INNER VARIATION

(Hopf Differentials and Uniqueness Theorem)

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Lecture 3 in Pisa

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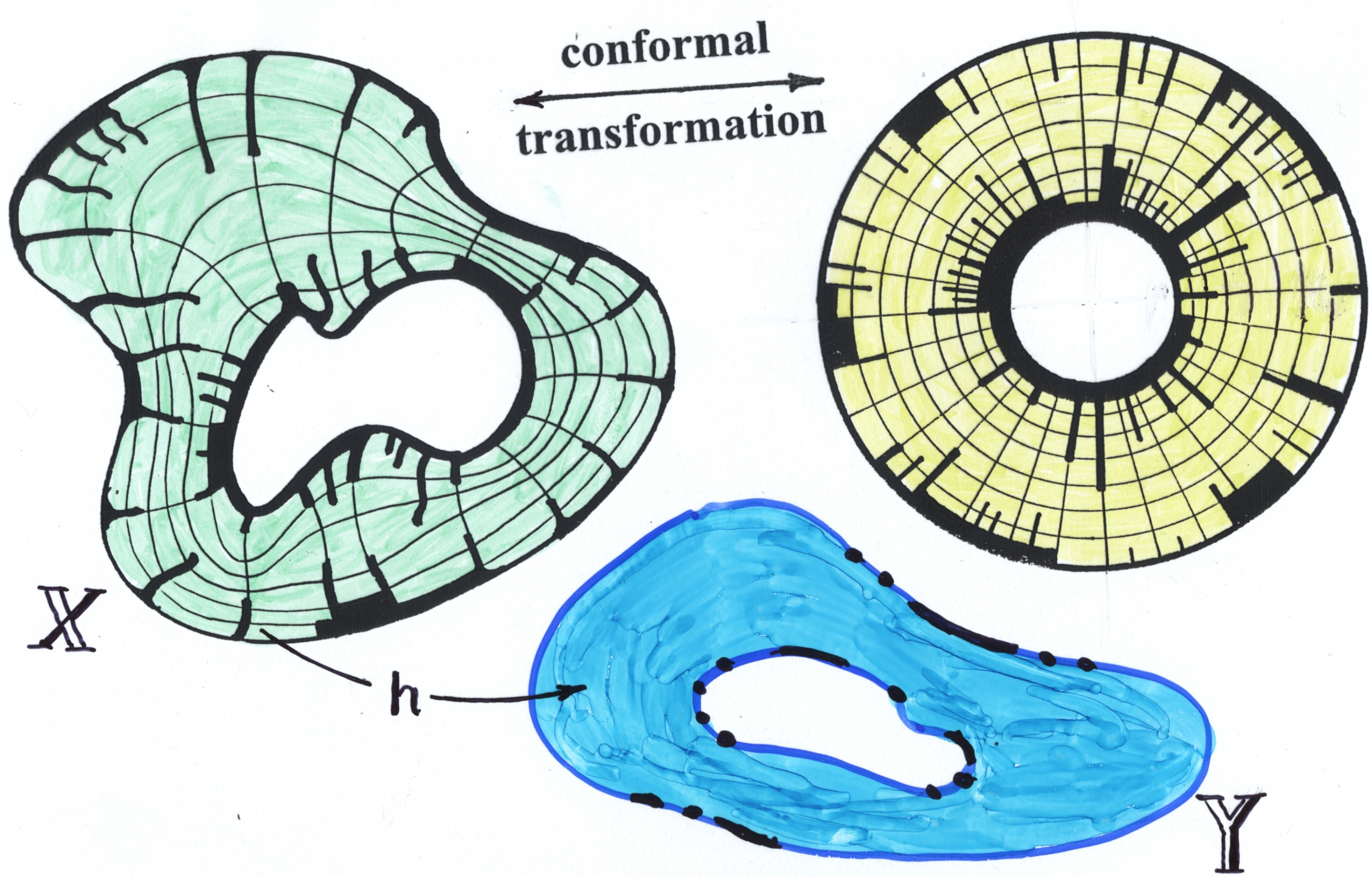
Jani comes back on stage

The inner variation (of the variables in $\mathbb{X}$) leads to the Hopf differential $h_z \overline{h_{\bar z}}\, \mathrm{d}z \otimes \mathrm{d}z$ and its trajectories. For a pair of doubly connected domains, in which $\mathbb{X}$ has finite conformal modulus, we establish the following principle:

A mapping $h \in \overline{\mathscr{H}_2}(\mathbb{X}, \mathbb{Y})$ is energy-minimal if and only if its Hopf-differential is analytic in $\mathbb{X}$ and real along $\partial \mathbb{X}$.

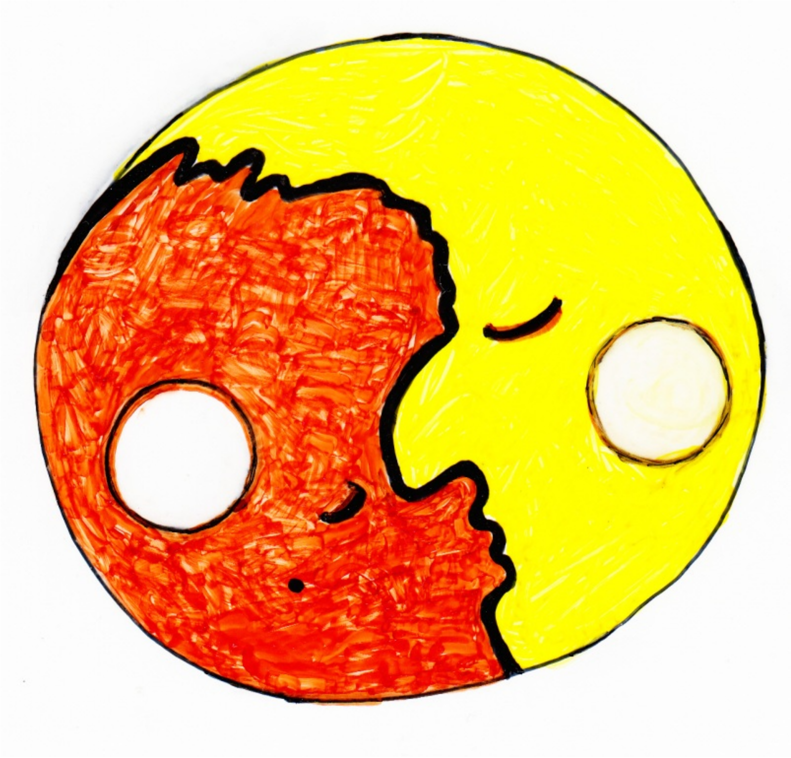
In general, the energy-minimal mappings may not be injective, in which case one observes the occurrence of cracks in $\mathbb{X}$. Nevertheless, cracks are triggered only by the points in $\partial \mathbb{Y}$ where $\mathbb{Y}$ fails to be convex. The general law of formation of cracks reads as follows:

Cracks propagate along vertical trajectories of the Hopf differential from $\partial \mathbb{X}$ toward the interior of $\mathbb{X}$ where they eventually terminate, so no crosscuts occur.



conformal
transformation
X
h
Y

A crosscut



Recollection

THEOREM ($\mathit{Existence}$) Consider a pair $(\mathbb{X}, \mathbb{Y})$ of nondegenerate multiply connected domains, in which $\mathbb{Y}$ is a Lipschitz domain. Then there exists $h : \mathbb{X} \to \overline{\mathbb{Y}}$ in the class $\overline{\mathscr{H}}_2(\mathbb{X}, \mathbb{Y}) \subset \mathscr{W}^{1,2}(\mathbb{X}, \mathbb{Y})$ such that

$$\mathscr{E}_{\mathbb{X}}[\boldsymbol{h}] = \inf\{\, \mathscr{E}_{\mathbb{X}}[\boldsymbol{f}] : \ \boldsymbol{f} \in \mathscr{H}_2(\mathbb{X}, \mathbb{Y}) \,\}$$

Definition. Let $\mathbb{X}$ and $\mathbb{Y}$ be bounded domains. A map $h \in \overline{\mathscr{H}}_2(\mathbb{X}, \mathbb{Y})$ such that

$$\mathscr{E}_{\mathbb{X}}[\boldsymbol{h}] = \inf\{\, \mathscr{E}_{\mathbb{X}}[\boldsymbol{f}] : \ \boldsymbol{f} \in \mathscr{H}_2(\mathbb{X}, \mathbb{Y}) \,\}$$

will hereafter be referred to as an ***energy-minimal map*** or, simply, ***minimal map***.

Uniqueness of the minimal map is tricky.

THEOREM (*Uniqueness*). Let $\mathbb{X}$ be a nondegenerate doubly connected domain and $\mathbb{Y}$ a bounded doubly connected Lipschitz domain. Then the energy-minimal map $h \in \overline{\mathscr{H}}_2(\mathbb{X}, \mathbb{Y})$ is unique up to a conformal change of variables in $\mathbb{X}$.

If one wants to find an easy and clear way of verifying whether a given map $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is energy-minimal, one must look into the Hopf differential $h_z \overline{h_{\bar{z}}}\, \mathrm{d}z \otimes \mathrm{d}z$. Here is the recipe.

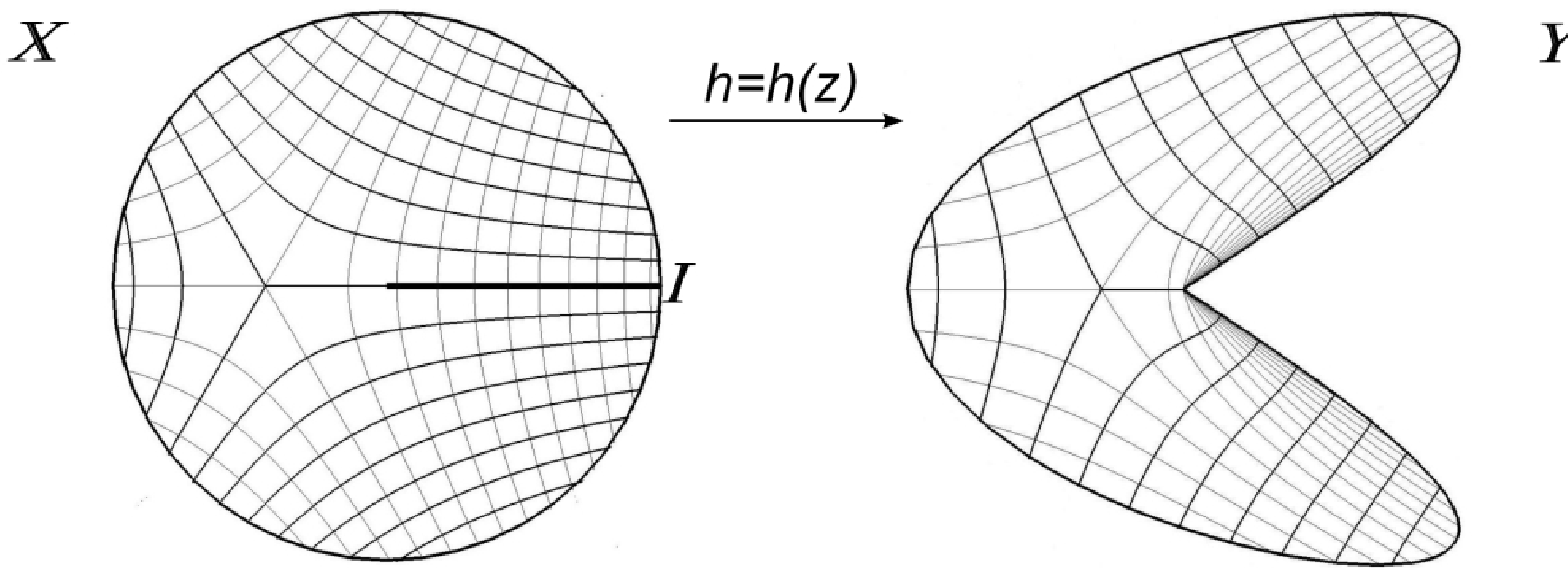
THEOREM (*Energy-minimal criterion*). Let $\mathbb{X}$ and $\mathbb{Y}$ be bounded doubly connected domains, $\mathbb{X}$ being nondegenerate. Then a mapping $h \in \overline{\mathscr{H}}_2(\mathbb{X}, \mathbb{Y})$ is energy-minimal if and only if its Hopf-differential is analytic and real along $\partial\mathbb{X}$.

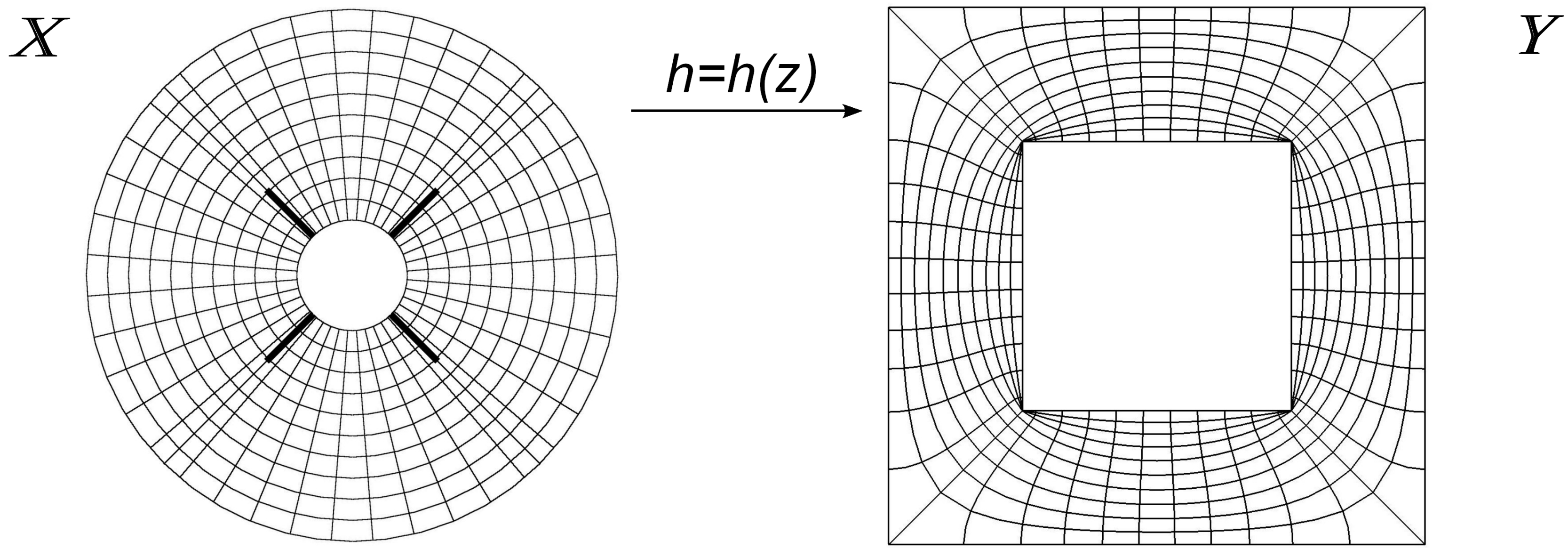
It is surprising that this very useful criterion has not been established before. The key observation in the proof of the **Uniqueness Theorem** is that the difference of two solutions to the same Hopf-Laplace equation has finite (not necessarily bounded) distortion on the set where at least one of the solutions has strictly positive Jacobian.

Propagation of cracks is an interesting phenomenon not only in modern theories of elasticity and plasticity, materials science or microscopic crystallographic defects found in real materials but also from mathematical point of view. Let $h \in \overline{\mathscr{H}}_2(\mathbb{X}, \mathbb{Y})$ be an energy-minimal map. It should be noted that in general h, being a limit of homeomorphisms from $\mathbb{X}$ onto $\mathbb{Y}$, has range in the closure of $\mathbb{Y}$, in symbols $h : \mathbb{X} \xrightarrow{\text{into}} \overline{\mathbb{Y}}$. The fact, referred to as ***partial harmonicity***, is that every energy-minimal map $h \in \overline{\mathscr{H}}_2(\mathbb{X}, \mathbb{Y})$ (or a solution to the Hopf Laplace equation) is a harmonic diffeomorphism of $h^{-1}(\mathbb{Y}) \subset \mathbb{X}$ onto $\mathbb{Y}$.

But there might be sets in $\mathbb{X}$ which are taken into a single point in $\partial\mathbb{Y}$.

Definition. Given a point $a \in \partial\mathbb{Y}$, the term *crack* (or a-crack) in $\mathbb{X}$ refers to any connected component of the set $\{ x \in \mathbb{X} : h(x) = a \}$.





When $\mathbb{Y}$ is at least Lipschitz regular then cracks in $\mathbb{X}$ emanate from $\partial\mathbb{X}$; are never reduced to a single point or any continuum in $\mathbb{X}$.

How to detect the cracks?

THEOREM. Let $h \in \overline{\mathscr{H}}_2(\mathbb{X}, \mathbb{Y})$ be an energy-minimal map, where $\mathbb{X}$ is a Jordan domain and $\mathbb{Y}$ a Lipschitz domain, both multiply connected. Suppose $\mathbb{Y}$ is convex at a boundary point $a \in \partial \mathbb{Y}$, meaning that the set $B \cap \mathbb{Y}$ is convex for some ball B centered at a. Then $a \notin h(\mathbb{X})$.

In spite of the impressive progress in the field, formation of cracks under energy-minimal deformations, to our knowledge, is not fully resolved. In this domain the best reference is the book by K. Bertram Broberg.

The problem clearly depends on the geometry of trajectories of the Hopf differential $h_z\overline{h_{\bar z}}\, \mathrm{d}z \otimes \mathrm{d}z$. In this lecture we give a detailed description of cracks in case of doubly connected domains. Accordingly, if the conformal modulus of $\mathbb{Y}$ (deformed configuration) is large relative to $\mathbb{X}$ then the Hopf differential $h_z\overline{h_{\bar z}}\, \mathrm{d}z \otimes \mathrm{d}z$ is real and negative along $\partial\mathbb{X}$. Consequently no cracks occur, even in the presence of points of concavity in the boundary of $\mathbb{Y}$.

If, on the other hand, the target is conformally very thin then the cracks are unavoidable. In this case the boundary components of $\mathbb{X}$ are horizontal trajectories along which the Hopf differential is positive. The cracks are born in $\partial\mathbb{X}$ and propagate along the vertical trajectories toward the interior of $\mathbb{X}$ where they eventually terminate. **Theoretical prediction of failure of bodies caused by cracks is a good motivation that should appeal to mathematical analysts and researchers in the engineering fields.**

Inner Variation and Hopf Differentials

For the convenience of the students we devote substantial part of this lecture to the inner variations and the associated Hopf differentials. Although the concept of inner variation has been used in the Calculus of Variations for a long time, some of its nuances are still to be scrutinized.

Throughout this section $\mathbb{X} \subset \mathbb{R}^2 \backsimeq \mathbb{C}$ is a bounded domain whose points, also called $variables$, will be denoted by $x = x_1 + ix_2$.

Change of variables The term change of variables in $\mathbb{X}$ refers to a $\mathscr{C}^\infty$-diffeomorphism $\Psi : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{X}$ which is continuous up to $\partial\mathbb{X}$.

Variation of variables in $\mathbb{X}$. This is a one-parameter family of change of variables $\{\Psi^{\varepsilon}\}_{-\varepsilon_\circ<\varepsilon<\varepsilon_\circ}$, such that

(i) The function $(\varepsilon, x) \rightsquigarrow \Psi^{\varepsilon}(x)$ is $\mathscr{C}^{\infty}$-smooth in $(-\varepsilon_\circ\, , \, \varepsilon_\circ) \times \mathbb{X}$

(ii) $\Psi^{0} = id$ in $\mathbb{X}$

Inner variation of a mapping . Let h be a function in the Sobolev space $\mathscr{W}^{1,2}(\mathbb{X}, \mathbb{C})$ and $\{\Psi^{\varepsilon}\}$ a variation of variables in $\mathbb{X}$. The family $h^\varepsilon = h \circ \Psi^\varepsilon$ is called an *inner variation* of h. Observe that all the mappings h^ε have the same range as h, which is one of the desired properties that motivates the use of inner variations. Additional assumption on the behavior of Ψ^ε near $\partial\mathbb{X}$ is needed in order to ensure that $h^\varepsilon \in \mathscr{W}^{1,2}(\mathbb{X}, \mathbb{C})$. In our applications, however, this property will always be satisfied.

Critical points. Choose and fix a variation $\{\Psi^{\varepsilon}\}_{-\varepsilon_\circ<\varepsilon<\varepsilon_\circ}$ of variables in $\mathbb{X}$. We say that $h \in \mathscr{W}^{1,2}(\mathbb{X}, \mathbb{C})$ is a *critical point* for $\{\Psi^{\varepsilon}\}$ if

$$\frac{\mathrm{d}\,\mathscr{E}_{\mathbb{X}}[h^{\varepsilon}]}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \equiv 0$$

Define $\lambda = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\Psi^{\varepsilon}$ and note that $D\lambda \equiv \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} D\Psi^{\varepsilon}$.

LEMMA. (*Integral form of the variational equation*). Every critical point for $\{\Psi^{\varepsilon}\}_{-\varepsilon_\circ<\varepsilon<\varepsilon_\circ}$ satisfies the equation

$$\Re e \iint_{\mathbb{X}} \overline{h_{\overline{z}}}\, h_z\, \lambda_{\overline{z}}\, \mathrm{d}z = 0\,, \quad \textit{where} \quad \lambda = \frac{\mathrm{d}\,\Psi^{\varepsilon}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}$$

PROOF

The peculiarity of our derivation of the inner variational equation lies in pointing out the relevance of the equation to the *Cauchy-Green stress tensor* of h and the *Ahlfors infinitesimal deformation operator* for λ. These connections were certainly overlooked in the literature; both are worth noting for prospective generalizations. For example, the same ideas work for the conformally invariant n-harmonic integrals in $\mathbb{R}^n$.

Let us begin with the composition rule $Dh^\varepsilon = Dh(\Psi^\varepsilon) \cdot D\Psi^\varepsilon$. We express the Hilbert-Schmidt norm of the differential matrix Dh^ε via scalar product of matrices

$$\begin{aligned} |Dh^\varepsilon|^2 = \langle Dh^\varepsilon \,|\, Dh^\varepsilon \rangle &= \langle Dh(\Psi^\varepsilon) \cdot D\Psi^\varepsilon \mid Dh(\Psi^\varepsilon) \cdot D\Psi^\varepsilon \rangle \\ &= \langle\, D^*h(\Psi^\varepsilon) \cdot Dh(\Psi^\varepsilon) \mid D\Psi^\varepsilon \cdot D^*\Psi^\varepsilon \,\rangle \end{aligned}$$

where D^* stands for the transposed differential. Then we integrate it over

$\mathbb{X}$ by substitution $\xi = \Psi^\varepsilon(z)$

$$\mathscr{E}_{\mathbb{X}}[h^\varepsilon] = \iint_{\mathbb{X}} \left\langle D^*h \cdot Dh \;\middle|\; \frac{D\Psi^\varepsilon \cdot D^*\Psi^\varepsilon}{\det D\Psi^\varepsilon} \right\rangle \mathrm{d}\xi$$

At this point one may recall the Cauchy-Green stress tensor of h

$$\mathbf{C}[h] = D^*h \cdot Dh$$

as well as Weyl's conformal tensor of Ψ^ε

$$\mathbf{W}[\Psi^\varepsilon] = \frac{D\Psi^\varepsilon \cdot D^*\Psi^\varepsilon}{\det D\Psi^\varepsilon}$$

Since $\Psi^0 = \mathrm{id}$, $\frac{\mathrm{d}\Psi^\varepsilon}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \equiv \lambda$ and $\frac{\mathrm{d}D\Psi^\varepsilon}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \equiv D\lambda$, the infinitesimal variation of $\mathbf{W}[\Psi^\varepsilon]$ at the identity map is readily computed as

$$\frac{\mathsf{d}\mathbf{W}[\Psi^\varepsilon]}{\mathsf{d}\varepsilon}\Big|_{\varepsilon=0} = D\lambda + D^*\lambda - (\mathsf{Tr}\, D\lambda)\, I$$

Here too, we recognize the Ahlfors infinitesimal deformation operator which defines the conformal component of the matrix $D\lambda$,

$$\mathbf{S}\lambda = \frac{1}{2}[D\lambda + D^*\lambda - (\mathsf{Tr}\, D\lambda)\, I] = \begin{bmatrix} \Re e\, \lambda_{\overline{z}} \;, & \Im m\, \lambda_{\overline{z}} \\ \Im m\, \lambda_{\overline{z}} \;, & -\Re e\, \lambda_{\overline{z}} \end{bmatrix}$$

The anticonformal component is afforded by the *Ahlfors' adjoint* operator

$$\mathbf{A}\lambda = \frac{1}{2}[D\lambda - D^*\lambda + (\mathsf{Tr}\, D\lambda)\, I] = \begin{bmatrix} \Re e\, \lambda_{z} \;, & -\Im m\, \lambda_{z} \\ \Im m\, \lambda_{z} \;, & \Re e\, \lambda_{z} \end{bmatrix}$$

Thus we have a pointwise orthogonal decomposition $D\lambda = \mathbf{S}\lambda + \mathbf{A}\lambda$. Now, a straightforward computation gives the variation of the energy integral

$$\frac{\mathsf{d}\,\mathscr{E}_{\mathbb{X}}[h^\varepsilon]}{\mathsf{d}\varepsilon}\,\Big|_{\varepsilon=0} = 2\iint_{\mathbb{X}} \Big\langle\, D^*h\,\cdot Dh\;\Big|\;\mathbf{S}\,\lambda\,\Big\rangle\,\mathsf{d}\xi$$

$$= \;8\,\Re e\iint_{\mathbb{X}} \overline{h_{\overline{z}}}\,h_z\,\lambda_{\overline{z}}\;\mathsf{d}z$$

because $\big\langle\, D^*h\,\cdot Dh\;\big|\;\mathbf{S}\,\lambda\,\big\rangle \;=\; 4\,\Re e\,(\overline{h_{\overline{z}}}\,h_z\,\lambda_{\overline{z}})$.

The Hopf-Laplace Equation , Homework 3

Every complex function $\lambda \in \mathscr{C}_0^\infty(\mathbb{X}, \mathbb{C})$ gives rise to a variation $\Psi^\varepsilon(z) = z + \varepsilon\lambda(z)$ of variables in $\mathbb{X}$ that is fixed at every point of $\partial\mathbb{X}$. This variation is legitimate for any function $h \in \mathscr{W}^{1,2}(\mathbb{X}, \mathbb{C})$. In particular, applying the integral form of the variational equation to such λ we infer via the classical Weyl's lemma that

PROPOSITION. A function $h \in \mathscr{W}^{1,2}(\mathbb{X}, \mathbb{C})$ that is critical for all inner variations must satisfy the *Hopf-Laplace equation* in $\mathbb{X}$,

$$\frac{\partial}{\partial \bar{z}} h_z \overline{h_{\bar{z}}} = 0 \ , \quad \text{(in the sense of distributions)}$$

equivalently,

$$h_z \overline{h_{\bar{z}}} = \varphi(z) \ , \quad \text{for some analytic function} \quad \varphi \in \mathscr{L}^1(\mathbb{X})$$

If one allows the critical mapping h to slip along an arc $\gamma \subset \partial \mathbb{X}$ then an additional equation on γ will emerge, which in turn yields a specific form of the analytic function φ. Let us illustrate it with an example of such situation when $\gamma = \partial \mathbb{X}$.

EXAMPLE. (Traction free critical solutions in an annulus) Consider a traction free problem in an annulus $\mathbb{X} = \{ z \,;\, r < |z| < R \}$; that is, allow $h : \partial \mathbb{X} \xrightarrow{\text{into}} \partial \mathbb{Y}$ to slide along the boundary circles of $\mathbb{X}$.

PROPOSITION. The Hopf-Laplace equation for a function $h \in \mathscr{W}^{1,2}(\mathbb{X}, \mathbb{C})$ that is critical for all inner variations in $\mathbb{X}$ takes the form

$$h_z \, \overline{h_{\overline{z}}} = \frac{c}{z^2} \,, \quad \text{for all} \quad z \in \mathbb{X}$$

where c is a real number.

Proof (optional). We expand the Hopf product of h into a Laurent series

$$h_z\,\overline{h_{\overline{z}}} = \varphi(z) = \sum_{n=-\infty}^{\infty} a_n\, z^n$$

Then we test the integral equation with the following variations of variables in $\mathbb{X}$,

$$\Psi^{\varepsilon}(z) = \; z \cdot \frac{1 + \varepsilon(a\bar{z}^k \; - \; \bar{a}z^k)}{[\,1 - \varepsilon^2(a\bar{z}^k \; - \; \bar{a}z^k)^2\,]^{1/2}}\,, \quad k = \pm 1, \pm 2, \ldots$$

where a can be any complex number and ε any sufficiently small real number. Evidently $|\Psi^{\varepsilon}(z)| \equiv |z|$, so $\Psi^{\varepsilon} : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{X}$ can easily be shown to be a $\mathscr{C}^\infty$ -diffeomorphism. A short computation gives

$$\lambda = \frac{\mathrm{d}\,\Psi^\varepsilon}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} = z(a\bar{z}^k - \bar{a}z^k) \quad \text{and} \quad \lambda_{\bar{z}} = k\,az\,\bar{z}^{k-1}$$

Put these values of $\lambda_{\bar{z}}$ into the equation to obtain

$$0 = \Re e \iint_{\mathbb{X}} \Big(\sum_{n=-\infty}^{\infty} a_n\, z^n \Big)\, k\, az\, \bar{z}^{k-1}\, \mathrm{d}z = \Re e(k\, a\, a_{k-2}) \iint_{\mathbb{X}} |z|^{2k-2}\, \mathrm{d}z$$

for every complex number a. This yields $a_{k-2} = 0$, except for $k = 0$. We just proved that $\varphi(z) = a_{_{-2}}\, z^{-2}$. To see that the coefficient $a_{_{-2}}$ is real we test the integral equation again, but with the following variation of variables in $\mathbb{X}$.

$$\Psi^{\varepsilon}(z) = \ z \cdot \frac{1 + \ i \ \log|z|}{[\,1 + \varepsilon^2 \log^2 |z|\,]^{1/2}} \quad \lambda = \frac{\mathsf{d}\,\Phi^{\varepsilon}}{\mathsf{d}\varepsilon}\,\Big|_{\varepsilon=0} = i\,z\log|z| \quad \text{and} \quad \lambda_{\bar z} \,=\, i\frac{z}{\bar z}$$

Then we find that

$$0 \,=\, \Re e \iint_{\mathbb{X}} \frac{a_{\_2}}{z^2}\, i\,\frac{z}{\bar z}\,\mathsf{d}z \,=\, -\Im m\,(a_{\_2}) \iint_{\mathbb{X}} |z|^{-2}\,\mathsf{d}z\ , \quad \text{hence}\ \ a_{\_2} \in \mathbb{R}$$

as desired.

Vertical and horizontal arcs of a quadratic differential

Let $\varphi(z)\,\mathrm{d}z \otimes \mathrm{d}z$ be a holomorphic quadratic differential in $\mathbb{X}$. A *vertical arc* is a $\mathscr{C}^\infty$-smooth curve $\gamma = \gamma(t)$, $a < t < b$, along which

$$[\dot\gamma(t)]^2 \varphi\big(\gamma(t)\big) < 0, \quad a < t < b.$$

A *vertical trajectory* of φ in $\mathbb{X}$ is a maximal vertical arc; that is, not properly contained in any other vertical arc. In exactly similar way are defined the *horizontal arcs* and *horizontal trajectories*, via the opposite inequality.

If $\mathbb{X}$ is a circular annulus $\mathbb{A} = A(r, R)$ and $\varphi(z)\,\mathrm{d}z \otimes \mathrm{d}z$ is real along its entire boundary then $\varphi(z) = cz^{-2}$, for some $c \in \mathbb{R}$. For $c > 0$ the concentric circles $\mathcal{C}_\rho = \{\, \rho\, e^{i\theta} \colon 0 \leqslant \theta < 2\pi \}$, $\rho \in [r, R]$, are the vertical trajectories, whereas the rays $\mathcal{R}^\theta = \{\, \rho\, e^{i\theta} \colon r < \rho < R\}$, $\theta \in [0, 2\pi)$, are the horizontal trajectories. For a negative c, this holds in reverse order. These two cases exhibit different behavior in regard to the formation of cracks.

Homeomorphic Solutions (recalled)

Kovalev, Onninen, T.I. arXiv:1006.5174)

Let $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ be a homeomorphism of Sobolev class $\mathscr{W}^{1,2}(\mathbb{X},\mathbb{Y})$ that satisfies the Hopf-Laplace equation

$$\frac{\partial}{\partial \overline{z}} \left(h_z \overline{h_{\overline{z}}} \right) = 0$$

Then h is a harmonic diffeomorphism.

Coffeeholics $\approx$ Coffeecolleagues

Jani **Tadeusz**

Coffee is the main ingredient in our proofs.

An integral identity

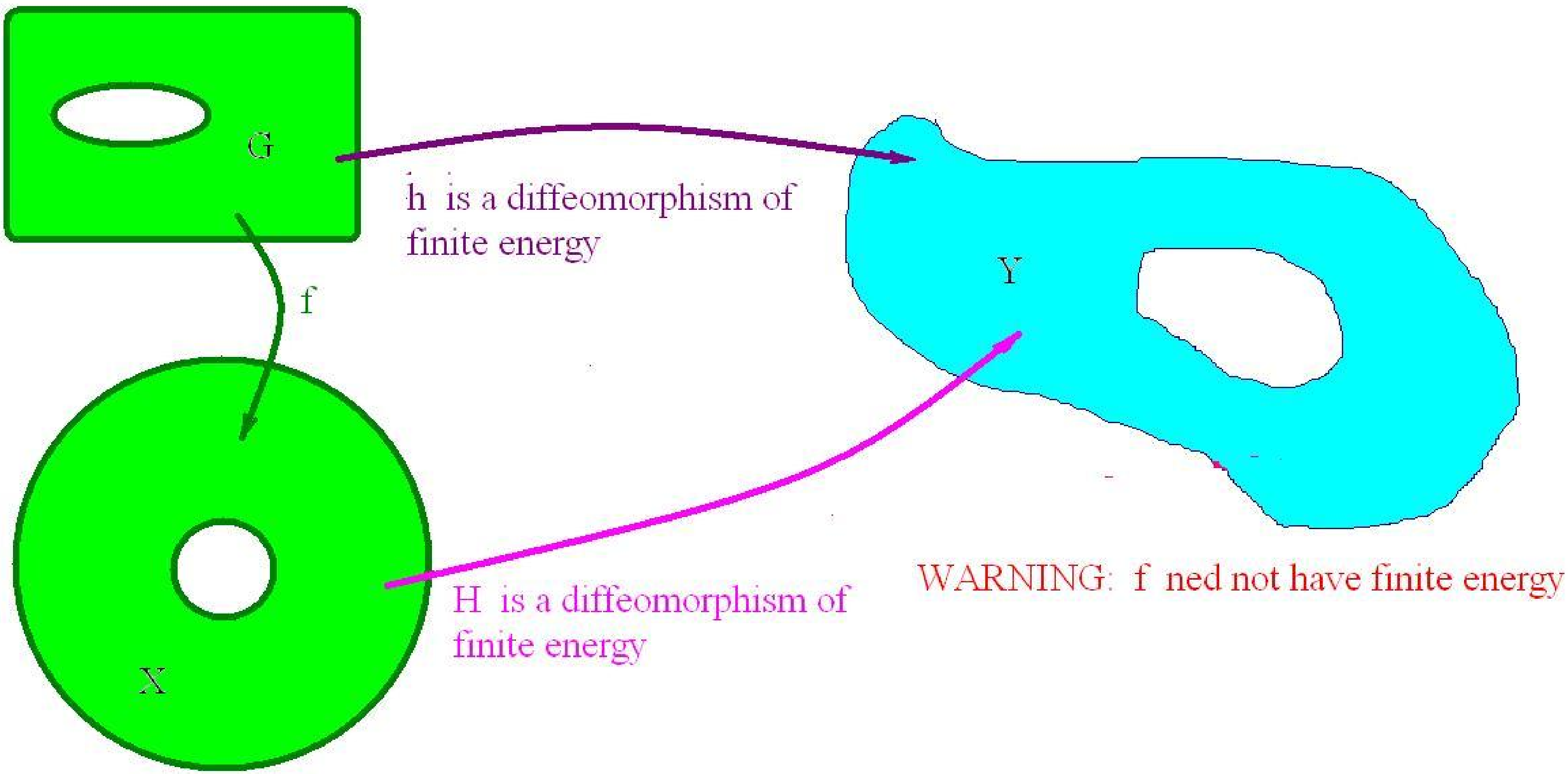
LEMMA. Let $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{G}$ be bounded domains in $\mathbb{C}$. Suppose that $h\colon \mathbb{G} \xrightarrow{\text{onto}} \mathbb{Y}$ and $H\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ are orientation preserving $\mathscr{C}^\infty$-diffeomorphisms of finite energy. Define $f = H^{-1} \circ h : \mathbb{G} \xrightarrow{\text{onto}} \mathbb{X}$. Then we have

$$\mathscr{E}_{\mathbb{X}}[H] - \mathscr{E}_{\mathbb{G}}[h] = 4 \iint_{\mathbb{G}} \left[\frac{|f_z - \gamma(z) f_{\bar{z}}|^2}{|f_z|^2 - |f_{\bar{z}}|^2} - 1 \right] |h_z h_{\bar{z}}| \, \mathrm{d}z$$

$$+ 4 \iint_{\mathbb{G}} \frac{(\,|h_z| - |h_{\bar{z}}|\,)^2 \cdot |f_{\bar{z}}|^2}{|f_z|^2 - |f_{\bar{z}}|^2} \, \mathrm{d}z$$

where

$$\gamma = \gamma(z) = \begin{cases} h_z \overline{h_{\bar{z}}}\, |h_z \overline{h_{\bar{z}}}|^{-1} & \text{if } h_z \overline{h_{\bar{z}}} \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$


G
h is a diffeomorphism of finite energy
Y
f
H is a diffeomorphism of finite energy
X
WARNING: f ned not have finite energy

PROOF. It is worth noting that $f : \mathbb{G} \xrightarrow{\text{onto}} \mathbb{X}$ need not have finite energy. The convergence of the integrals, not obvious at the first glance, is a consequence of the finite energy condition imposed on the mappings h and H.

We begin with the chain rule applied to $H = h \circ f^{-1} \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$,

$$\frac{\partial H(w)}{\partial w} = h_z(z)\frac{\partial f^{-1}}{\partial w} + h_{\bar{z}}(z)\overline{\frac{\partial f^{-1}}{\partial \overline{w}}}$$
$$\frac{\partial H(w)}{\partial \bar{w}} = h_z(z)\frac{\partial f^{-1}}{\partial \bar{w}} + h_{\bar{z}}(z)\overline{\frac{\partial f^{-1}}{\partial w}}$$

where $w = f(z)$. We express the complex partial derivatives of $f^{-1} \colon \mathbb{X} \to$

$\mathbb{X}$ at w in terms $f_z(z)$ and $f_{\bar{z}}(z)$ at $z = f^{-1}(w)$,

$$\frac{\partial f^{-1}}{\partial w} = \frac{\overline{f_z(z)}}{J(z,f)} \quad \text{and} \quad \frac{\partial f^{-1}}{\partial \bar{w}} = -\frac{f_{\bar{z}}(z)}{J(z,f)} \tag{0}$$

Note that the Jacobian determinant $J(z,f) = |f_z|^2 - |f_{\bar{z}}|^2$ is strictly positive. These expressions yield

$$\frac{\partial H}{\partial w} = \frac{h_z \overline{f_z} - h_{\bar{z}} \overline{f_{\bar{z}}}}{|f_z|^2 - |f_{\bar{z}}|^2} \quad \text{and} \quad \frac{\partial H}{\partial \bar{w}} = \frac{h_{\bar{z}} f_z - h_z f_{\bar{z}}}{|f_z|^2 - |f_{\bar{z}}|^2}$$

Next we compute the energy of H over the set $f(\mathbb{G}) = \mathbb{X}$ by substitution

$w = \chi(z)$,

$$\mathscr{E}_{f(\mathbb{G})}[H] = 2 \iint_{f(\mathbb{G})} \left(|H_w|^2 + |H_{\bar{w}}|^2 \right) \, \mathsf{d}w$$
$$= 2 \iint_{\mathbb{G}} \frac{|h_z \overline{f_z} - h_{\bar{z}} \overline{f_{\bar{z}}}|^2 + |h_{\bar{z}} f_z - h_z f_{\bar{z}}|^2}{|f_z|^2 - |f_{\bar{z}}|^2} \, \mathsf{d}z.$$

On the other hand, the energy of h over the set $\mathbb{G}$ equals

$$\mathscr{E}_{\mathbb{G}}[h] = 2 \iint_{\mathbb{G}} \left(|h_z|^2 + |h_{\bar{z}}|^2 \right) \, \mathsf{d}z$$

The desired formula follows by subtracting these two integrals,

$$
\begin{aligned}
\mathscr{E}_{\mathbb{X}}[H] - \mathscr{E}_{\mathbb{G}}[h] &= 4 \iint_{\mathbb{G}} \frac{\left(|h_z|^2 + |h_{\bar z}|^2\right) \cdot |f_{\bar z}|^2 - 2\,\mathrm{Re}\left[h_z \overline{h_{\bar z} f_z} f_{\bar z}\right]}{|f_z|^2 - |f_{\bar z}|^2}\, \mathsf{d}z \\
&= 4 \iint_{\mathbb{G}} \frac{2|h_z h_{\bar z}| \cdot |f_{\bar z}|^2 - 2\,\mathrm{Re}\left[h_z \overline{h_{\bar z} f_z} \chi_{\bar z}\right]}{|f_z|^2 - |f_{\bar z}|^2}\, \mathsf{d}z \\
&+ 4 \iint_{\mathbb{G}} \frac{(\,|h_z| - |h_{\bar z}|\,)^2 \cdot |f_{\bar z}|^2}{|f_z|^2 - |f_{\bar z}|^2}\, \mathsf{d}z \\
&= 4 \iint_{\mathbb{G}} \left[\frac{|f_z - \gamma(z) f_{\bar z}|^2}{|f_z|^2 - |f_{\bar z}|^2} - 1\right] |h_z h_{\bar z}|\, \mathsf{d}z \\
&+ 4 \iint_{\mathbb{G}} \frac{(\,|h_z| - |h_{\bar z}|\,)^2 \cdot |f_{\bar z}|^2}{|f_z|^2 - |f_{\bar z}|^2}\, \mathsf{d}z
\end{aligned}
$$

Some Free Lagrangians

Recall that a free Lagrangian for a pair of domains $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ is a nonlinear differential n -form $L(x, h, Dh)\, \mathrm{d}x$ whose integral mean over $\mathbb{X}$ depends only on the homotopy class of a homeomorphism $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$. Here are a few of them for the annular domains $\mathbb{X} = \mathbb{A} = \{x \in \mathbb{C} \colon r < |x| < R\}$ and $\mathbb{Y} = \mathbb{A}^* = \{y \in \mathbb{C} \colon r_* < |y| < R_*\}$, where we make use of polar coordinates ρ and θ

$$z = \rho e^{i\theta}\,, \qquad 0 \leqslant \rho < \infty \quad \text{and} \quad 0 \leqslant \theta < 2\pi\,.$$

The normal (radial) and tangential (angular) derivatives of a Sobolev mapping f are defined by

$$f_N(z) := \frac{\partial f(\rho e^{i\theta})}{\partial \rho}, \qquad \rho = |z|$$

and

$$f_T(z) := \frac{1}{\rho}\frac{\partial f(te^{i\theta})}{\partial \theta}, \qquad \rho = |z|\,.$$

The Jacobian determinant of f is

$$J(\cdot, f) = J_f = |f_z|^2 - |f_{\bar z}|^2 = \operatorname{Im} \overline{f_N} f_T.$$

- Pullback of a 2-form in $\mathbb{Y}$ via an orientation preserving homeomorphism

$h \in \mathscr{H}(\mathbb{X},\mathbb{Y}) \cap \mathscr{W}^{1,2}(\mathbb{X},\mathbb{Y})$ is a free Lagrangian

$$\iint_{\mathbb{X}} N(|h|)\, J(x,h)\, \mathrm{d}x = \iint_{\mathbb{Y}} N(|y|)\, \mathrm{d}y\,.$$

- Normal differentiation gives rise to a free Lagrangian for $h \in \mathscr{H}(\mathbb{X},\mathbb{A}^*) \cap \mathscr{W}^{1,1}(\mathbb{X},\mathbb{A}^*)$ defined by

$$\begin{aligned} \left| \iint_{\mathbb{X}} A(|h|) \frac{|h|_N}{|x|}\, \mathrm{d}x \right| &= 2\pi \left| \int_r^R A\big(|h|\big) \frac{\partial |h|}{\partial \rho}\, \mathrm{d}\rho \right| \\ &= 2\pi \left| \int_{r_*}^{R_*} A(\tau)\, \mathrm{d}\tau \right|\,. \end{aligned}$$

- A dual free Lagrangian for $h \in \mathscr{H}(\mathbb{A},\mathbb{Y}) \cap \mathscr{W}^{1,1}(\mathbb{A},\mathbb{Y})$ arises from

tangential differentiation

$$\left|\iint_{\mathbb{A}} B\big(|x|\big)\,\mathsf{Im}\frac{h_T}{h}\,\mathsf{d}x\right| = \left|\int_r^R B(t)\left(\int_{|x|=t}\frac{\partial\,\mathsf{Arg}\,h}{\partial\theta}\,\mathsf{d}\theta\right)\mathsf{d}t\right|$$

$$= 2\pi\left|\int_r^R B(t)\,\mathsf{d}t\right|$$

Normal and Tangential Components of the Distortion Function

Geometric function theory is concerned with the distortion function

$$K^f = \frac{|Df|^2}{J_f} = \frac{2\left(|f_z|^2 + |f_{\bar{z}}|^2\right)}{J_f}.$$

We may decompose it as $K^f = K_N^f + K_T^f$, where (using polar coordinates) the *normal* and *tangential distortions* of f are defined by the rules

$$K_N^f := \frac{|f_z + \frac{\bar{z}}{z} f_{\bar{z}}|^2}{J_f} = \frac{|f_N|^2}{J_f}$$

$$K_T^f := \frac{|f_z - \frac{\bar{z}}{z} f_{\bar{z}}|^2}{J_f} = \frac{|f_T|^2}{J_f}$$

By convention, these two quotients are understood as 0 whenever the numerator vanishes. Naturally, they assume the value $+\infty$ if the Jacobian vanishes but the numerator does not. For a mapping $f \in \mathscr{W}^{1,1}_{\mathrm{loc}}$ the quantities f_N, f_T, and J_f are finite a.e. and, therefore, K^f_N and K^f_T are well defined measurable functions in the domain of definition of f.

Two Lemmas

Using free Lagrangians we obtain sharp inequalities for $\mathscr{L}^2$- averages of the distortion functions

LEMMA 1. (Estimate of the normal component)
Let $\mathbb{X}$ be a bounded doubly connected domain that separates the origin 0 from ∞, and let $\mathbb{A}^* = A(r_*, R_*)$ be a circular annulus. If $h \in \mathscr{H}(\mathbb{X}, \mathbb{A}^*) \cap \mathscr{W}^{1,2}(\mathbb{X}, \mathbb{A}^*)$ then

$$\iint_{\mathbb{X}} \frac{|h_N|^2}{J_h} \frac{\mathrm{d}z}{|z|^2} \geqslant 2\pi \log(R_*/r_*) \quad , \qquad \frac{|h_N|^2}{J_h} = K_N^h .$$

Proof (optional)

We have

$$2\pi \log(R_*/r_*) = 2\pi \left| \int_{r_*}^{R_*} \frac{\mathsf{d}\tau}{\tau} \right| \leqslant \iint_{\mathbb{X}} \frac{|h_N|}{|h|} \frac{\mathsf{d}z}{|z|}$$

and

$$\left(\iint_{\mathbb{X}} \frac{|h_N|}{|h|} \frac{\mathsf{d}z}{|z|} \right)^2 \leqslant \iint_{\mathbb{X}} \frac{|h_N|}{J_h} \frac{\mathsf{d}z}{|z|^2} \iint_{\mathbb{X}} \frac{J_h}{|h|^2} \, \mathsf{d}z$$

$$= \iint_{\mathbb{X}} \frac{|h_N|}{J_h} \frac{\mathsf{d}z}{|z|^2} \iint_{\mathbb{A}^*} \frac{\mathsf{d}y}{|y|^2}$$

LEMMA 2. (Estimate of the tangential component)
Let $\mathbb{A} = A(r,R)$ be a circular annulus and $\mathbb{Y}$ a bounded doubly connected domain of finite conformal modulus. If $h \in \mathscr{H}(\mathbb{A},\mathbb{Y}) \cap \mathscr{W}^{1,2}(\mathbb{A},\mathbb{Y})$, then

$$\iint_{\mathbb{A}} \frac{|h_T|^2}{J_h} \frac{\mathrm{d}z}{|z|^2} \geqslant 2\pi \frac{\log^2(R/r)}{\operatorname{Mod}\mathbb{Y}} \;, \qquad \frac{|h_T|^2}{J_h} = K_T^h.$$

Proof (0ptional) There exists a conformal transformation $F\colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{A}^*$, of $\mathbb{Y}$ onto an annulus $\mathbb{A}^* = \{z\colon 0 < r_* < |z| < R_*\}$. We define $g = F \circ h\colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{A}^*$. Since F is conformal

$$\iint_{\mathbb{A}} \frac{|h_T|^2}{J_h} \frac{\mathrm{d}z}{|z|^2} = \iint_{\mathbb{A}} \frac{|g_T|^2}{J_g} \frac{\mathrm{d}z}{|z|^2}$$

We have

$$2\pi \log R/r = 2\pi \left| \int_r^R \frac{\mathsf{d}t}{t} \right| \leqslant \iint_{\mathbb{A}} \frac{|g_T|}{|g|} \frac{\mathsf{d}z}{|z|} \,.$$

Now, it follows by Hölders inequality that

$$\begin{aligned}
\left(\iint_{\mathbb{A}} \frac{|g_T|}{|g|} \frac{\mathsf{d}z}{|z|} \right)^2 &\leqslant \iint_{\mathbb{A}} \frac{|g_T|^2}{J_g} \frac{\mathsf{d}z}{|z|^2} \iint_{\mathbb{A}} \frac{J_g}{|g|^2} \\
&= \iint_{\mathbb{A}} \frac{|g_T|^2}{J_g} \frac{\mathsf{d}z}{|z|^2} \iint_{\mathbb{A}^*} \frac{\mathsf{d}y}{|y|^2} \,.
\end{aligned}$$

Returning to Hopf Laplace Equation

LEMMA 3. Let $\mathbb{A} = A(r, R)$ be a circular annulus, $0 < r < R < \infty$, and $\mathbb{Y}$ a bounded doubly connected domain. Suppose $h \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{A}, \mathbb{Y})$ satisfies the Hopf-Laplace equation

$$h_z \overline{h_{\bar z}} \equiv \frac{c}{z^2} \qquad \text{in } \mathbb{A}\text{ , and } J_h \geqslant 0 \text{ almost everywhere}$$

where $c \in \mathbb{R}$ is a constant. Then we have point-wise inequalities

$$\begin{cases} |h_N|^2 \leqslant J_h, & \text{if } c \leqslant 0 \\ |h_T|^2 \leqslant J_h, & \text{if } c \geqslant 0 \end{cases}$$

. ***PROOF.*** The complex Hopf-Laplace equation reduces to the system of two real equations,

$$|h_N|^2 - |h_T|^2 = \frac{4c}{|z|^2}$$

$$\mathrm{Re}(\overline{h_N} h_T) = 0.$$

Recall that $J_h = \mathrm{Im}\, \overline{h_N}\, h_T \geqslant 0$ which in view of $\mathrm{Re}(\overline{h_N} h_T) = 0$ reads as

$$J_h = |h_N||h_T|$$

Combining these identities yields the point-wise estimates of the distortion functions.

Hopf solutions are energy-minimal

Throughout this section $\mathbb{X}$ and $\mathbb{Y}$ are Lipschitz doubly connected domains.

PROPOSITION. Suppose the Hopf-differential $h_z\overline{h_{\bar z}}\,\mathrm{d}z \otimes \mathrm{d}z$ defined for $h \in \overline{\mathscr{H}_2}(\mathbb{X}, \mathbb{Y})$ is holomorphic and real along $\partial\mathbb{X}$. Then

$$\mathscr{E}_{\mathbb{X}}[h] = \inf\{\, \mathscr{E}_{\mathbb{X}}[g] \colon g \in \overline{\mathscr{H}_2}(\mathbb{X}, \mathbb{Y}) \,\}\,. \tag{-14}$$

Furthermore, h is a unique minimizer (up to the conformal change of variables in $\mathbb{X}$) within the class $\overline{\mathscr{H}_2}(\mathbb{X}, \mathbb{Y})$.

We shall only sketch the proof in case of positive Hopf differential.

PROPOSITION. Suppose that a Hopf-differential $h_z \overline{h_{\bar z}}\, \mathrm{d}z \otimes \mathrm{d}z$, defined for $h \in \overline{\mathscr{H}}_2(\mathbb{X}, \mathbb{Y})$, is holomorphic and real positive along $\partial \mathbb{X}$. Then h is an energy-minimal.

PROOF. A conformal transformation of $\mathbb{X}$ onto an annulus $\mathbb{A} = \{z \colon r < |z| < R\}$ induces an isometry of $\overline{\mathscr{H}}_2(\mathbb{X}, \mathbb{Y})$ onto $\overline{\mathscr{H}}_2(\mathbb{A}, \mathbb{Y})$. Thus we may assume that $\mathbb{X} = \mathbb{A}$, so as to apply the equality

$$h_z \overline{h_{\bar z}} = \frac{c}{z^2} \qquad c \in \mathbb{R} \setminus \{0\}\,.$$

The assumption that $\varphi(z)\, \mathrm{d}z \otimes \mathrm{d}z$ is positive along $\partial \mathbb{A}$ simply means that $c < 0$.

We write $\mathbb{G} := h^{-1}(\mathbb{Y})$. In view of Theorem the mapping $h \colon \mathbb{A} \to \overline{\mathbb{Y}}$ is a harmonic diffeomorphism from $\mathbb{G} \subset \mathbb{A}$ onto $\mathbb{Y}$. Let $H \colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{Y}$ be an orientation preserving $\mathscr{C}^\infty$-diffeomorphism. We denote

$$f = H^{-1} \circ h \colon \mathbb{G} \xrightarrow{\text{onto}} \mathbb{A}.$$

Applying Lemma, we have

$$\begin{aligned}
\mathscr{E}_{\mathbb{A}}[H] - \mathscr{E}_{\mathbb{G}}[h] &= 4\,|c| \iint_{\mathbb{G}} \left[\frac{|f_z + \frac{z}{\bar z} f_{\bar z}|^2}{|f_z|^2 - |f_{\bar z}|^2} - 1\right] \frac{\mathrm{d}z}{|z|^2} \\
&\quad + 4 \iint_{\mathbb{G}} \frac{(|h_z| - |h_{\bar z}|)^2\,|f_{\bar z}|^2}{|f_z|^2 - |f_{\bar z}|^2}\,\mathrm{d}z \\
&= 4\,|c| \iint_{\mathbb{G}} \left[K_N^f - 1\right] \frac{\mathrm{d}z}{|z|^2} \\
&\quad + 4 \iint_{\mathbb{G}} \frac{(|h_z| - |h_{\bar z}|)^2\,|f_{\bar z}|^2}{J_f}\,\mathrm{d}z.
\end{aligned}$$

Before estimating the right hand side we will show that

$$\mathscr{E}_{\mathbb{A}}[h] = \mathscr{E}_{\mathbb{G}}[h] + 4\,|c| \iint_{\mathbb{A}\setminus\mathbb{G}} \frac{\mathrm{d}z}{|z|^2}. \tag{-15}$$

In view of Lemma $J_h = 0$ in $\mathbb{A} \setminus \mathbb{G}$. Since $c < 0$, by Lemma, $|h_N| = 0$ in $\mathbb{A} \setminus \mathbb{G}$. Therefore, $|Dh|^2 = |h_T|^2$ in $\mathbb{A} \setminus \mathbb{G}$. On the other hand, in view of, we have $|h_T|^2 = -4c|z|^{-2}$ in $\mathbb{A} \setminus \mathbb{G}$. Therefore

$$\iint_{\mathbb{A}\setminus\mathbb{G}} |Dh|^2 = -4\,c \int_{\mathbb{A}\setminus\mathbb{G}} \frac{\mathrm{d}}{|z|^2}.$$

Combining with we arrive at the identity

$$\mathscr{E}_{\mathbb{A}}[H] - \mathscr{E}_{\mathbb{A}}[h] = 4\,|c| \left[\iint_{\mathbb{G}} K_N^f(z)\,\mathsf{d}z - \iint_{\mathbb{A}} \frac{\mathsf{d}z}{|z|^2} \right] + 4 \iint_{\mathbb{G}} \frac{(|h_z| - |h_{\bar z}|)^2\, |f_{\bar z}|^2}{J_f}\,\mathsf{d}z.$$

According to Lemma

$$\iint_{\mathbb{G}} K_N^f \geqslant 2\pi \log\left(R/r\right) = \iint_{\mathbb{A}} \frac{\mathsf{d}z}{|z|^2}.$$

Therefore, if $H \colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{Y}$ is a $\mathscr{C}^\infty$-diffeomorphism, we can write

$$\mathscr{E}_{\mathbb{A}}[H] - \mathscr{E}_{\mathbb{A}}[h] \geqslant 4 \iint_{\mathbb{G}} \frac{(|h_z| - |h_{\bar z}|)^2\, |f_{\bar z}|^2}{J_f}\,\mathsf{d}z \geqslant 0.$$

The last inequality follows from the fact that f preserves the orientation. Hence $\mathscr{E}_{\mathbb{A}}[H] \geqslant \mathscr{E}_{\mathbb{A}}[h]$ for an arbitrary $H \in \overline{\mathscr{H}_2}(\mathbb{A}, \mathbb{Y})$, meaning that h is an energy-minimal map.

The distortion of the difference of two solutions

(the key ingredient in the proof of uniqueness).

Suppose $h, H \in \mathscr{W}^{1,2}(\mathbb{X}, \mathbb{C})$, $J_h \geqslant 0$ and $J_H \geqslant 0$, have the same *Hopf-product*,

$$h_z \overline{h}_{\bar{z}} = H_z \overline{H}_{\bar{z}} = \varphi(z) \neq 0 \quad \text{almost everywhere (not necessarily analytic).}$$

Consider the difference

$$F(z) = H(z) = h(z) \in \mathscr{W}^{1,2}(\mathbb{X}, \mathbb{C}).$$

We have

$$
\begin{aligned}
h_{\bar{z}}\overline{F}_z &= h_{\bar{z}}(\overline{H}_z - \overline{h}_z) = h_{\bar{z}}\overline{H}_z - \overline{\varphi} = h_{\bar{z}}\overline{H}_z - \overline{H}_z H_{\bar{z}} \\
&= \overline{H}_z(h_{\bar{z}} - H_{\bar{z}}) = -F_{\bar{z}}\overline{H}_z
\end{aligned}
$$

where we notice that

$$
\begin{aligned}
|h_{\bar{z}}|^2 &\leqslant |h_{\bar{z}}||h_z| = |\varphi||H_{\bar{z}}|^2 \\
&\leqslant |H_z||H_{\bar{z}}| = |\varphi|
\end{aligned}
$$

Hence $|\varphi|^2|F_z|^2 \geqslant |\varphi|^2|F_{\bar{z}}|^2$ so $J_F \geqslant 0$ almost everywhere. Next we introduce the Beltrami distortion coefficients

$$
k_h(z) = \frac{|h_{\bar{z}}|}{|h_z|} \leqslant 1 \quad \text{and} \quad k_H(z) = \frac{|H_{\bar{z}}|}{|H_z|} \leqslant 1.
$$

We find that

$$|F_{\bar z}| = k_F(z)|F_z| \quad \text{where } k_F(z) = \sqrt{k_h(z)\,k_H(z)} \leqslant 1.$$

Indeed, we have

$$|h_{\bar z}|^2|F_z|^2 = |H_z|^2|F_{\bar z}|^2 \ , \quad \text{where } |h_{\bar z}|^2 = k_h|\varphi| \quad \text{and} \quad k_H|H_z|^2 = |\varphi|.$$

Hence

$$k_h k_H|F_z|^2|\varphi| = k_H|h_{\bar z}|^2|F_z|^2 = k_H|H_z|^2|F_{\bar z}|^2 = \varphi\,|F_{\bar z}|^2$$

and therefore

$$|F_{\bar z}| = \sqrt{k_h k_H}|F_z| = k_F(z)|F_z|.$$

Note that

$$k_F(z) \begin{cases} < 1 & \text{whenever } J_h + J_H \neq 0 \\ = 1 & \text{whenever } J_h + J_H = 0. \end{cases}$$

In particular, F has finite distortion whenever $J_h \neq 0$ or $J_H(z) \neq 0$

$$|DF|^2 = 2\left(|F_z|^2 + |F_{\bar{z}}|^2\right) = 2\,\frac{1 + k_h k_H}{1 - k_h k_H} J_F\ .$$

Two examples

Let us illustrate how Theorem works for mappings $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ between doubly connected domains. Thus we shall look at the Hopf differential $\mathfrak{h}_z \overline{\mathfrak{h}_{\bar z}}\, \mathrm{d}z \otimes \mathrm{d}z$ to check as to whether it is real along $\partial \mathbb{X}$. The two examples here also serve to show a delicate difference between Hopf differentials being positive or negative. We can, and do, assume without affecting the results that the domain $\mathbb{X}$ is an annulus . Thus in either case we are dealing with the solutions to the Hopf-Laplace equation

$$\mathfrak{h}_z \overline{\mathfrak{h}_{\bar z}} = \frac{c}{z^2}\,, \quad \text{in an annulus} \quad \mathbb{X}$$

CASE $c > 0$, Hopf differentials are negative along $\partial \mathbb{X}$. This is the case in which no cracks emerge. Consider the following infinite series of orthogonal

$$\mathfrak{h}(z) = -\frac{2}{R}\log|z| \; + \; \frac{R^2-1}{R}\sum_{n=1}^{\infty}\frac{z^n \; - \; \bar{z}^{-n}}{n\,R^n}$$
$$= \; \big(R - \frac{1}{R}\big)\,\mathsf{Log}\frac{R\,z\bar{z} - z}{R - z} \; - \; 2R\log|z|$$

The series converges in the closed annulus $\mathbf{A} = \{z\,;\,R^{-1} \leqslant |z| \leqslant R\,\}$, except for two boundary points $z = R^{\pm 1}$. Her the symbol Log stands for the continuous branch of logarithm in $\mathbb{C}+ = \{\xi \in \mathbb{C}\,;\; \Re e\,\xi > 0\,\}$ that is specified by $\mathsf{Log}\,1 = 0$. Observe that the expression $\xi = \frac{R\,z\bar{z}-z}{R-z}$ takes values in $\mathbb{C}+$, whenever $R^{-1} \leqslant |z| \leqslant R$ and $z \neq R^{\pm 1}$. Clearly $\mathfrak{h}(z) = 0$ for $|z| = 1$ and we have the following identity $\mathfrak{h}(1/z) = -\mathfrak{h}(\bar{z})$. Elementary geometric considerations show that $\mathfrak{h}$ takes the open annulus $\mathbb{A}_R = \{z\,;1 < |z| < R\,\}$ homeomorphically into a simply connected domain with puncture at the origin which is contained in a horizontal strip

$$\mathfrak{h}(\mathbb{A}_R) \subset \left\{ \zeta\, ;\; |\Im m\, \zeta| < \frac{\pi}{2}\Big(R - \frac{1}{R}\Big) \right\}$$

Passing to the limit as $R \to \infty$, we obtain harmonic homeomorphism $\mathfrak{h}^\infty = z - \bar{z}^{-1}$ of $\mathbb{A}_\infty = \{z\,; 1 < |z| < \infty\}$ onto the punctured complex plane $\mathbb{C}_\circ$. Next observe that the function $\xi(z) = \frac{R\, z\bar{z} - z}{R - z}$ agrees with the Möbius transformation $\xi(z) = \frac{R\,\rho^2 - z}{R - z}$ when restricted to any circle $\mathcal{C}_\rho = \{z\,;\; |z| = \rho\}$. Thus the image of $\mathcal{C}_\rho$ under $\xi = \xi(z)$ is a circle in $\mathbb{C}+$. We now observe that Log takes circles in $\mathbb{C}+$ into strictly convex smooth Jordan curves. The curves $\mathfrak{h}(\mathcal{C}_\rho)\,, 1 < \rho < R$, resemble a family of ellipses with common focus at the origin. But the image of the outer circle, $\rho = R$, looks more like a parabola, but it has been flattened to fit into the horizontal strip.

Checking the Hopf-Laplace Equation The Hopf differential $\mathfrak{h}_z\overline{\mathfrak{h}_{\bar z}}\, \mathrm{d}z \otimes \mathrm{d}z$ is real and negative on every circle $\mathcal{C}_\rho\,,\quad R^{-1} < \rho < \mathbb{R}$. These circles are horizontal trajectories while rays are vertical trajectories. Precisely, we have

$$\mathfrak{h}_z\overline{h_{\bar z}} \;=\; \frac{1}{z^2}\,, \qquad \text{for all } \; R^{-1} \leqslant |z| \leqslant R\,, \;\text{ except for }\; z = R^{\pm 1}$$

Indeed, a straightforward differentiation shows that

$$\mathfrak{h}_z = \frac{Rz-1}{R-z}\,\frac{1}{z}, \quad \text{and} \quad \overline{\mathfrak{h}_{\bar z}} = \frac{R-z}{Rz-1}\,\frac{1}{z}\,,$$

whence the equation. The Jacobian determinant

$$J(z,\mathfrak{h}) = |\mathfrak{h}_z|^2 - |\mathfrak{h}_{\bar z}|^2 \;=\; \frac{(R^2-1)(|z|^2-1)}{|R-z|^2|Rz-1|^2}$$

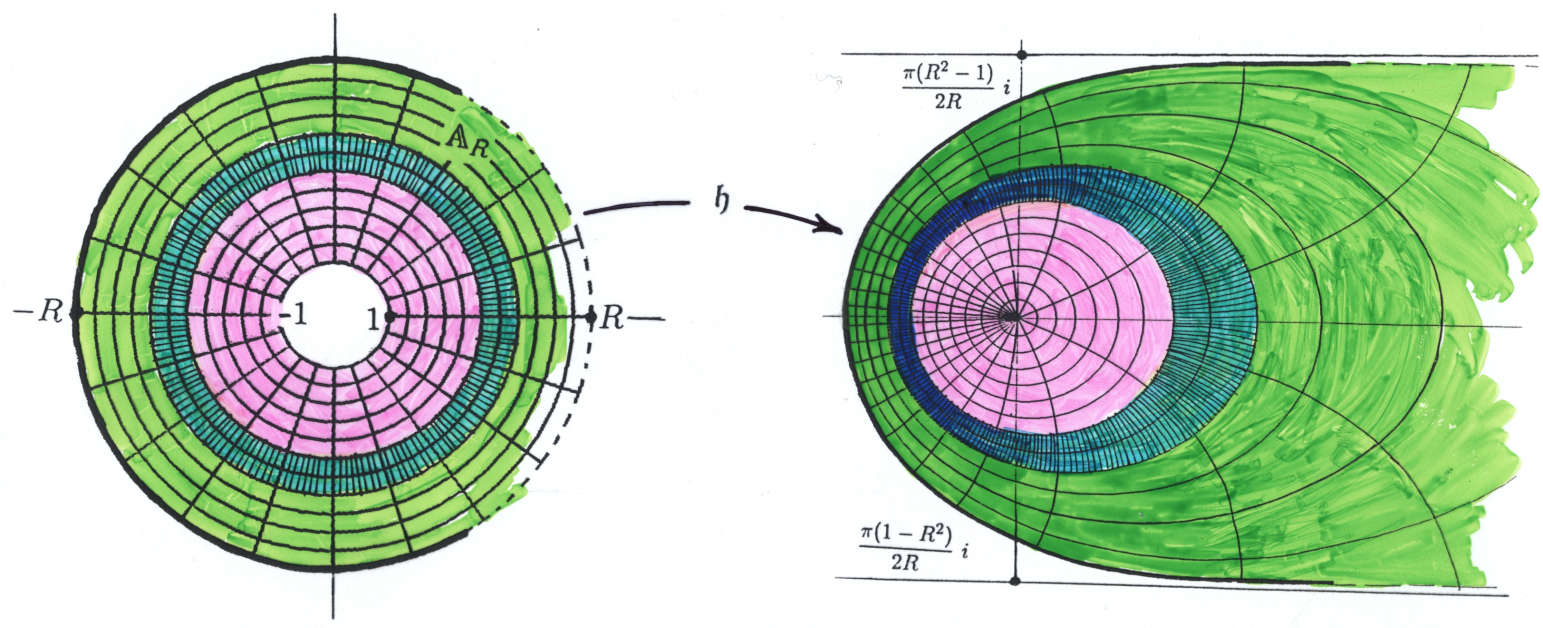
changes sign when crossing the unit circle.

In general, finding the energy-minimal homeomorphism between designated domains is not a trivial matter. Sometimes it comes unplanned, like in the above example, in which the Hopf equation yields:

PROPOSITION. Denote by $\mathcal{A}_\rho = \mathfrak{h}(\mathbb{A}_\rho)$, where $1 < \rho < R$. Among all homeomorphisms $f\colon \mathbb{A}_\rho \xrightarrow{\text{onto}} \mathcal{A}_\rho$ the minimum Dirichlet energy is attained for $f = \mathfrak{h}$, uniquely up to a rotation of $\mathbb{A}_\rho$. The minimum of energy equals

$$2 \iint_{\mathbb{A}_\rho} \left(|\mathfrak{h}_z|^2 + |\mathfrak{h}_{\bar z}|^2 \right) = \frac{4\,\pi \,\log\, \rho}{R^2} + \frac{2\,\pi (R^2-1)^2}{R^2} \log \frac{R^2\rho^2 - 1}{R^2\rho^2 - \rho^4}$$

REMARK. The mapping $\mathfrak{h}$ also represents the energy-minimal deformation of any sub-annulus $\mathbb{A}_{r_2} \setminus \mathbb{A}_{r_1}$ onto a doubly connected shell $\mathcal{A}_{r_2} \setminus \mathcal{A}_{r_1}$ where $1 < r_1 < r_2 < \mathbb{R}$. The energy $\mathscr{E}_{\mathbb{A}_{r_2} \setminus \mathbb{A}_{r_1}}[\mathfrak{h}] = \mathscr{E}_{\mathbb{A}_{r_2}}[\mathfrak{h}] - \mathscr{E}_{\mathbb{A}_{r_1}}[\mathfrak{h}]$.



CASE. $c < 0$, Hopf differentials are positive along $\partial\mathbb{X}$.
This is the case in which cracks may, though need not, emerge. The utility of this approach is illustrated by the quick proof of the following

THEOREM. Let $\mathbb{X} = \mathbb{A}(r, R)$ and $\mathbb{Y} = \mathbb{A}(r_*, R_*)$ be planar annuli.
<u>Case 1.</u> If

$$\frac{R_*}{r_*} \geqslant \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right)$$

then the harmonic homeomorphism

$$\mathfrak{h}(z) = \frac{r_*}{2}\left(\frac{z}{r} + \frac{r}{\bar{z}}\right), \quad \mathfrak{h} : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$$

has the smallest energy among all homeomorphisms $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, and is unique up to a rotation of $\mathbb{A}$.
<u>Case 2.</u> If

$$\frac{R_*}{r_*} < \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right)$$

then the infimum energy among all homeomorphisms $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is not attained. Let a radius $r < \sigma < R$ be determined by the equation

$$\frac{R_*}{r_*} = \frac{1}{2}\left(\frac{R}{\sigma} + \frac{\sigma}{R}\right)$$

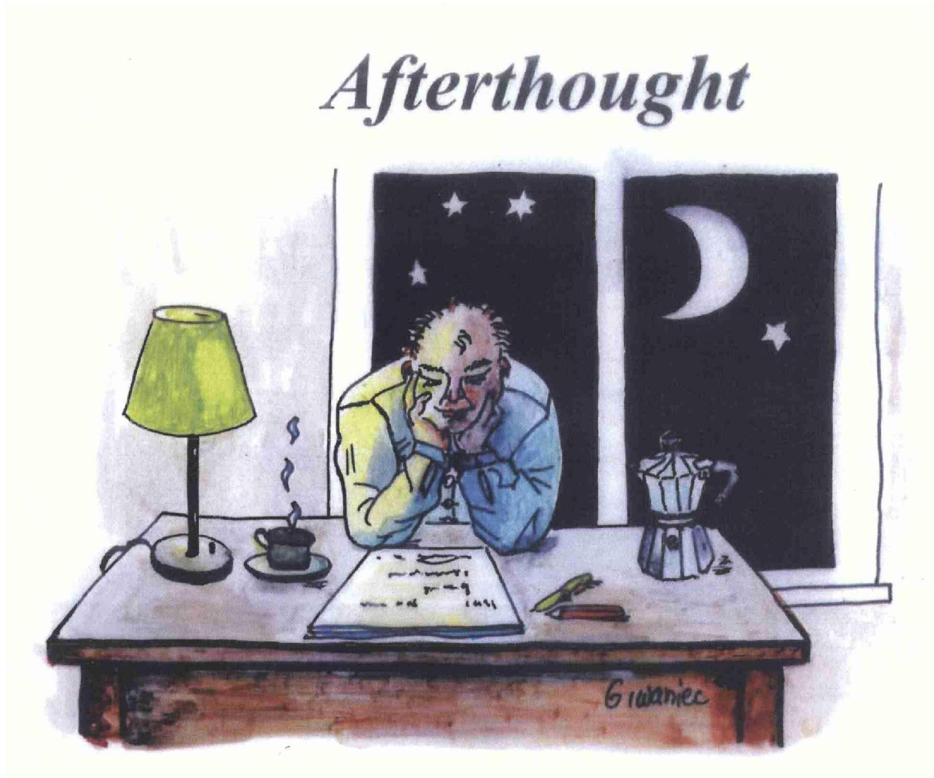
Then the following mapping

$$\mathfrak{h}(z) = \begin{cases} r_* \frac{z}{|z|} & r < |z| \leqslant \sigma \quad \textit{cracks along the rays } [r, \rho]\, e^{i\,\theta} \\ \\ \frac{r_*}{2}\left(\frac{z}{\sigma} + \frac{\sigma}{\bar{z}}\right) & \sigma \leqslant |z| < R \quad \textit{harmonic diffeomorphism} \end{cases}$$

has smallest energy within the class $\overline{\mathscr{H}}(\mathbb{X}, \mathbb{Y})$. This energy-minimal map is unique up to a rotation of $\mathbb{X}$.

PROOF. The proof is immediate from Theorem once we notice that

$$\mathfrak{h}_z\, \overline{\mathfrak{h}_{\bar{z}}}\ \mathrm{d}z \otimes \mathrm{d}z = -\frac{r_*^2}{4} \frac{\mathrm{d}z \otimes \mathrm{d}z}{z^2}\,, \quad \text{in either case}$$



A mediocre idea that generates enthusiasm will go further than a great idea that inspires no one."

- Quote on Enthusiasm by Mary Kay Ash quotes.