EXISTENCE OF TRACTION FREE MINIMAL DEFORMATIONS

(within the Strong Limits of Sobolev Homeomorphisms)

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You bet *Jani Onninen* had hands in it.

Direct Method

This is a general method for constructing a proof of the existence of a minimizer for a given functional, the method introduced by Stanisław Zaremba and David Hilbert around 1900.

Consider an energy integral of the form

$$\mathscr{E}[\boldsymbol{h}] = \int_{\mathbb{X}} \mathbf{E}(\boldsymbol{x}, \boldsymbol{h}, \boldsymbol{D}\boldsymbol{h})\, \mathrm{d}\boldsymbol{x}$$

defined for mappings $h : \mathbb{X} \to \mathbb{Y}$ in a subset $\mathscr{H} \subset \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, $p > 1$ (a reflexive Banach space), where $\mathbb{X}$ and $\mathbb{Y}$ are domains in $\mathbb{R}^n$ and

$$\mathbf{E} : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{n \times n} \to \mathbb{R}$$

The direct method may be applied to such a functional by showing

1 $\mathscr{E}$ is bounded from below,

2 Any minimizing sequence for $\mathscr{E}$ is bounded, and

3 $\mathscr{E}$ is weakly sequentially lower semi-continuous. Precisely, for any sequence $\{h_j\} \subset \mathscr{H}$ weakly converging to h it holds that $h \in \mathscr{H}$ and $\mathscr{E}[h] \leqslant \liminf \mathscr{E}[h_j]$

Showing sequential lower semi-continuity is usually the most difficult part when applying the direct method.

Finding the proper class $\mathscr{H} \subset \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ for a specific minimization problem is the heart of the matter.

Notation and Preliminaries

Domains.

We shall study a pair of bounded domains $\mathbb{X}\,,\ \mathbb{Y} \subset \mathbb{R}^n \quad (\mathbb{R}^2 \simeq \mathbb{C})$ of the same connectivity $1 \leqslant \ell < \infty$. This amounts to saying that each of the complements $\mathbb{R}^n \setminus \mathbb{X}$ and $\mathbb{R}^n \setminus \mathbb{Y}$ consists of ℓ disjoint closed connected sets.

$\mathbb{X}_1, \mathbb{X}_2, ..., \mathbb{X}_\ell$ -the components of $\mathbb{R}^n \setminus \mathbb{X}$

$\mathbb{Y}_1, \mathbb{Y}_2, ..., \mathbb{Y}_\ell$ -the components of $\mathbb{R}^n \setminus \mathbb{Y}$

Their boundaries are exactly the components of $\partial\mathbb{X}$ and $\partial\mathbb{Y}$, respectively;

$\partial\mathbb{X}_1, \partial\mathbb{X}_2, ..., \partial\mathbb{X}_\ell$ -the components of $\partial\mathbb{X}$

$\partial\mathbb{Y}_1, \partial\mathbb{Y}_2, ..., \partial\mathbb{Y}_\ell$ -the components of $\partial\mathbb{Y}$

Adding any number of sets in $\{\mathbb{X}_1, \mathbb{X}_2, ..., \mathbb{X}_\ell\}$ to $\mathbb{X}$ results in a domain.

Doubly connected domains .

For a doubly connected domain $\mathbb{X}$ we reserve special notation, $\mathbb{X}_I$ and $\mathbb{X}_O$, for the bounded and unbounded components of $\mathbb{R}^n \setminus \mathbb{X}$, respectively. Thus $\partial\mathbb{X}$ consists of two continua $\partial\mathbb{X}_I = \partial_I\mathbb{X}$ and $\partial\mathbb{X}_O = \partial_O\mathbb{X}$, referred to as $inner$ and $outer\ boundaries$.

Boundary Correspondence .

Every homeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ gives rise to a one-to-one correspondence between boundary components of $\mathbb{X}$ and boundary components of $\mathbb{Y}$ which, upon a suitable arrangement of indices, will be designated as

$$h : \partial\mathbb{X}_\nu \rightsquigarrow \partial\mathbb{Y}_\nu \ , \quad \text{for} \quad \nu = 1, ..., \ell$$

Precisely this means that the cluster set of $\partial\mathbb{X}_\nu$ under h is the boundary component $\partial\mathbb{Y}_\nu$. The components will be so ordered that the outer boundary of $\mathbb{X}$ corresponds to the outer boundary of $\mathbb{Y}$.

The class $\mathscr{H}(\mathbb{X},\mathbb{Y})$.

It consists of all orientation preserving homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ which satisfy the boundary correspondence (as above) .

$\mathscr{W}^{1,p}(\mathbb{X},\mathbb{Y})$ *-Sobolev space of mappings*

$$\boldsymbol{h = (h^1, h^2, ..., h^n) : \mathbb{X} \xrightarrow{\text{into}} \overline{\mathbb{Y}}}$$

$$\boldsymbol{Dh = \left[\frac{\partial h^i}{\partial x_j}\right] \in \mathscr{L}^p(\mathbb{X}, \mathbb{R}^{n\times n})}$$

Note that $h \in \mathscr{W}^{1,p}(\mathbb{X},\mathbb{Y})$ has range in the closure of the target domain;

$$\boldsymbol{h} : \mathbb{X} \xrightarrow{\text{into}} \overline{\mathbb{Y}} \quad (\text{ most often } \mathbb{Y} \subset h(\mathbb{X}) \subset \overline{\mathbb{Y}} \text{)}$$

.

The Space $\mathscr{H}_p(\mathbb{X},\mathbb{Y})$ *and Relevant Classes*

Various classes of Sobolev mappings will be considered. Here they are:

- $\mathscr{H}_p(\mathbb{X},\mathbb{Y}) = \mathscr{H}(\mathbb{X},\mathbb{Y}) \cap \mathscr{W}^{1,p}(\mathbb{X},\mathbb{Y})\,, \quad 1 < p < \infty\,.$

- $\overline{\mathscr{H}_p}(\mathbb{X},\mathbb{Y})$ is the closure of $\mathscr{H}_p(\mathbb{X},\mathbb{Y})$ in strong topology of $\mathscr{W}^{1,p}(\mathbb{X},\mathbb{Y})$

- $\widetilde{\mathscr{H}_p}(\mathbb{X},\mathbb{Y})$ is the closure of $\mathscr{H}_p(\mathbb{X},\mathbb{Y})$ in weak topology of $\mathscr{W}^{1,p}(\mathbb{X},\mathbb{Y})$

- $\widehat{\mathscr{H}_p}(\mathbb{X},\mathbb{Y}) \subseteq \widetilde{\mathscr{H}_p}(\mathbb{X},\mathbb{Y})$ stands for the family of all weak $\mathscr{W}^{1,p}(\mathbb{X},\mathbb{Y})$ -limits of sequences $\{h_j\} \subset \mathscr{H}_p(\mathbb{X},\mathbb{Y})$.

- $\mathit{Diff}_p(\mathbb{X},\mathbb{Y})$ consists of $\mathscr{C}^\infty$-diffeomorphisms in $\mathscr{H}_p(\mathbb{X},\mathbb{Y})$. The fact is that in planer domains $\overline{\mathscr{H}_p}(\mathbb{X},\mathbb{Y}) = \overline{\mathit{Diff}_p}(\mathbb{X},\mathbb{Y})$.

Remark.

In general, the family of all weak limits of a set in a Banach space need not be weakly closed. That is why we have only the inclusion $\widehat{\mathscr{H}_p}(\mathbb{X},\mathbb{Y}) \subsetneq \widetilde{\mathscr{H}_p}(\mathbb{X},\mathbb{Y})$. However if $\mathbb{X}$ and $\mathbb{Y}$ are planar Lipschitz domains, both not simply connected, then for $p \geqslant 2$ we have

$$\widehat{\mathscr{H}_p}(\mathbb{X},\mathbb{Y}) = \widetilde{\mathscr{H}_p}(\mathbb{X},\mathbb{Y}) = \overline{\mathscr{H}}_p(\mathbb{X},\mathbb{Y}) \quad (\,\overline{\mathscr{H}}_2(\mathbb{X},\mathbb{Y}) = \overline{\mathscr{H}}_2(\overline{\mathbb{X}},\overline{\mathbb{Y}})).$$

The notation $\overline{\mathscr{H}}_2(\overline{\mathbb{X}},\overline{\mathbb{Y}})$ stands for the class of strong limits in the Sobolev space $\mathscr{W}^{1,2}(\mathbb{X},\mathbb{Y})$ of homeomorphisms $h_j : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$.

Homework 2. ***Let $\mathscr{H}$ be a subset of a Banach space $\mathscr{W}$. Denote by $\overline{\mathscr{H}}$ the closure of $\mathscr{H}$ in norm topology of $\mathscr{W}$; that is, the collection of all strong limits of sequences in $\mathscr{H}$. Denote by $\widehat{\mathscr{H}}$ the collection of all weak limits of sequences in $\mathscr{H}$. Suppose that $\widehat{\mathscr{H}} = \overline{\mathscr{H}}$.***

Show that the (sequentially) weak closure and the strong closure of $\mathscr{H}$ are the same

DEFINITION. The (sequentially) weak closure of $\mathscr{H}$ is the smallest set $\widetilde{\mathscr{H}} \supset \mathscr{H}$ which is (sequentially) weakly closed. A set is (sequentially) weakly closed if any weak limit of elements in this set belongs to this set. Note that every closed set (in weak topology) is sequentially closed. The converse is not generally true.

Mazur's Lemma

The lemma tells us that any weakly convergent sequence in a Banach space has a sequence of convex combinations of its members that converges strongly to the same limit, and is used to prove that

Weakly closed convex sets and closed convex sets are the same thing

Remark. However, in our application the set $\mathscr{H} \subset \mathscr{W}$ will consists of homeomorphisms in the Sobolev space $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, which is not convex.

The Classes $\mathscr{H}_p(\mathbb{X}, \mathbb{Y}) \subset \overline{\mathscr{H}_p}(\mathbb{X}, \mathbb{Y}) \subset \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$

Recall that $\mathscr{H}_p(\mathbb{X}, \mathbb{Y})$ consists of homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in the Sobolev space $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$. We have seen (the Nitsche phenomenon) that for the viable theory of energy-minimal deformations we ought to sacrifice injectivity.

This is the challenge.

We invoke the strong closure $\overline{\mathscr{H}_p}(\mathbb{X}, \mathbb{Y})$ of $\mathscr{H}_p(\mathbb{X}, \mathbb{Y}) \subset \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$.

The existence of energy-minimal deformations within $\overline{\mathscr{H}_p}(\mathbb{X}, \mathbb{Y})$ will follow once we show that ***the weak limit of Sobolev homeomorphisms*** (*applied to an energy minimizing sequence*) ***can also be obtained as strong limit***.

From now on $\mathbb{X}$ and $\mathbb{Y}$ are planar Lipschitz domains

I am a bi-Lipschitz image of a $\mathscr{C}^\infty$-smooth domain, but I am not Lipschitz. Inward cusps and logarithmic spirals emerged.

The Weak and Strong Closures of Sobolev Homeomorphisms are the Same Thing

Let homeomorphisms $h_j \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ between planar ℓ-connected Lipschitz domains, $2 \leqslant \ell < \infty$, converge *weakly* in $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ to $h \in \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, $2 \leqslant p < \infty$. Then there exists a sequence of

$$\mathscr{C}^\infty\text{-}\mathbf{diffeomorphisms}\;\; f_j \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}, \qquad f_j \in h + \mathscr{W}^{1,p}_\circ(\mathbb{X}, \mathbb{Y})$$

converging *strongly* in $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ to h.

This also holds for simply connected Lipschitz domains provided $p > 2$. However, the simply connected variant of this theorem with $p = 2$ requires that the mappings be fixed at one interior point or at three boundary points.

Traction-Free Problem

Let $\mathbb{X}$ and $\mathbb{Y}$ be ℓ-connected bounded Lipschitz domains , $\ell \geqslant 2$. Consider the energy functional

$$\mathscr{E}_{\mathbb{X}}[h] = \iint_{\mathbb{X}} \mathbf{E}(x, h, Dh) \, \mathrm{d}x \,, \qquad \text{for Sobolev mappings} \quad h \in \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$$

Here $\mathbf{E} : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{2\times 2} \to \mathbb{R}_+$ satisfies the usual Carathéodory regularity conditions, coercivity $|\xi|^p \succcurlyeq \mathbf{E}(\cdot, \cdot, \xi) \succcurlyeq |\xi|^p$, $p \geqslant 2$, and Morrey's quasiconvexity. Then there exists $\hbar \in \overline{\mathscr{H}_p}(\mathbb{X}, \mathbb{Y})$ such that

$$\mathscr{E}_{\mathbb{X}}[\hbar] = \inf_{h \in \mathscr{H}_p(\mathbb{X}, \mathbb{Y})} \iint_{\mathbb{X}} \mathbf{E}(x, h, Dh) \, \mathrm{d}x$$

This also holds for mappings that are fixed on some portion of $\partial\mathbb{X}$, while slipping along the rest of $\partial\mathbb{X}$ is allowed.

Monotone Mappings

It is a matter of topological routine to see that all the above mentioned $\mathscr{H}$ -classes consist of *monotone mappings.*

Monotone mappings were born in the work of C.B. Morrey *The Topology of Path Surfaces*, Amer. Journ. Math., 1935 [p.26].

We combine Morrey's original definition with a theorem of G.T. Whyburn, see page 2 in *Analytic topology*, AMS, Providence, R.I. (1963).

See also T. Radó, *Length and Area*, American Mathematical Society, New York, 1948, for further reading about monotone mappings.

DEFINITION. A continuous mapping $f : \mathbf{X} \overset{\text{onto}}{\longrightarrow} \mathbf{Y}$ between compact metric spaces is monotone if for each connected set $C \subset \mathbf{Y}$ its preimage $f^{-1}(C)$ is connected in $\mathbf{X}$.

THEOREM (*Kuratowski-Lacher*) Let $\mathbf{X}$ and $\mathbf{Y}$ be compact Hausdorff spaces, $\mathbf{Y}$ being locally connected. Suppose we are given a sequence of monotone mappings $f_k : \mathbf{X} \xrightarrow{\text{onto}} \mathbf{Y}$ converging uniformly to a mapping $f : \mathbf{X} \to \mathbf{Y}$, then $f : \mathbf{X} \xrightarrow{\text{onto}} \mathbf{Y}$ is monotone.

Equicontinuity and Monotonicity

LEMMA (*Equicontinuity*) **Let $\mathbb{X}$ and $\mathbb{Y}$ be planar Lipschitz domains of connectivity $\ell \geqslant 2$. Then, every $h \in \mathscr{H}^{1,2}_{\text{weak lim}}(\mathbb{X}, \mathbb{Y})$ extends to a continuous monotone map $h : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$. The boundary map $h : \partial\overline{\mathbb{X}} \xrightarrow{\text{onto}} \partial\overline{\mathbb{Y}}$ is also monotone. Moreover,**

$$|\boldsymbol{h}(\boldsymbol{x_1}) - \boldsymbol{h}(\boldsymbol{x_2})|^2 \leqslant \frac{\boldsymbol{C}(\mathbb{X}, \mathbb{Y})}{\log\left(\boldsymbol{e} + \frac{\operatorname{diam} \mathbb{X}}{|\boldsymbol{x_1} - \boldsymbol{x_2}|}\right)} \iint_{\mathbb{X}} |\boldsymbol{D}\boldsymbol{h}|^2, \quad \boldsymbol{x_1}, \boldsymbol{x_2} \in \overline{\mathbb{X}}.$$

Proof. It is known that a homeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ between Lipschitz domains in the Sobolev space $\mathscr{W}^{1,2}(\mathbb{X}, \mathbb{Y})$ extends continuously up to the boundaries. And it is topologically clear that a continuous extension of a homeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ results in monotone mappings $h : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ and $h : \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$. These properties carry over to the uniform limits of homeomorphisms. Thus it only remains to justify the uniform estimates. Local estimates like this are well known for monotone mappings of Sobolev class $\mathscr{W}^{1,2}(\mathbb{X}, \mathbb{Y})$

T. Iwaniec, P. Koskela and J. Onninen, ***Mappings of finite distortion: monotonicity and continuity*****, Invent. Math. 144 (2001), no. 3, 507531.**

T. Iwaniec and G. Martin, ***Geometric function theory and non-linear analysis*****, Oxford University Press, New York, 2001.**

The global inequality follows by the standard method of extending h beyond the boundaries. It is at this point that the Lipschitz regularity of the domains is required. We leave the routine details to the students.

REMARK. Similar arguments, including the extension procedure, apply to a pair of simply connected Lipschitz domains if the mappings are fixed at some interior point ($h(x_\circ) = y_\circ$ for some $x_\circ \in \mathbb{X}$ and $y_\circ \in \mathbb{Y}$) or at three boundary points ($h(x_1) = y_1$, $h(x_2) = y_2$, $h(x_3) = y_3$).

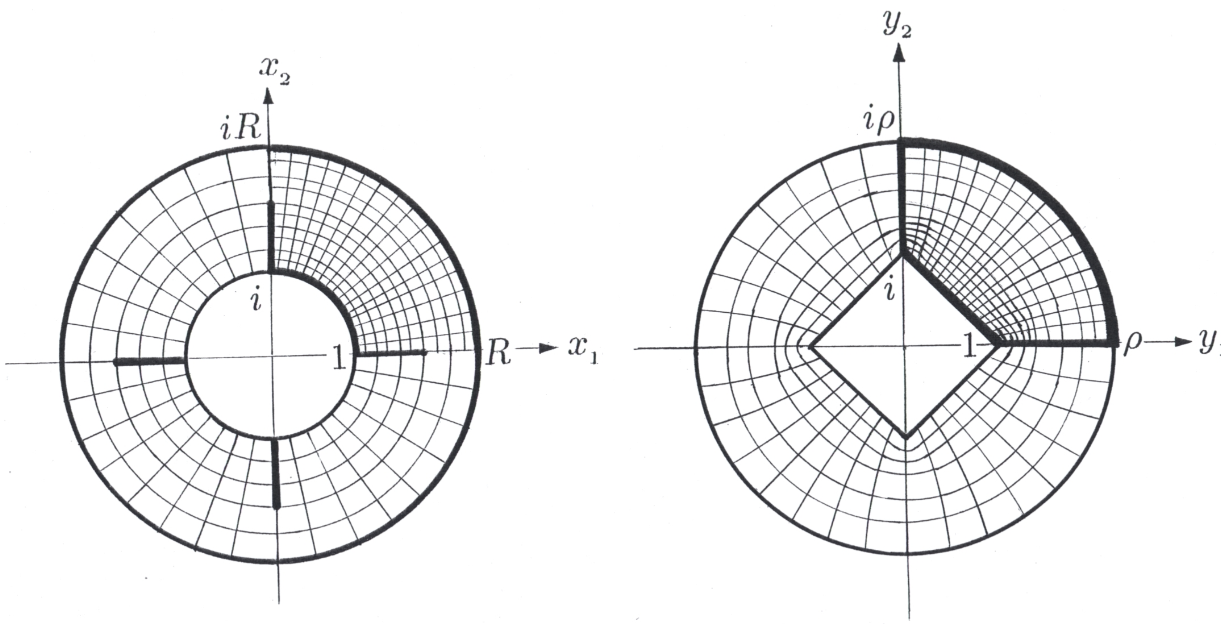
The Dirichlet and the p-Harmonic Integrals

Let $h = u + i\,v \,:\, \Omega \to \mathbb{C}$. The guiding examples of the energy integrals will be:

$$\mathscr{D}[h] = \iint_\Omega |Dh|^2 \;=\; \iint_\Omega \left(\, |\nabla u|^2 + |\nabla v|^2 \,\right) \;<\; \infty$$

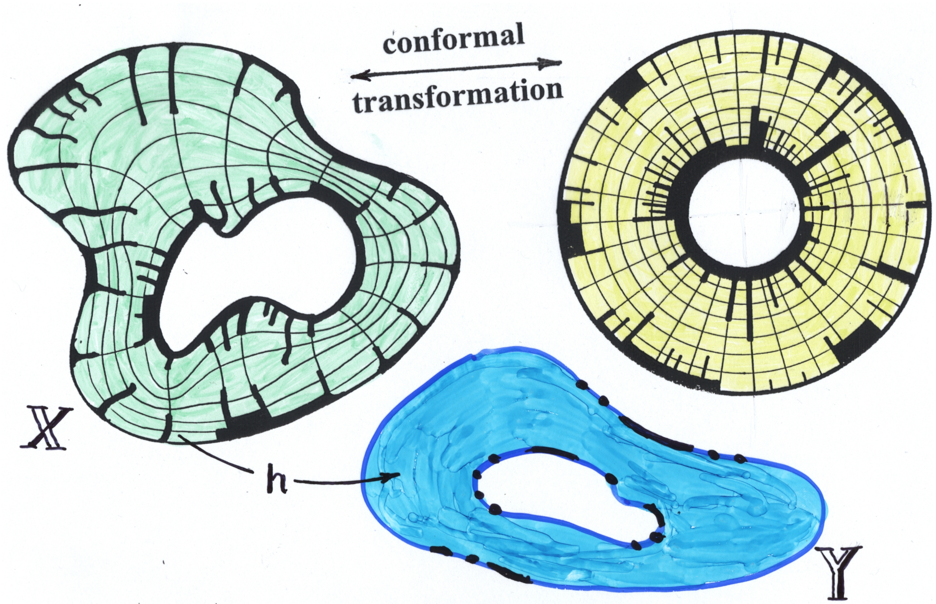
$$\mathscr{D}_p[h] = \iint_\Omega \left(\, |\nabla u|^p + |\nabla v|^p \,\right) \;<\; \infty \;\;,\;\; 1 < p < \infty$$

How to make a square-hole washer using minimal energy



The minimal map $h \in \mathscr{L}ip(\overline{\mathbb{X}}, \overline{\mathbb{Y}})$, $h \notin \mathscr{C}^1(\mathbb{X}, \mathbb{Y})$ consists of harmonic quadrilateral maps. It is ***a strong*** $\mathscr{W}^{1,2}$***- limit of diffeomorphisms*** , and a diffeomorphism (no cracks) if $1 < R \sim 1$. See ***Homework 2.***

Cracks in Doubly Connected Domains
(no interior fractures)



Cracks propagate from $\partial\mathbb{X}$ along vertical trajectories of the Hopf differential $h_z\,\overline{h_{\bar z}}\,\mathrm{d}z \otimes \mathrm{d}z$. A crack terminates in the interior of $\mathbb{X}$. The map h is energy minimal iff $h_z\,\overline{h_{\bar z}}\,\mathrm{d}z \otimes \mathrm{d}z$ is analytic in $\mathbb{X}$ and real along $\partial\mathbb{X}$.

Lipschitz Regularity

In spite of occurrence of cracks, we have

Every energy-minimal deformation for the Dirichlet integral within the class $\overline{\mathcal{H}_2}(\mathbb{X},\mathbb{Y})$ is locally Lipschitz continuous, but not necessarily $\mathscr{C}^1$ -smooth

This is a corollary from ***Cristina, Kovalev, Onninen, T.I.*** arXiv:1011.5934. For more general Lipschitz regularity results concerning inner variational equations see ***Kovalev, Onninen, T.I.*** arXiv:1109.0720, and the forthcoming Lecture 4.

Existence of Harmonic Homeomorphisms

Theorem. (*Koh, Kovalev, Onninen, T.I. Invent. Math. 2011*) ***Among all homeomorphisms*** $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ ***between bounded doubly connected domains such that***

$$\operatorname{Mod} \mathbb{X} \leqslant \operatorname{Mod} \mathbb{Y}$$

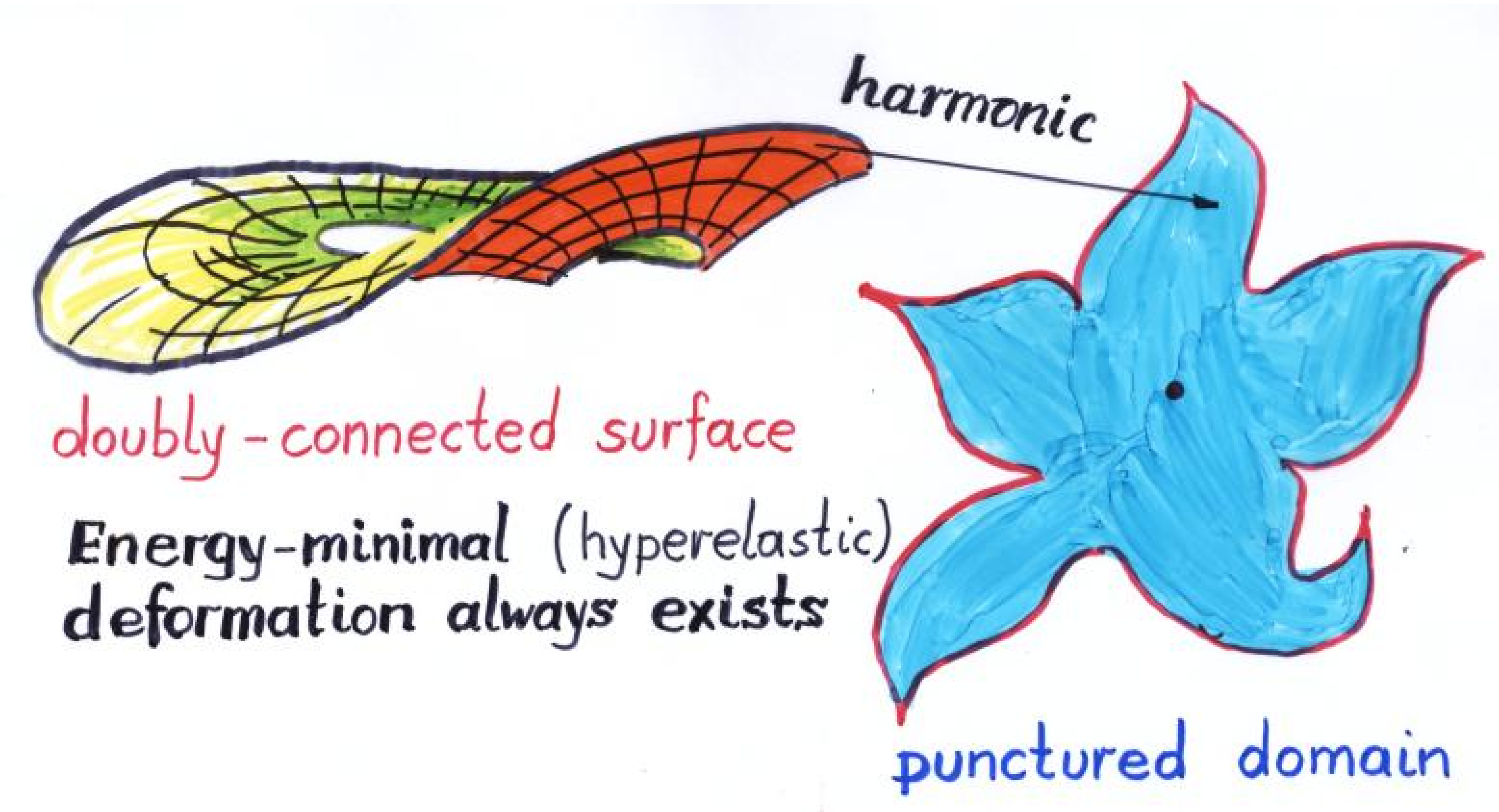
there exists a harmonic diffeomorphism. This map has smallest Dirichlet energy and, as such, is *unique up to conformal automorphisms of* $\mathbb{X}$.

Homeomorphisms of Smallest Mean Distortion
$\mathscr{L}^1$-variant of the Teichmüller map

$$2\,\|K_f\|_{\mathscr{L}^1(\mathbb{Y})} \;=\; \iint_{\mathbb{Y}} \frac{|Df|^2}{\mathsf{det}\,Df} \qquad \left(= \iint_{\mathbb{X}} |Dh|^2\right)$$

THEOREM. ***Let $\mathbb{X}$ and $\mathbb{Y}$ be bounded doubly connected domains in $\mathbb{C}$ such that $\operatorname{Mod}\mathbb{X} \leqslant \operatorname{Mod}\mathbb{Y}$. Among all homeomorphisms $f\colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ there exists, unique up to a conformal change of variables in $\mathbb{X}$, mapping of smallest $\mathscr{L}^1$-norm of the distortion.***

Doubly connected membrane in $\mathbb{R}^3$



No coffee no theorems; this is not a question?

Proving the Weak=Strong Theorem

Let homeomorphisms $h_j \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ ***between planar*** ℓ***-connected Lipschitz domains,*** $2 \leqslant \ell < \infty$ ***, converge weakly in*** $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$***,*** $2 \leqslant p < \infty$ ***, to*** $h \in \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$***. Then there exists a sequence of*** $\mathscr{C}^\infty$***-diffeomorphisms***

$$f_j \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}, \qquad f_j \in h + \mathscr{W}_\circ^{1,p}(\mathbb{X}, \mathbb{Y})$$

converging strongly in $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ ***to*** h***. This also holds for simply connected Lipschitz domains when*** $p > 2$ ***, even for*** $p = 2$ ***but under a suitable normalization; say, if the mappings are fixed at one interior point or at three boundary points.***

The class $\widetilde{\mathscr{H}_p}(\mathbb{X}, \mathbb{Y}) \subset \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$

The class $\widehat{\mathscr{H}_p}(\mathbb{X}, \mathbb{Y})$ of weak $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$-limits of homeomorphisms in $\mathscr{H}_p(\mathbb{X}, \mathbb{Y})$ will eventually turn out to be the weak closure of $\mathscr{H}_p(\mathbb{X}, \mathbb{Y})$. This is not obvious at all , see ***Homework 2***. Recall, our goal is to show that

$$\widehat{\mathscr{H}_p}(\mathbb{X}, \mathbb{Y}) \subseteqq \overline{\mathscr{H}_p}(\mathbb{X}, \mathbb{Y}) \quad , \quad p \geqslant 2$$

LEMMA. Every $h \in \mathscr{H}_p(\mathbb{X}, \mathbb{Y})$, $p \geqslant 2$, extends as a continuous monotone map $h \colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$. The boundary map $h \colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ is also monotone. There is a uniform bound of the modulus of continuity of h in terms of its $\mathscr{L}^p$- energy. Hence weak convergence in $\mathscr{H}_p(\mathbb{X}, \mathbb{Y})$ yields uniform convergence to the effect that all of the above properties hold for $h \in \widetilde{\mathscr{H}_p}(\mathbb{X}, \mathbb{Y})$

p-Harmonic Mappings in $\mathbb{C}\,, 1 < p < \infty$

Let $h = u + i\, v : \Omega \to \mathbb{C}\,,\quad \Omega -$ bounded simply connected domain

$$\mathscr{E}_p[h] = \iint_\Omega \left(\, |\nabla u|^p \, + \, |\nabla v|^p \, \right) \; < \; \infty$$

Such a map is said to be p-harmonic if $\triangle_p u \; = \; \triangle_p v \; = \; 0\,.$
PROPOSITION . To every $h \in \mathscr{C}(\overline{\Omega}) \cap \mathscr{W}^{1,p}(\Omega)$ there corresponds unique p-harmonic $\widetilde{h} \in \mathscr{C}(\overline{\Omega}) \cap \mathscr{W}^{1,p}(\Omega)$ such that $\widetilde{h} = h$ on $\partial\Omega$ and $\widetilde{h} \in h + \mathscr{W}_\circ^{1,p}(\Omega)$, which yields $\mathscr{E}_p[\widetilde{h}] \leqslant \mathscr{E}_p[h]$, equality occurs iff $\widetilde{h} \equiv h$

If Ω is a Jordan domain and h takes $\partial\Omega$ homeomorphically onto a convex curve, then $\widetilde{h}$ is a $\mathscr{C}^\infty$-diffeomorphism. The latter is an adaptation of the celebrated Radó-Kneser-Choquet theorem to p-harmonic setting by Alessandrini-Sigalotti

p-**Harmonic Hurwitz's Theorem** **(Onninen, I.)**

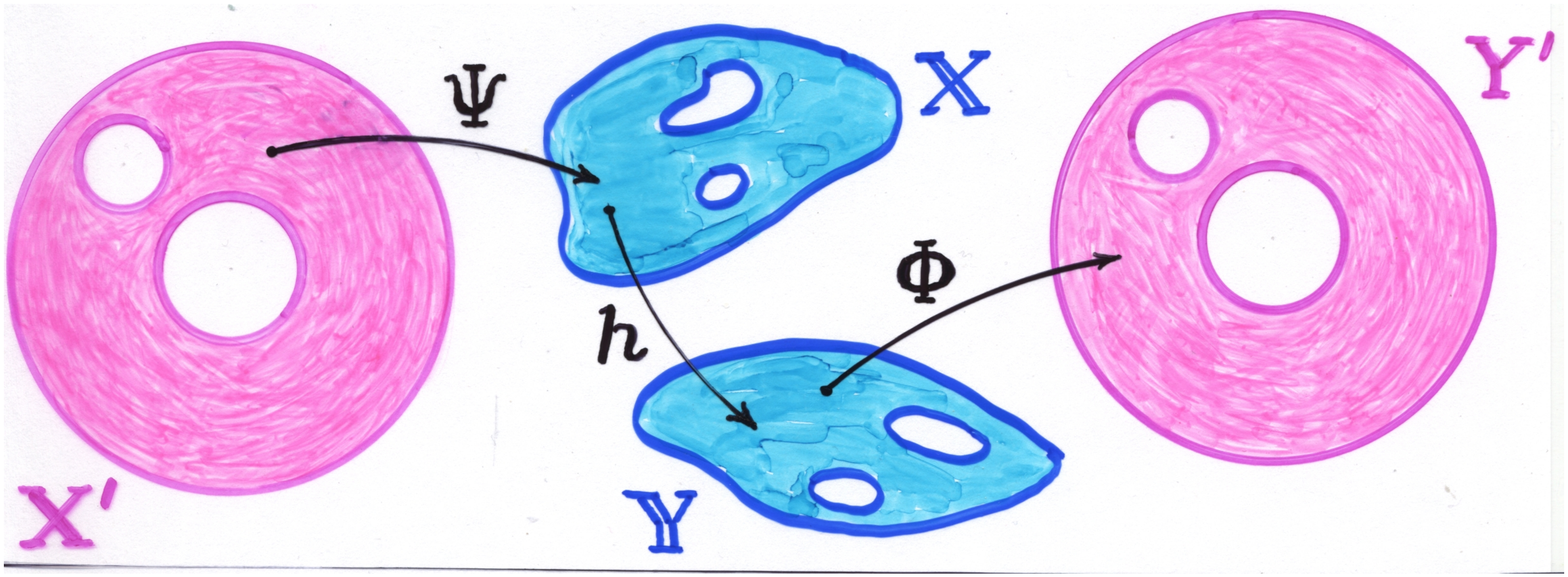
Suppose p-harmonic homeomorphisms

$$h_n = u_n + i\, v_n : \Omega \to \mathbb{C}\,, \quad \triangle_p u_n = \triangle_p v_n = 0$$

converge (c-uniformly) to h. Then either h is a p -harmonic homeomorphism or its Jacobian determinant $J(z, h) \equiv 0$ in Ω.

$h_n(x + \, i\, y) = x \, + \frac{i}{n} y \; \longrightarrow \; x$ in a rectangle

Step 1. Transfiguration to Circular Domains



$$\mathscr{W}^{1,p}(\mathbb{X},\mathbb{Y}) \xrightarrow{\quad\Psi_\sharp\quad} \mathscr{W}^{1,p}(\mathbf{X}',\mathbb{Y}) \xrightarrow{\quad\Phi_\sharp\quad} \mathscr{W}^{1,p}(\mathbf{X}',\mathbf{Y}')$$

$\Psi_\sharp(h) = g = h \circ \Psi$

continuous linear isomorphism

$\Phi_\sharp(g) = f = \Phi \circ g$

nonlinear bounded bijection

The problem lies in the continuity of the nonlinear operator induced by a bi-Lipschitz transformation of the target domain $\mathbb{Y}$.

In spite of discouraging example (by Piotr Hajłasz) we proved that:

PROPOSITION

Given any bounded ℓ-connected Lipschitz domain $\mathbb{Y} \subset \mathbb{R}^2$, there exists a bi-Lipschitz map $\Phi \colon \mathbb{R}^2 \xrightarrow{\text{onto}} \mathbb{R}^2$ which takes $\mathbb{Y}$ onto a circular (Schottky) domain $\mathbf{Y}'$ such that the induced composition maps:

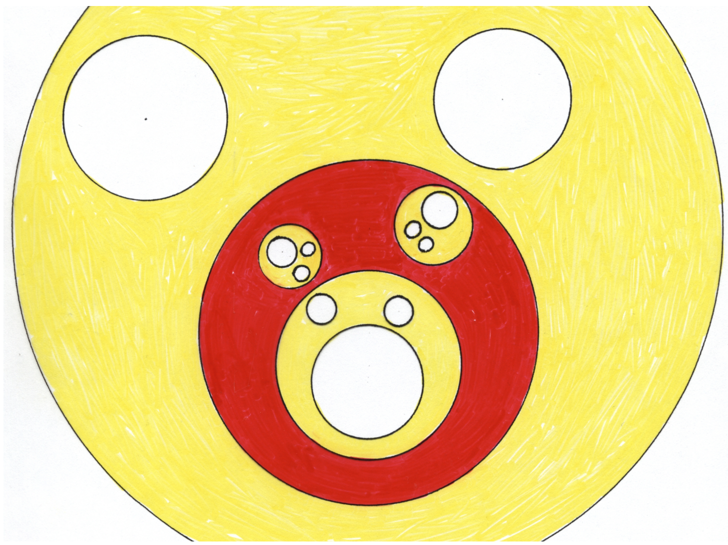
$$\Phi_\sharp \colon \mathscr{W}^{1,p}(\Omega, \mathbb{Y}) \to \mathscr{W}^{1,p}(\Omega, \mathbf{Y}'), \qquad \Phi_\sharp(g) = \Phi \circ g$$

and its inverse

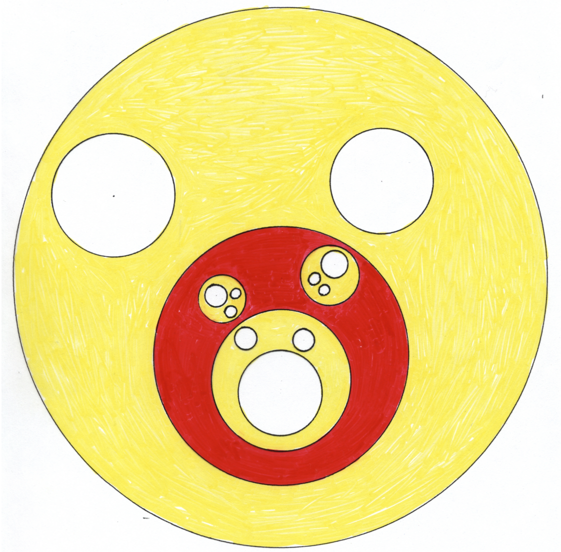
$$\Phi_\sharp^{-1} \colon \mathscr{W}^{1,p}(\Omega, \mathbf{Y}') \to \mathscr{W}^{1,p}(\Omega, \mathbb{Y}), \qquad \Phi_\sharp^{-1}(f) = \Phi^{-1} \circ f$$

are continuous for all $1 < p < \infty$, and all domains $\Omega \subset \mathbb{R}^2$.

Step 2. The extension $\widehat{h} : \overline{\mathbb{X}}_+ \xrightarrow{\text{onto}} \overline{\mathbb{Y}}_+$



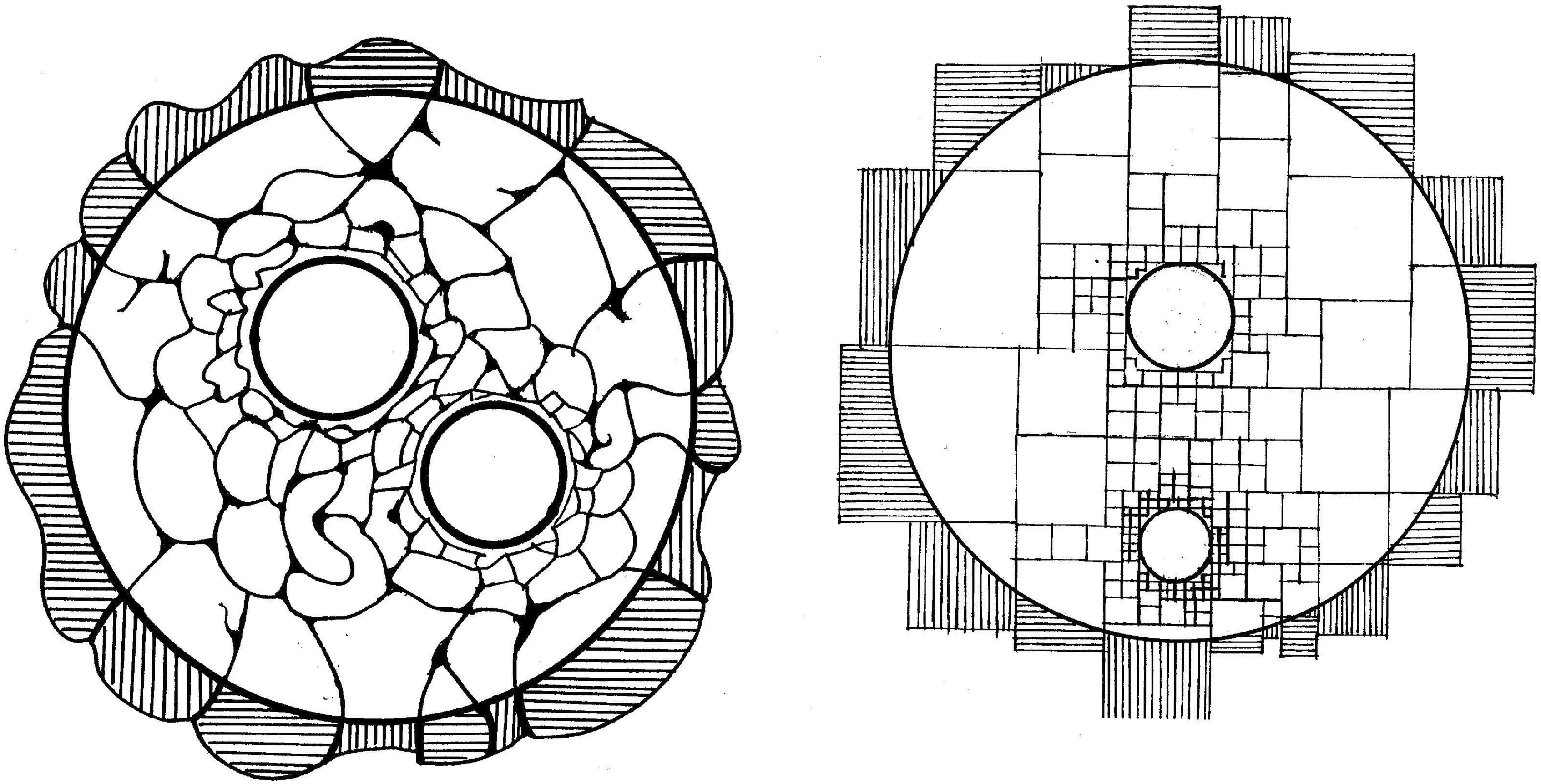
$-\widehat{h} \longrightarrow$

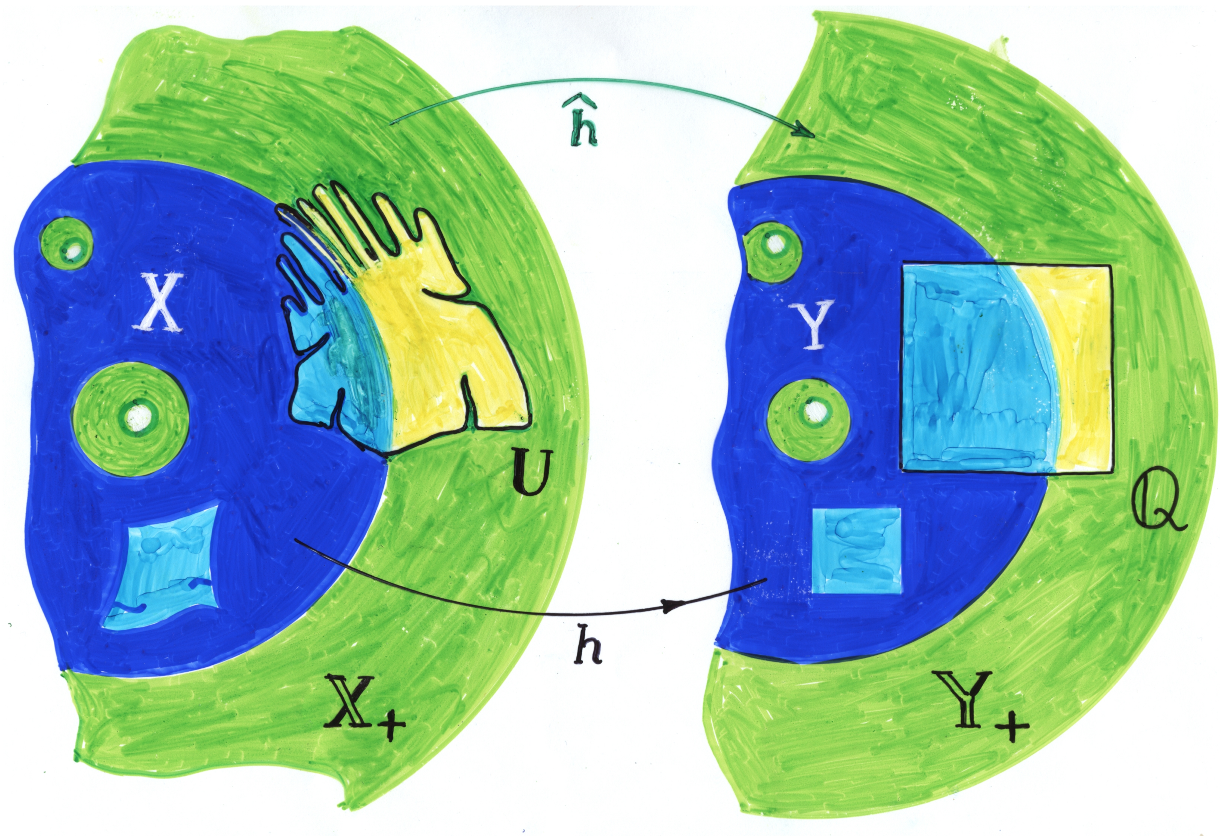


$$\iint_{\mathbb{X}_+} |D\widehat{h}|^p \preccurlyeq \iint_{\mathbb{X}} |Dh|^p \,.$$ Hence, when $h_k \rightharpoonup h$, we have

$$\iint_{\mathbb{X}} \varphi(x) J(x, h_k) \mathrm{d}x \longrightarrow \iint_{\mathbb{X}} \varphi(x) J(x, h) \mathrm{d}x \;, \quad \text{for} \quad \varphi \in \mathscr{L}^\infty(\mathbb{X})$$

Step 3. Meshes of open squares in $\mathbb{Y}$ and the corresponding open cells in $\mathbb{X}$





$U = h^{-1}(\mathbb{Q} \cap \overline{\mathbb{Y}}) = \{x \in \mathbb{X}\,;\; h(x) \in \mathbb{Q} \cap \overline{\mathbb{Y}}\,\}$ **open simply connected**

Step 4. p-Harmonic Replacement in a Cell

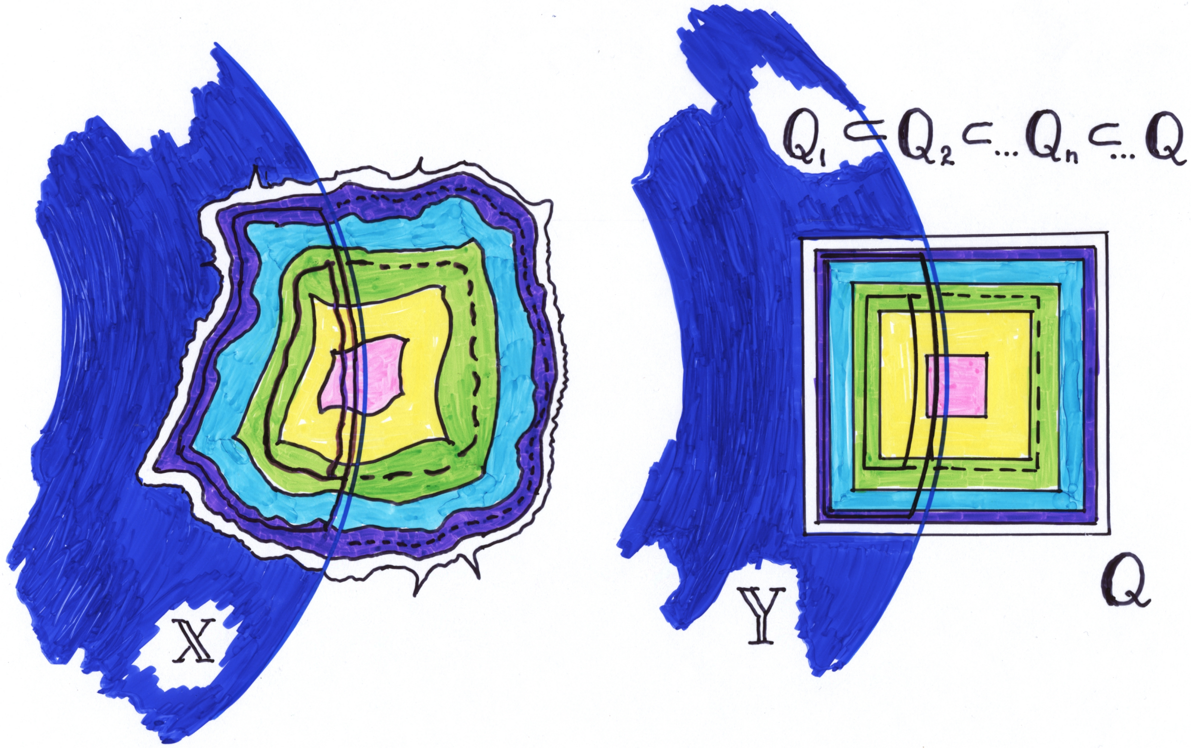
PROPOSITION. Let $h \in \widetilde{\mathscr{H}_p}(\mathbb{X}, \mathbb{Y})$ and $U \subset \mathbb{X}$ be a cell. Then there exists $h^* \colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$, $h^* = h \colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$, such that

(i) $h^* \in \widetilde{\mathscr{H}_p}(\mathbb{X}, \mathbb{Y})$

(ii) $h^* = h \colon \overline{\mathbb{X}} \setminus U \xrightarrow{\text{onto}} \overline{\mathbb{Y}} \setminus (\mathbb{Q} \cap \mathbb{Y})$

(iii) $h^* \colon U \xrightarrow{\text{onto}} \mathbb{Q} \cap \mathbb{Y}$ is a p-harmonic diffeomorphism

(iv) $\displaystyle\iint_{\mathbb{X}} |\nabla h^*|^p \leqslant \iint_{\mathbb{X}} |\nabla h|^p$



Proof sketch

Having in the disposal weakly converging homeomorphisms $h_n \rightharpoonup h$, the p-harmonic replacement $h^* \colon U \xrightarrow{\text{onto}} \mathbb{Q} \cap \mathbb{Y}$ is constructed as limit of p-harmonic extensions of the boundary homeomorphisms $h_n^* \colon \partial U_n \xrightarrow{\text{onto}} \partial \mathbb{V}_n$.

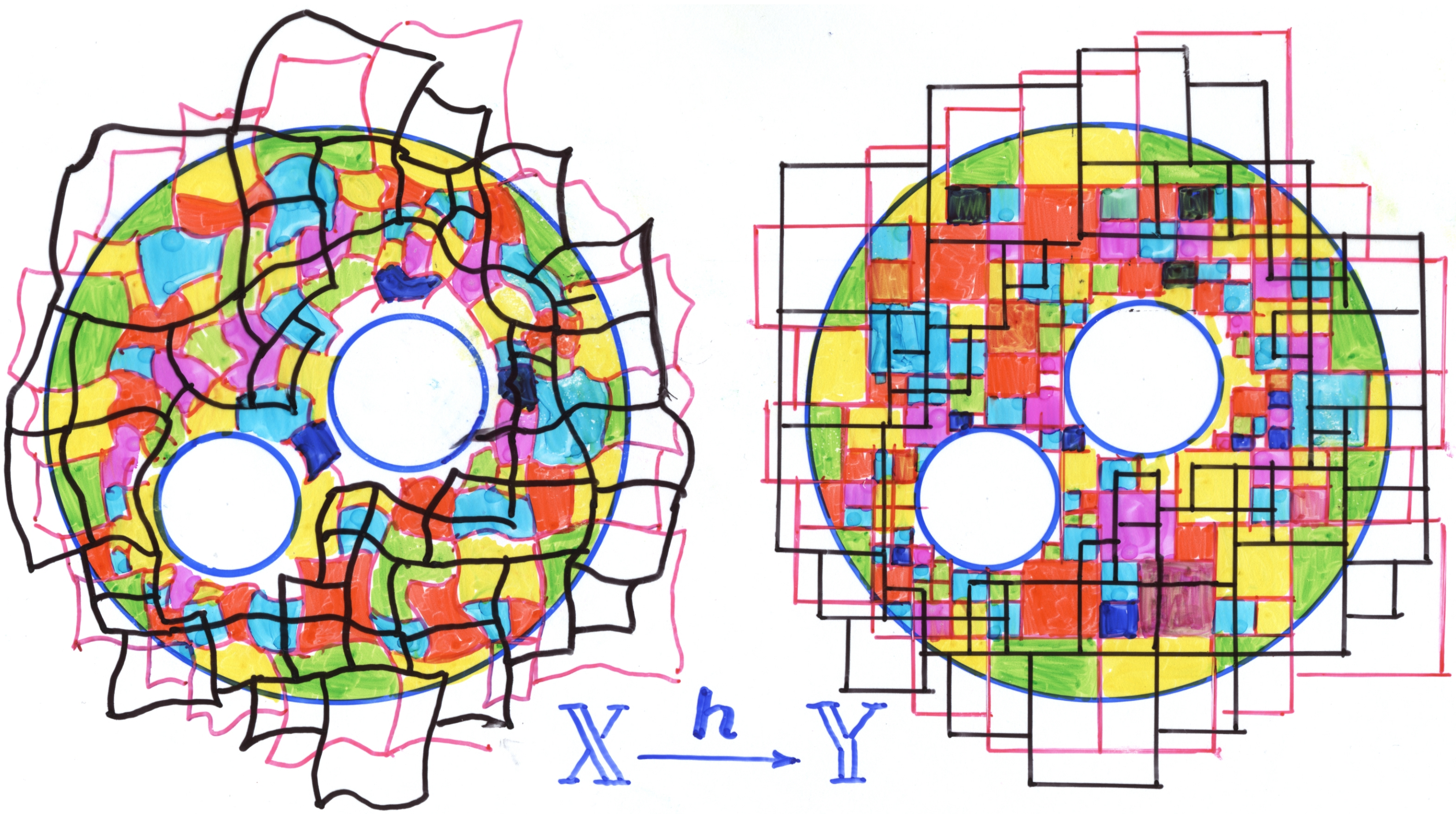
Parempi ruuvi löysällä kuin monta liian tiukalla.

(*One screw loose is better than too many too tight*)

Remember, however:

Failure to prepare details of the proof is a preparation for failure. And this is not the only punishment for laziness; there is also success of others.

$\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ - *incommensurate meshes*



The union $\bigcup \mathscr{A} \cup \bigcup \mathscr{B} \cup \bigcup \mathscr{C}$ must cover $\mathbb{Y}$ and its outer boundary

Step 5. p-harmonic replacement in all $\mathscr{A}$-cells

The $\mathscr{A}$ -cells provide us with a mapping $h_{\mathcal{A}} \colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$, such that

(i) $h_{\mathcal{A}} \in \widetilde{\mathscr{H}_p}(\mathbb{X}, \mathbb{Y})\,, \quad h_{\mathcal{A}} = h \colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$

(ii) $h_{\mathcal{A}} \colon \bigcup \mathscr{A}_{\text{cells}} \xrightarrow{\text{onto}} \bigcup (\mathscr{A}_{\text{squares}} \cap \mathbb{Y})$ is a p-harmonic diffeomorphism.

$h_{\mathcal{A}} = h$ elsewhere.

(iii) $\displaystyle\iint_{\mathbb{X}} |\nabla h_{\mathcal{A}}|^p \leqslant \iint_{\mathbb{X}} |\nabla h|^p$.

(iv) $\|h_{\mathscr{A}} - h\|_{\mathscr{W}^{1,p}(\mathbb{X})} \leqslant \frac{1}{3}\,\varepsilon$ (***by Clarkson's convexity argument***)

Step 6. Further replacements

We mimic the construction of $h_{\mathscr{A}}$ using the mesh $\mathscr{B}$ instead. In this procedure $h_{\mathscr{A}}$ plays the role of the original map h. We obtain a map $h_{\mathscr{A}\mathscr{B}} \in \widetilde{\mathscr{H}_p}(\mathbb{X}, \mathbb{Y})$. This in turn, with the aid of $\mathscr{C}$-cells, advances to a mapping $h_{\mathscr{A}\mathscr{B}\mathscr{C}}$.

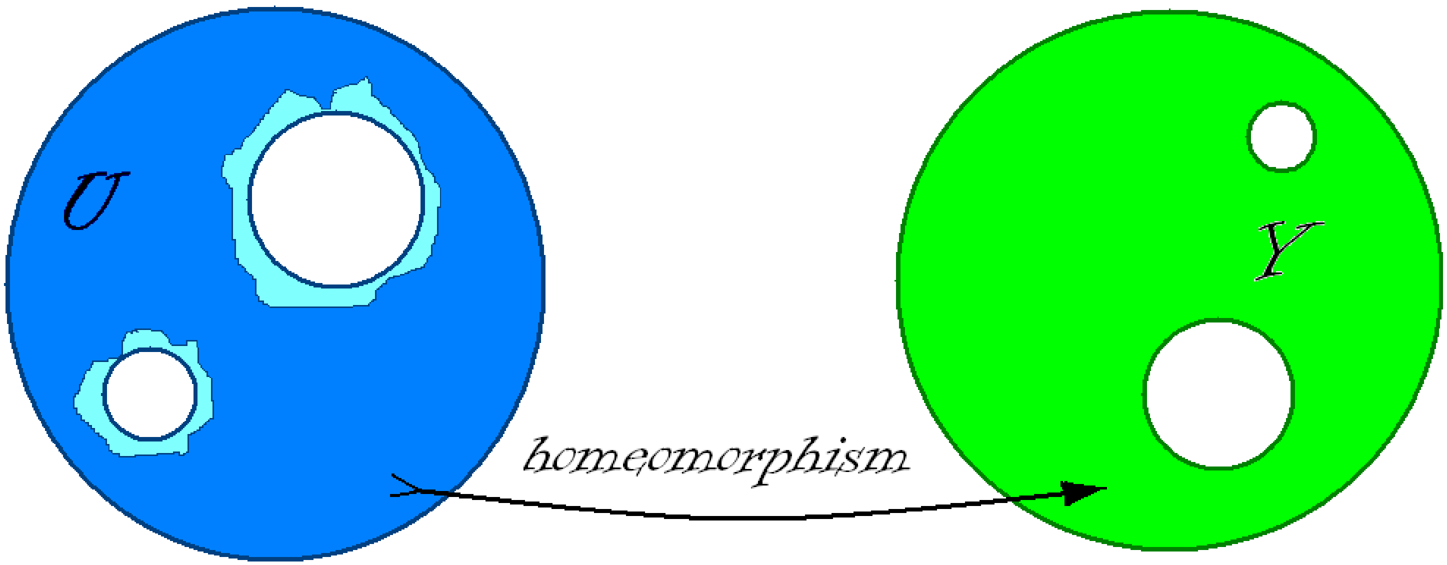
Upon three consecutive replacements in the $\mathscr{A}$-cells, $\mathscr{B}$-cells and $\mathscr{C}$-cells we arrive at the mapping $H = h_{\mathscr{A}\mathscr{B}\mathscr{C}}$ that satisfies the following:

LEMMA. Let $\mathbb{X}$ and $\mathbb{Y}$ be circular domains whose outer boundary circles are $\mathfrak{X}$ and Υ, respectively, and let $h \in \widetilde{\mathscr{H}_p}(\mathbb{X}, \mathbb{Y})$. Consider the open region in $\mathbb{X}$

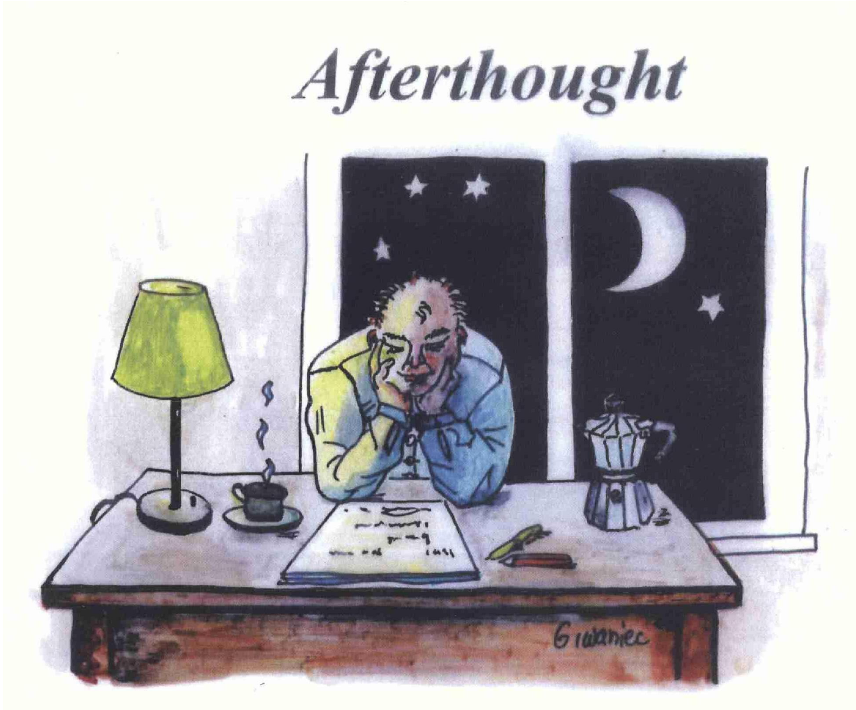
$\Omega = h^{-1}(\mathbb{Y} \cup \Upsilon) = \{x \in \mathbb{X} \colon h(x) \in \mathbb{Y} \cup \Upsilon\}$

Then for every $\epsilon > 0$ there exists $H \in \widetilde{\mathscr{H}_p}(\mathbb{X}, \mathbb{Y})$, such that

$$\|H - h\|_{\mathscr{W}^{1,p}(\mathbb{X})} \leqslant \epsilon$$



$$\mathcal{U} = \{x \in \mathbb{X} : h(x) \in \mathbb{Y} \cup \partial_\circ \mathbb{Y}\}\,, \quad H : \mathcal{U} \xrightarrow{\text{onto}} \mathbb{Y} \text{ is injective,}$$



Every block of stone has a statue inside it and it is the task of the sculptor to discover it." - Michelangelo, 1475-1564 .

Thank You for Listening Lecture 2