *of the set by the***

author. These studies are cleverly applied in the present paper to establish isoperimetric property of the hypersphere, which is proved in the class of sets of finite perimeter. The proof is essentially based on an important compactness criterion and on rather hidden theorem concerning the comparison between the perimeter of a generic set and the perimeter of another set obtained by normalization and symmetrization with respect to a hyperplane. The result is important, but especially interesting is the original methodology of this work, which is a starting point for new type of variational isoperimetric problems. The Committee particularly recommend publication of the paper of Dr. De Giorgi in the series **Memoirs of the Accademia dei Lincei.**"

$*$ January 11, 1958. On behalf of the Committee, *Mauro Picone*

Helping Hands

Leonid Kovalev **Jani Onninen**

In fact Leonid and Jani are the major players in our lectures

Motto

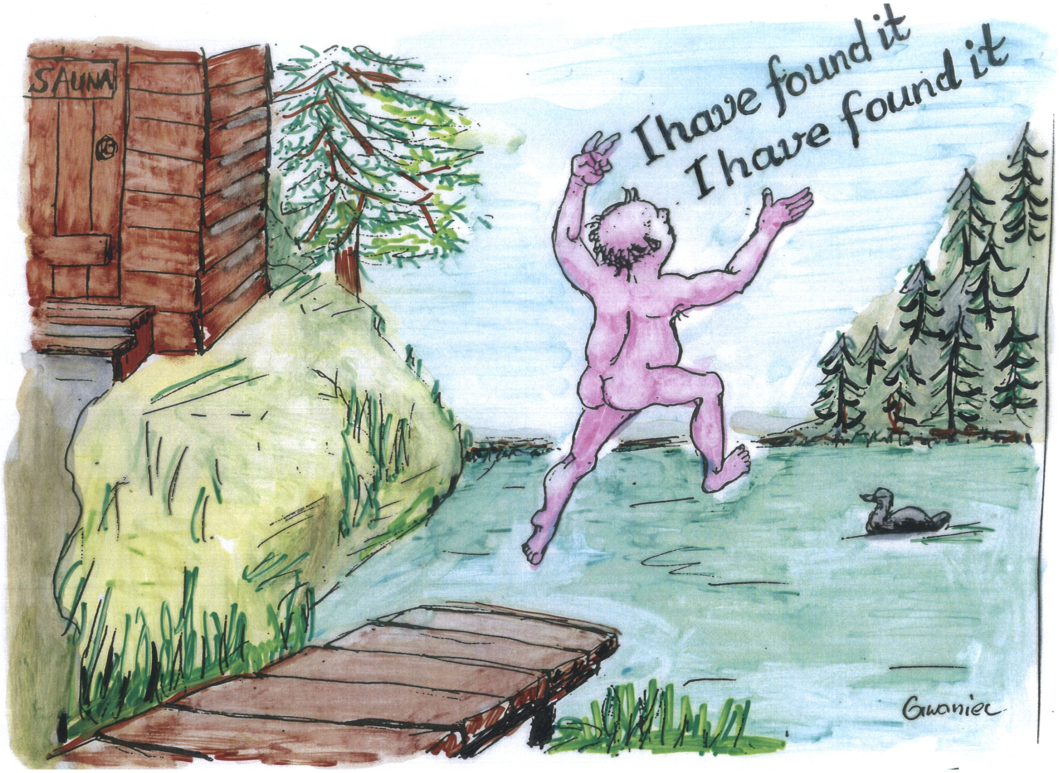
Sharing humor
and laughing with others
is more effective (powerful)
than laughing alone;

it strengthens friendship
and triggers positive feelings
among adversaries.

Heart-felt welcome

Tadeusz

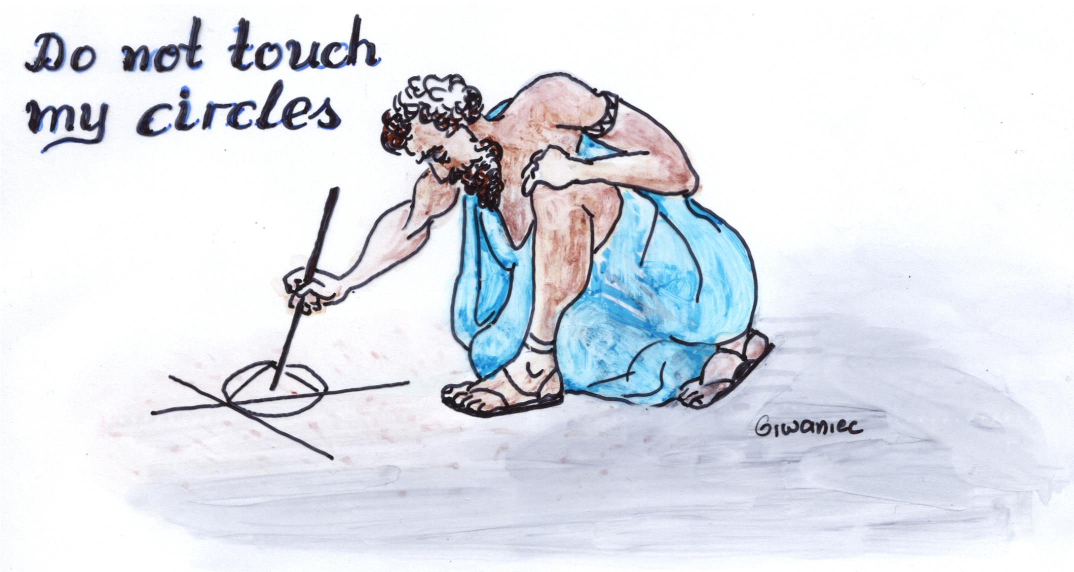
Tadeusz of SyracUSA/ Suomalinen



Blue lakes and sky,
white snow and white nights of the Finnish summer

Archimedes of Siracusa (287BC - 212BC)

Father of the application of scientific knowledge

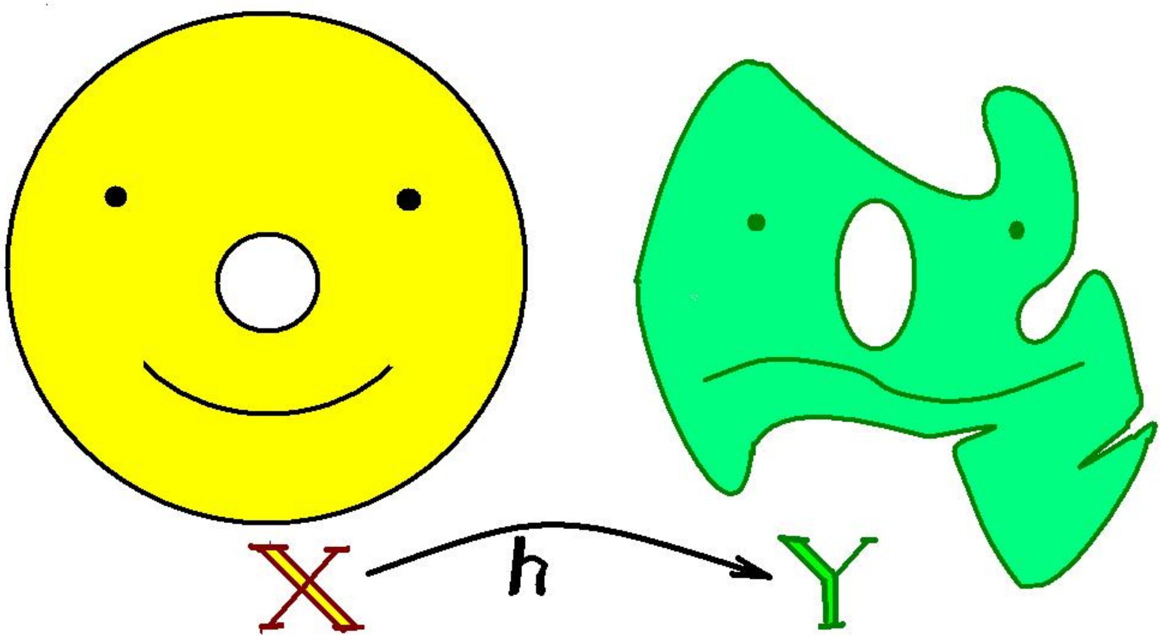


Quasiconformal Geometry and Nonlinear Elasticity share compelling beauty through variational integrals

- *Tadeusz, SiracUSA*

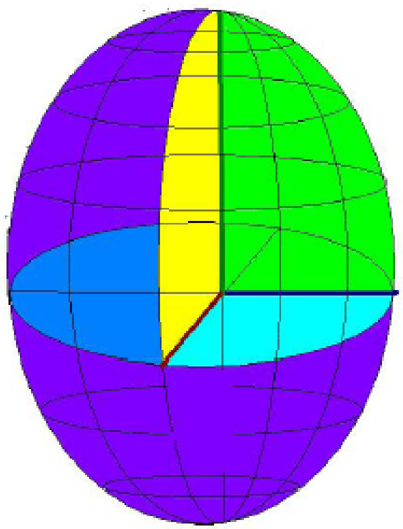
Mappings $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$

OUR LOGO

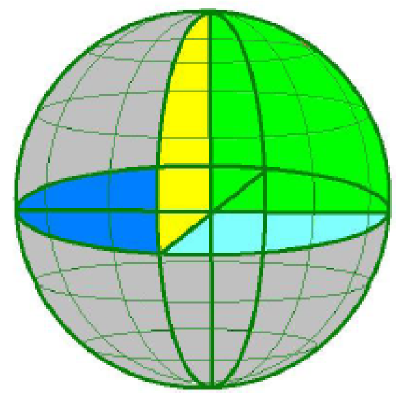


Quasiconformal Mappings

$$h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}\,, \quad y = h(x)$$



$$Dh(x) \quad : \quad \mathbf{T}_x\mathbb{X} \xrightarrow{\text{onto}} \mathbf{T}_y\mathbb{Y}$$



$$\{\xi \in \mathbf{T}_x\mathbb{X}\ ;\ \langle G(x)\xi\,,\,\xi\rangle = constant\}$$

$D^*h(x)\cdot Dh(x) = J(x,h)^{2/n}\,\mathbf{G}(x)$ - ***Beltrami Equation***

The n-dimensional Cauchy-Riemann system

$$D^*h(x)\ Dh(x) = J(x,h)^{2/n}\mathbf{I} \quad , \quad \text{conformal mappings}$$

Beltrami system for $\mathbf{G}$ conformal mappings

$$D^*h(x)\ Dh(x) = J(x,h)^{2/n}\mathbf{G}(x)$$

$\mathbf{G}(x)$ -***distortion tensor***, det $\mathbf{G}(x) \equiv 1$

In dimension $n = 2$ these equations are linear.

$$h_{\bar{z}} \ = \ \mu(z)h_z$$

The Energy Integrals

$$\mathscr{E}_{\mathbf{G}}[h] = \int_{\mathbb{X}} \langle\, \mathbf{G}^{-1}(x) D^*h\,,\, D^*h\,\rangle^{n/2}\,\mathsf{d}x\;, \quad \mathbf{G} - \text{conformal}$$

$$\mathscr{E}_{\mathbf{I}}[h] = \int_{\mathbb{X}} |Dh(x)|^n\,\mathsf{d}x\;, \quad -\text{conformal energy}$$

For every homeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of finite energy we have

$$\mathscr{E}_{\mathbf{G}}[h] = \int_{\mathbb{X}} \langle\, \mathbf{G}^{-1}(x) D^*h\,,\, D^*h\,\rangle^{n/2}\,\mathsf{d}x$$

$$\geqslant n^{n/2} \int_{\mathbb{X}} J(x,h)\,\mathsf{d}x = n^{n/2}|\mathbb{Y}|$$

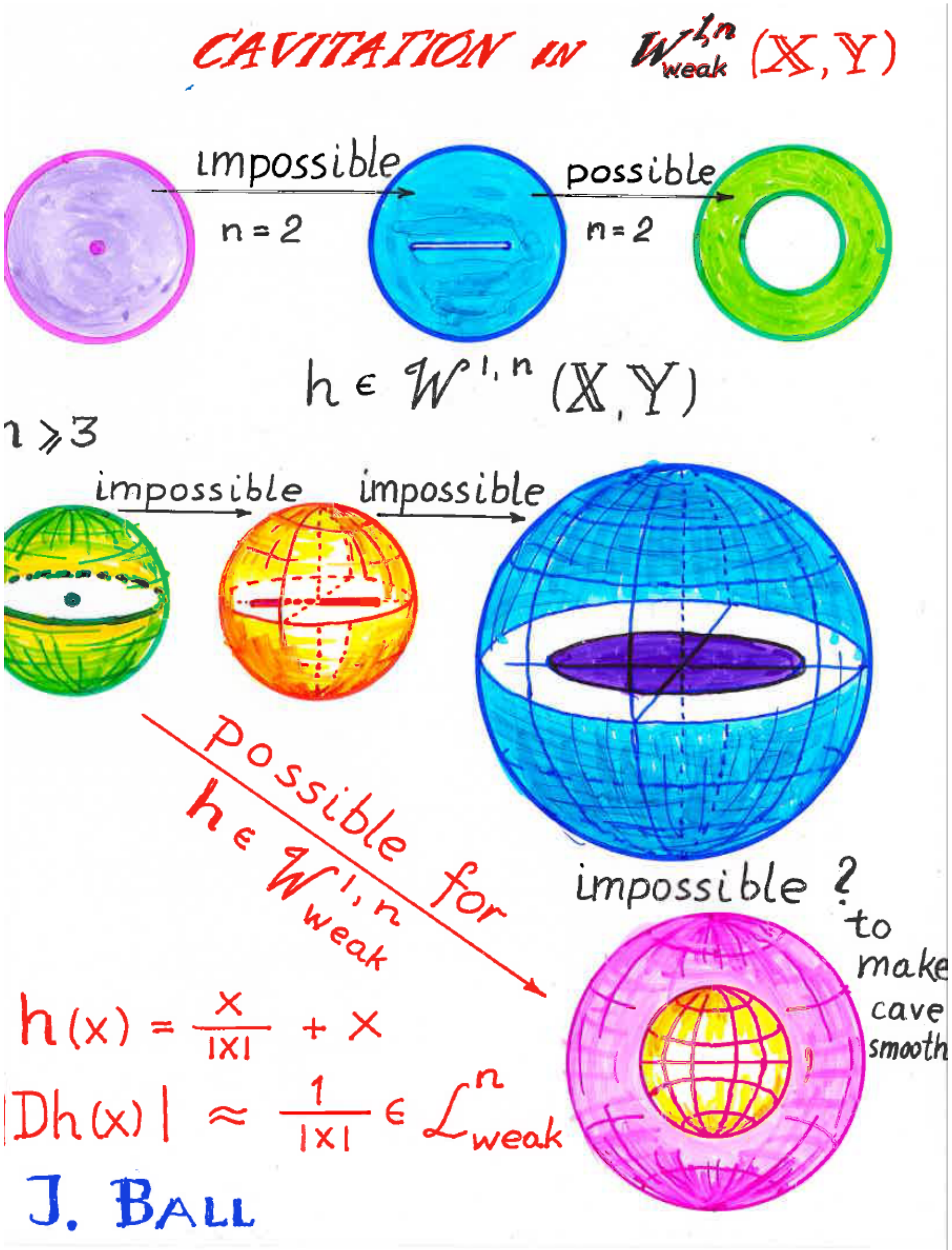
Equality occurs if and only if h satisfies the Beltrami system

$$D^*h(x)\ Dh(x) = J(x,h)^{2/n}\mathbf{G}(x) \quad , \quad \mathbf{G}\text{−conformal solutions}$$

In dimension $n \geqslant 3$, no G -conformal solution may exist (even locally), but the energy-minimal solutions (usually) do exist. In this latter case we have

$$\mathscr{E}_{\mathbf{G}}[h] \ > \ n^{n/2}|\mathbb{Y}|$$

The existance and geometric features of the G -energy minimal maps are challenging problems in Geometric Function Theory.


CAVITATION IN $W^{1,n}_{weak}(\mathbb{X},\mathbb{Y})$
impossible
n = 2
possible
n = 2
$h \in W^{1,n}(\mathbb{X},\mathbb{Y})$
n ⩾ 3
impossible
impossible
possible for $h \in W^{1,n}_{weak}$
impossible ?
to make cave smooth
$h(x) = \frac{x}{|x|} + x$
$|Dh(x)| \approx \frac{1}{|x|} \in \mathcal{L}^n_{weak}$
J. BALL

Nonlinear Hyperelasticity
(brief description)

One enquires into deformations $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of smallest energy

$$\mathscr{E}[h] = \int_{\mathbb{X}} \mathbf{E}(x, h, Dh)\, \mathrm{d}x \,, \qquad \mathbf{E} : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{m \times n} \to \mathbb{R}$$

where the accustomed hypothesis is Morrey's Quasiconvexity. This yields lower semicontinuity of the energy functional

$$\int \mathbf{E}(x, h, Dh)\, \mathrm{d}x \;\leqslant \liminf \int \mathbf{E}(x, h_k, Dh_k)\, \mathrm{d}x$$

whenever $h_k \rightharpoonup h$, weakly in $\mathscr{W}^{1,p}(\mathbb{X})$.

Special (more practical) cases are ***polyconvex functionals***. They are convex with respect to subdeterminants $D_{\boxplus}h$ of the differential matrix Dh

$$\mathscr{E}[h] = \int_{\mathbb{X}} \mathbf{E}(x, h, D_{\boxplus}h)\, \mathsf{d}x$$

Such are ***neohookean energy*** integrals like this

$$\mathscr{E}[h] = \int_{\mathbb{X}} \Big(|Dh(x)|^n + \tfrac{1}{J(x,h)} \Big)\, \mathsf{d}x$$

In quasiconformal geometry, a fundamental example of polyconvex functionals is furnished by the $\mathscr{L}^1$ -norm of the ***inner distortion*** function of the inverse map $f = h^{-1} : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$,

$$\mathscr{E}[f] = \int_{\mathbb{Y}} \mathbf{K}_I(y,f)\ \ \mathsf{d}y = \int_{\mathbb{Y}} \frac{|D^\sharp f(y)|^n}{[J(y,f)]^{n-1}}\ \ \mathsf{d}y\ = \int_{\mathbb{X}} |Dh(x)|^n\, \mathsf{d}x$$

Such mappings have originated in the paper by V. Šverák and T. Iwaniec, "***On mappings with integrable dilatation***" Proceedings of AMS vol. 118, no. 1 (1993), pp. 181-188. Those of smallest $\mathscr{L}^1$ - norm of the distortion are the analogues of the planar Teichmüller mappings. In the forthcoming lectures, the basic concepts developed for the Teichmüller theory (quadratic differentials and their trajectories) will also come into play in our approach to extremal harmonic maps.

Dirichlet Energy ($n = 2$)

The Dirichlet (or conformal) energy of a Sobolev mapping $h \in \mathscr{W}^{1,2}(\mathbb{X})$ is given by

$$\mathscr{E}_{\mathbb{X}}[h] = \iint_{\mathbb{X}} |Dh|^2 \; = \; 2 \iint_{\mathbb{X}} \left(|h_z|^2 + |h_{\bar{z}}|^2 \right)$$

The first variation of $\mathscr{E}$ results in the Euler-Lagrange equation.

$$\Delta h = 4 h_{z\bar{z}} = 0$$

Hopf-Laplace Equation

In contrast, the inner variation leads to a nonlinear equation

$$\frac{\partial}{\partial \bar{z}}\left(h_z\,\overline{h_{\bar{z}}}\right) = 0, \qquad h_z\,\overline{h_{\bar{z}}} = \phi, \quad (\phi \text{ is analytic})$$

for mapping in the Sobolev space $\mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{X})$. This equation will henceforth be referred to as the *Hopf-Laplace equation*, **also known as** *energy-momentum* **or** *equilibrium* **equation.**

THEOREM. ***Every homeomorphism $h \in \mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{X})$ which satisfies the Hopf-Laplace equation is in fact a harmonic diffeomorphism.***

Lecture 1, Harmonic Mappings

Radó-Kneser-Choquet Theorem

A harmonic map $h:\mathbb{X}\to\mathbb{C}$ of a Jordan domain, which extends continuously as a monotone map of $\partial\mathbb{X}$ onto a boundary of a convex region $\mathbb{Y}$ is a $\mathscr{C}^\infty$ -diffeomorphism of $\mathbb{X}$ onto $\mathbb{Y}$.

See: P. Duren, ***Harmonic mappings in the plane***, *Cambridge Tracts in Mathematics, 156. Cambridge University Press, Cambridge, 2004.*

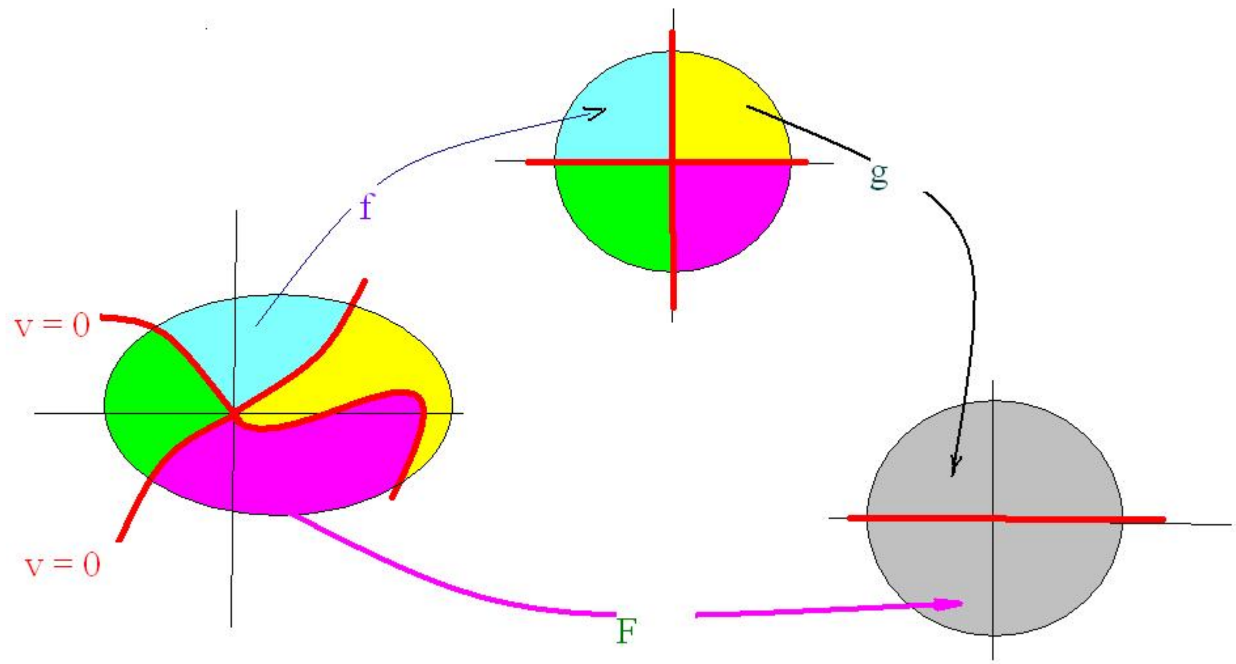
THEOREM. Every harmonic homeomorphism $h = u + i\,v : \mathbb{X} \to \mathbb{C}$ is a $\mathscr{C}^\infty$-diffeomorphism

Proof. Suppose to the contrary that $h(0) = 0$ and $J_h(0) = 0$. Thus $\nabla u(z)$ and $\nabla v(z)$ are linearly dependent at $z = 0$. With the aid of a rotation of h (multiplying it by a suitable complex number) we are reduced to the case when $\nabla v(0) = 0$. Now consider a real-valued harmonic conjugate function $w = w(z)$ (locally defined near 0) such that

$F(z) = w(z) + i\,v(z)$ is analytic and $F(0) = 0$

The Cauchy-Riemann equations $w_x = v_y$ and $w_y = -v_x$ yield $F'(0) = 0$. Thus for some $k \geqslant 2$ we have

$$F(z) = g \circ f = f^k\,,\; f = f(z) \quad \text{is a conformal homeomorphism}\,, \quad f'(0) \neq 0$$

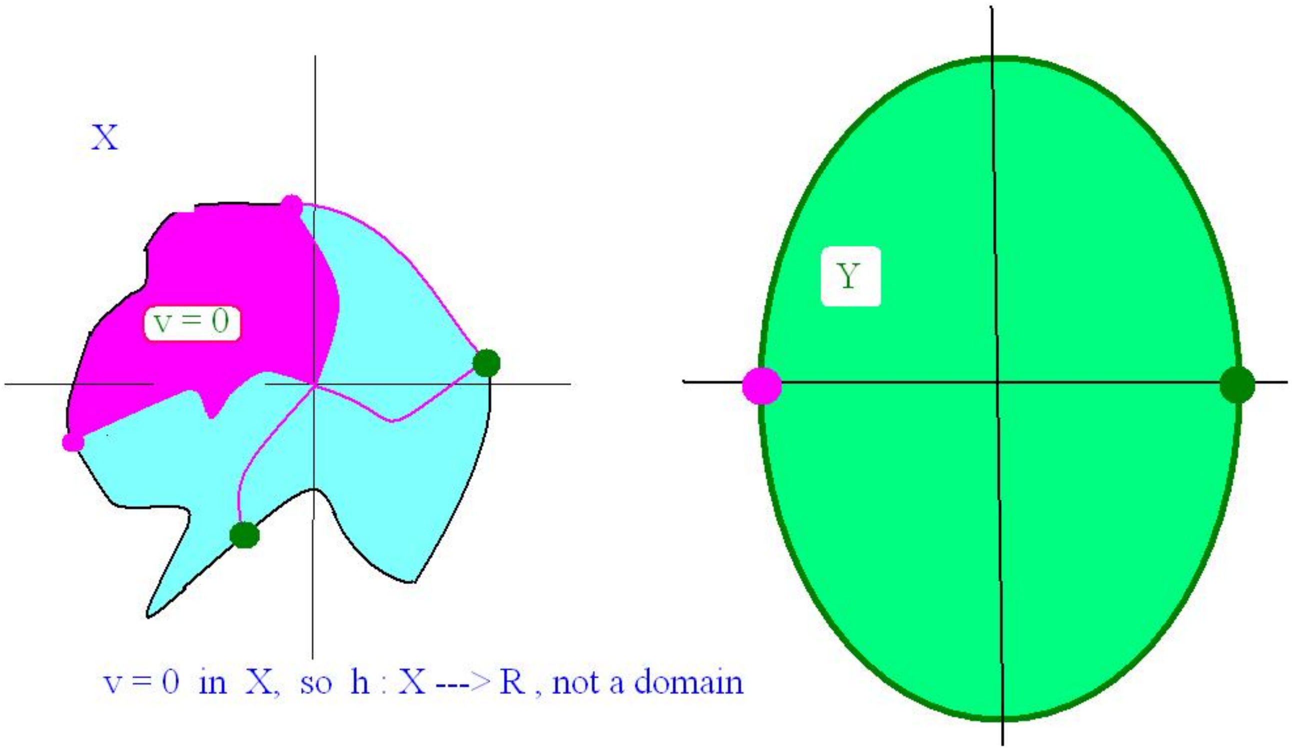


$$v = \Im m\, h$$

, **This configuration of branches is impossible for a homeomorphism. Indeed h cannot take 3 (or more) disjoint branches of the level set $\{v = 0\}$ into one line.**

Proof of Radó-Kneser-Choquet Theorem

By max/min principle $h : \mathbb{X} \overset{\text{into}}{\longrightarrow} \mathbb{Y}$. We need only show that $h = u + i\,v : \mathbb{X} \to \mathbb{C}$ is a local homeomorphism; equivalently, that $J_h(z) \neq 0$. As before, assume that $h(0) = 0$ and $J_h(0) = 0$, say $\nabla v(0) = 0$.



Target is convex.

THEOREM. ***If the infimum Dirichlet energy within the class $\mathscr{H}_2(\mathbb{X}, \mathbb{Y})$ is attained, then the energy-minimal map is harmonic.***
Proof. By harmonic replacement argument (at the blackboard).

THEOREM. ***For Lipschitz multiply connected domains (or simply connected + normalization), the infimum Dirichlet energy within the class $\overline{\mathscr{H}_2}(\mathbb{X}, \mathbb{Y})$ is attained at some $\hbar \in \overline{\mathscr{H}_2}(\mathbb{X}, \mathbb{Y})$. If $\hbar$ is harmonic, then it belongs to $\mathscr{H}_2(\mathbb{X}, \mathbb{Y})$.***

Proof. $J_\hbar \geqslant 0$, so $|\hbar_z| \geqslant |\overline{\hbar_{\bar z}}|$, where $\hbar_z$ and $\overline{\hbar_{\bar z}}$ are analytic. We rule out the case $\hbar_z \equiv 0$, because otherwise $\hbar \equiv const$ which is not the energy minimal map. Thus the analytic function $\hbar_z$ has isolated zeros in $\mathbb{X}$ outside which the ratio $\left|\frac{\overline{\hbar_{\bar z}}}{\hbar_z}\right| \leqslant 1$. Consequently, the zeros of $\hbar_z$ are removable singularities of the meromotphic function $\frac{\overline{\hbar_{\bar z}}}{\hbar_z}$. We conclude that outside the zeros $|\frac{\overline{\hbar_{\bar z}}}{\hbar_z}| < 1$, or $J_\hbar \equiv 0$. The latter case is impossible for mappings in $\overline{\mathscr{H}_2}(\mathbb{X}, \mathbb{Y})$. The former case tells us that $\hbar$ is a local homeomorphism outside the zeros of $\hbar_z$. This, by elementary topology, is possible for $\hbar \in \overline{\mathscr{H}_2}(\mathbb{X}, \mathbb{Y})$ only when $\hbar \in \mathscr{H}_2(\mathbb{X}, \mathbb{Y})$.

We just established the following principle for the Dirichlet energy:

The loss of harmonicity comes exactly with the loss of injectivity.

Coffee is proof that

God loves mathematicians

and wants us

to be happy

Benjamin Franklin (paraphrase)

Existence of Energy-Minimal Homeomorphisms

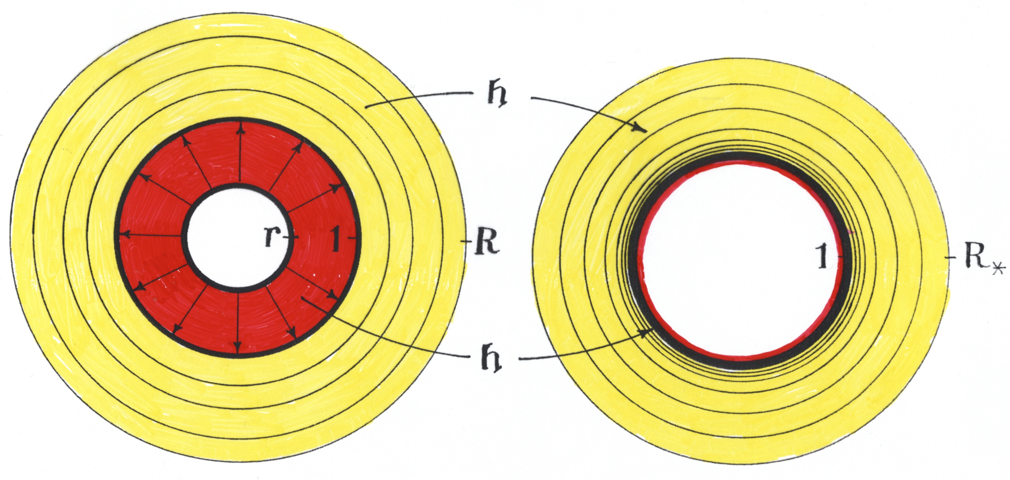
(K. Astala, G. Martin, T.I. Arch. Rat. Mech. Anal. 2010)

An energy–minimal homeomorphism
$h: \ A(r,R) \xrightarrow{\text{onto}} A(r_*,R_*)$ ***between annuli exists iff***

$$\frac{R_*}{r_*} \geqslant \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right)$$

A straightforward proof of this (*so called Nitsche bound*) has been established *(J. Onninen and T.I. Memoirs of AMS, 2011)* via the concept of *free Lagrangians*.

Below the Nitsche bound, $\frac{R_*}{r_*} \leqslant \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right)$



$$\mathfrak{h}(z) = \begin{cases} \frac{z}{|z|}\,, & r \leqslant |z| \leqslant 1 \quad \left(\begin{smallmatrix}\text{collapsing into} \\ \text{concave boundary}\end{smallmatrix}\right) \\ \frac{1}{2}\left(z + \frac{1}{\bar{z}}\right)\,, & 1 < |z| < R \quad \text{elastic response} \end{cases}$$

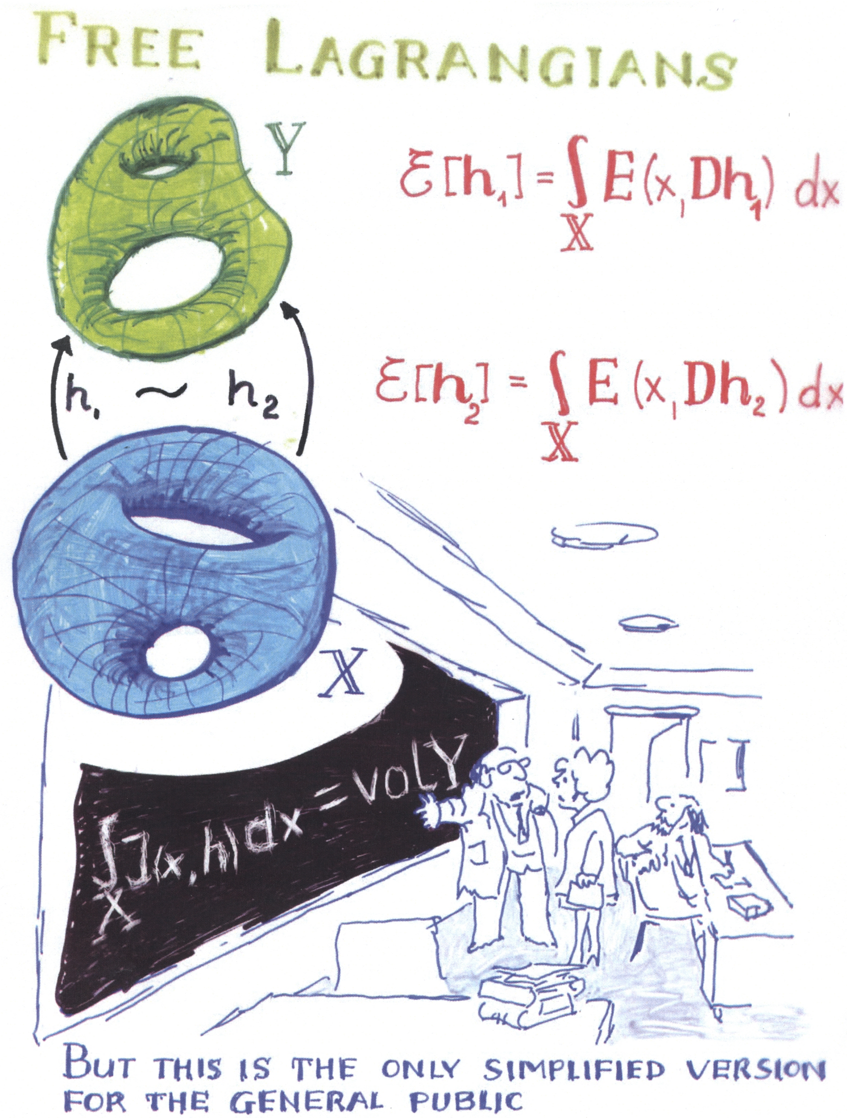
This energy-minimal map is a $\mathscr{W}^{1,2}$ -strong limit of homeomorphisms. Here we scaled the target annulus by setting $r_* = 1 < R_* = \frac{1}{2}\left(R + \frac{1}{R}\right)$, for some number $R > 1$. Consequently, the domain annulus is given the inner and outer radii $r \leqslant 1 < R$.

The Nitsche Conjecture

In this latter example the energy-minimal deformation fails to be harmonic homeomorphism. Actually there is no harmonic homeomorphism at all. In the early 1960's German-American mathematician Johannes C.C. Nitsche raised a question of existence of harmonic homeomorphisms between annuli. This fascinating problem is deeply rooted in the theory of doubly connected minimal surfaces. Nitsche's conjecture, which is now a theorem (*L. Kovalev, J. Onninen, T.I.*, JAMS 2011), asserts that

A harmonic homeomorphism exists iff :

$$\frac{R_*}{r_*} \geqslant \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right)$$


FREE LAGRANGIANS
Y
$\mathcal{E}[h_1] = \int_X E(x, Dh_1)\, dx$
$h_1 \sim h_2$
$\mathcal{E}[h_2] = \int_X E(x, Dh_2)\, dx$
X
$\int_X J(x, h)\, dx = \mathrm{vol}\, Y$
BUT THIS IS THE ONLY SIMPLIFIED VERSION
FOR THE GENERAL PUBLIC

Free Lagrangians

We are about to introduce the basic concept in the study of homeomorphisms with smallest energy, a

Given two domains Ω and Ω^* in $\mathbb{R}^n$, we shall consider orientation preserving homeomorphisms $h : \Omega \to \Omega^*$ in a suitable Sobolev class $\mathscr{W}^{1,p}(\Omega, \Omega^*)$ so that a given energy integral

$$I[h] = \int_\Omega \mathcal{E}(x, h, Dh)\, dx$$

is well defined. The term *Free Lagrangian* pertains to a differential n-form $\mathcal{E}(x, h, Dh)\, dx$ whose integral depends only on the homotopy class of $h : \Omega \xrightarrow{\text{onto}} \Omega^*$. The example of the Jacobian determinant is pretty obvious;

we state it as

$$\int_\Omega J(x,h)\,dx = |\Omega^*|$$

This identity holds for all orientation preserving homeomorphisms of Sobolev class $\mathscr{W}^{1,n}(\Omega,\Omega^*)$. Many more differential expressions enjoy a property such as this. In the next three lemmas we collect examples of free Lagrangians for homeomorphisms $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ of annuli $\mathbb{A} = \{x\,; \quad r < |x| < R\}$ and $\mathbb{A}^* = \{x\,; \quad r_* < |x| < R_*\}$.

Lemma Let $\Phi : [r_*, R_*] \to \mathbb{R}$ be any integrable function. Then the n-form

$$\Phi(|h|)\,dh^1 \wedge \ldots \wedge dh^n$$

is a free Lagrangian.

Precisely, we have

$$\int_{\mathbb{A}} \Phi(|h|) J(x,h)\, dx = \omega_{n-1} \int_{r_*}^{R_*} \tau^{n-1}\, \Phi(\tau)\, d\tau$$

for every orientation preserving homeomorphism $h \in \mathscr{W}^{1,n}(\mathbb{A}, \mathbb{A}^*)$.
This is none other than a general formula of integration by substitution.
Lemma. The following differential n-form

$$\sum_{i=1}^{n} \frac{x_i\, dx_1 \wedge \ldots \wedge dx_{i-1} \wedge d|h| \wedge dx_{i+1} \wedge \ldots \wedge dx_n}{|h|\, |x|^n} = \frac{(d|h|) \wedge \star d|x|}{|h|\, |x|^{n-1}}$$

is a free Lagrangian in the class of all homeomorphisms $h \in \mathscr{W}^{1,1}(\mathbb{A}, \mathbb{A}^*)$ preserving the order of the boundary components of the annuli $\mathbb{A}$ and $\mathbb{A}^*$.

Precisely, we have

$$\int_{\mathbb{A}} \frac{d|h| \ \wedge \star d|x|}{|h|\,|x|^{n-1}} = \text{Mod}\,\mathbb{A}^*$$

Another free Lagrangians, dual to the above, relies on the topological degree.

Lemma. The following differential n-form

$$\sum_{i=1}^{n} \frac{h^i\, dh^1 \wedge \ldots \wedge dh^{i-1} \wedge d|x| \wedge dh^{i+1} \wedge \ldots \wedge dh^n}{|x|\,|h|^n} \;=\; \frac{d|x|}{|x|} \wedge h^\sharp \omega$$

is a free Lagrangian on the class of all orientation preserving homeomorphism $h \in \mathscr{W}^{1,n-1}(\mathbb{A}, \mathbb{A}^*)$. Precisely, ω is the standard volume form in $\mathbb{S}^{n-1}$

$$\omega = \sum_{i=1}^{n} (-1)^i \frac{y_i\, dy_1 \wedge \ldots \wedge dy_{i-1} \wedge dy_{i+1} \wedge \ldots \wedge dy_n}{|y|^n}\,, \quad \int_{\mathbb{A}} \frac{d|x|}{|x|} \wedge h^\sharp \omega = \text{Mod}\,\mathbb{A}$$

Proof Below The Nitsche Configuration

$\frac{R_*}{r_*} \leqslant \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right)$. The target annulus $\mathbb{A}^*$ is too thin, a portion of $\mathbb{A}$ has to be hammered into the inner circle of $\mathbb{A}^*$. Let us invoke normal (radial) and tangential (angular) derivatives

$$|Dh|^2 = |h_N|^2 + |h_T|^2 \ , \qquad J_h = \text{Im}\,\left(h_T\,\overline{h_N}\right) \leqslant |h_N|\;|h_T|$$

We begin with an obvious inequality

$$\left(\frac{|h|\,|h_N|}{\sqrt{|h|^2-1}} - |h_T|\right)^2 \geqslant 0 \ , \quad \text{equivalently:}$$

$$\begin{aligned} |Dh|^2 \quad & \geqslant \left|\frac{h_T}{h}\right|^2 + 2\,|h_N|\;|h_T|\,\sqrt{1-|h|^{-2}} \\ & \geqslant \frac{2}{|x|}\,\text{Im}\left(\frac{h_T}{h}\right) - \frac{1}{|x|^2} + 2J_h\,\sqrt{1-|h|^{-2}} \end{aligned}$$

Each of the three terms is a free Lagrangian. Therefore,

$$\int_{\mathbb{A}} |Dh|^2 \;\geqslant\; \int_{\mathbb{A}} \left[\frac{2}{|x|} \operatorname{Im}\left(\frac{h_T}{h}\right) - \frac{1}{|x|^2} + 2J_h \sqrt{1-|h|^{-2}} \right] \mathrm{d}x$$

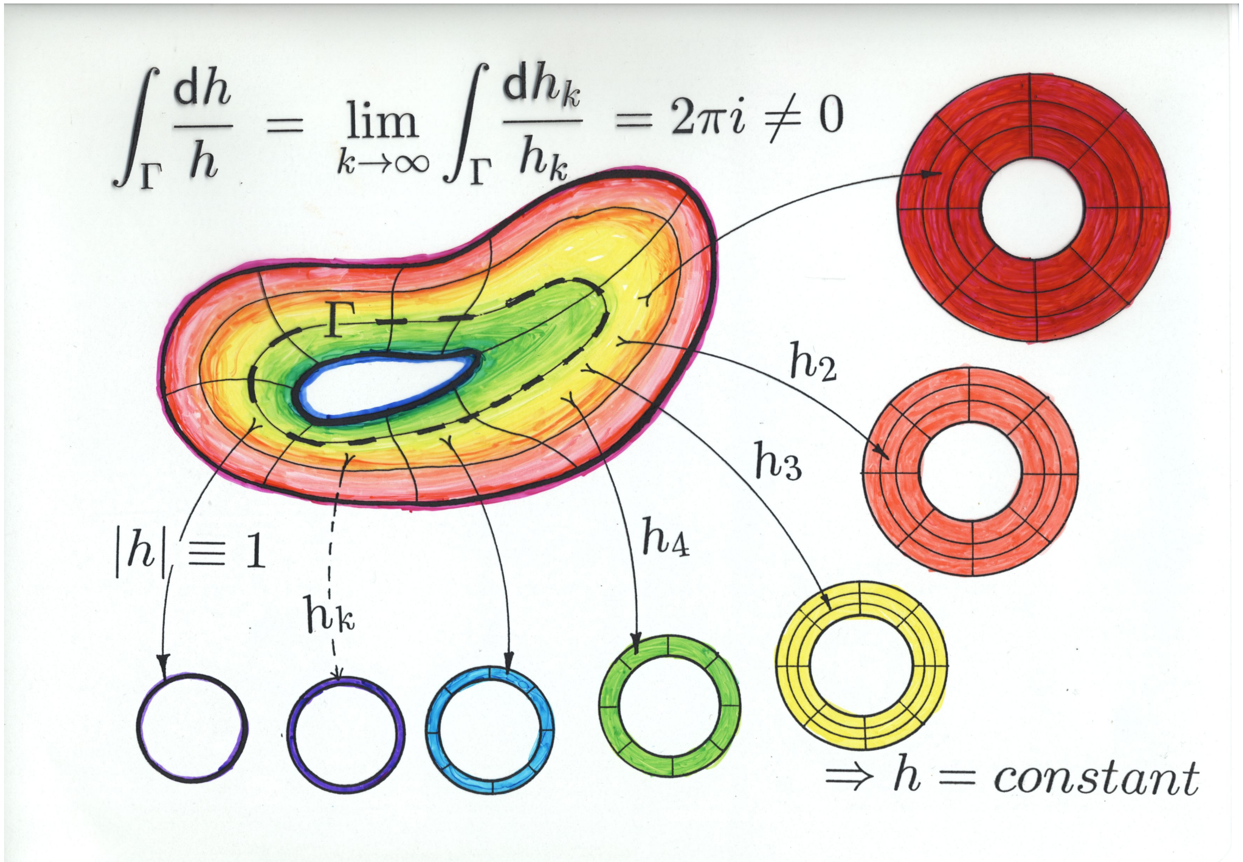
$$= 4\pi \log \frac{R}{r} - 2\pi \log \frac{R}{r} + 2\pi \left[\tau\sqrt{\tau^2-1} - \log(\tau + \sqrt{\tau^2-1}) \right]_{\tau=1}^{R_*}$$

$$= 2\pi \log \frac{1}{r} + 2\pi R_* \sqrt{R_*^2 - 1}$$

The first term represents the energy of the hammering map $z \mapsto \frac{z}{|z|}$; that is, $R = R_* = 1$. The second term represents the energy of the critical Nitsche map $z \mapsto \frac{1}{2}\left(z + {}^1\!/\bar{z}\right)$; that is, $r = 1$. The above energy-minimal map is unique up to the rotation, by backwards analysis of the inequalities. REMARK. Any energy-minimal map must satisfy the Hopf-Laplace equation. Indeed, in spite of irregularity of the above minimal map, we have

$$h_z \cdot \overline{h_{\bar{z}}} = \frac{-1}{4z^2}\,, \quad \text{an analytic function}$$

About the Nitsche Conjecture



Failure of Radial Symmetry

Theorem. (*Onninen, T.I.* Mem. Amer. Math. Soc. 2011). For each $n \geqslant 4$, there are annuli $\mathbb{A}(1,R)$ and $\mathbb{A}^*(1,R_*)$ in $\mathbb{R}^n$ such that the infimum of the n-harmonic energy

$$\mathscr{E}[h] = \int_{\mathbb{A}} |Dh(x)|^n \, dx$$

among homeomorphisms is smaller than the energy of any radial map.

Genuine mathematics does not abide in complexity but somewhere in the unlimited beauty.

Let me paraphrase Luciano Pavarotti:

Learning mathematics by only reading about it is like making love by e-mail

Thanks for listening