Lipschitz Regularity

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Inner Variation
(brief description)

We establish Lipschitz regularity of solutions of nonlinear first-order PDEs that arise from inner variation of numerous energy integrals for mappings $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ between two designated domains in $\mathbb{C}$. Even in the simplest model case of the Dirichlet energy the *inner-stationary solutions* need not be differentiable everywhere; the Lipschitz continuity is the best possible. But the proofs, even in the Dirichlet case, turn out to rely on topological arguments. The appeal to the inner-stationary solutions in this context is motivated by the classical problems in the theory of harmonic mappings and some hyperelastic material models; specifically, Neo-Hookean solids.

Dirichlet Integral
(brief description)

One enquires into deformations $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of smallest stored energy

$$\mathscr{D}_{\mathbb{X}}[h] = \iint_{\mathbb{X}} |Dh|^2 = 2 \iint_{\mathbb{X}} \left(|h_z|^2 + |h_{\bar z}|^2\right) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \ , \quad z = x_1 + i\, x_2$$

Hereafter $h_z = \frac{\partial h}{\partial z}$ and $h_{\bar z} = \frac{\partial h}{\partial \bar z}$ are complex partial derivatives of h. The first variation of $\mathscr{E}$ results in the Euler-Lagrange equation,

$$\Delta h = 4 h_{z \bar z} = 0$$

Hopf-Laplace Equation

In contrast, the inner variation leads to a nonlinear equation

$$\frac{\partial}{\partial \bar{z}}\left(h_z\,\overline{h_{\bar{z}}}\right) = 0, \quad \textbf{equivalently,} \quad h_z\,\overline{h_{\bar{z}}} = \phi, \quad (\phi \textbf{ is analytic})$$

for mapping in the Sobolev space $\mathscr{W}^{1,2}_{\text{loc}}(\mathbb{X})$. This equation will henceforth be referred to as the *Hopf-Laplace equation*, also known as *energy-momentum* or *equilibrium* equation.

THEOREM. ***Every homeomorphism $h \in \mathscr{W}^{1,2}_{\text{loc}}(\mathbb{X})$ which satisfies the Hopf-Laplace equation is in fact a harmonic diffeomorphism.***

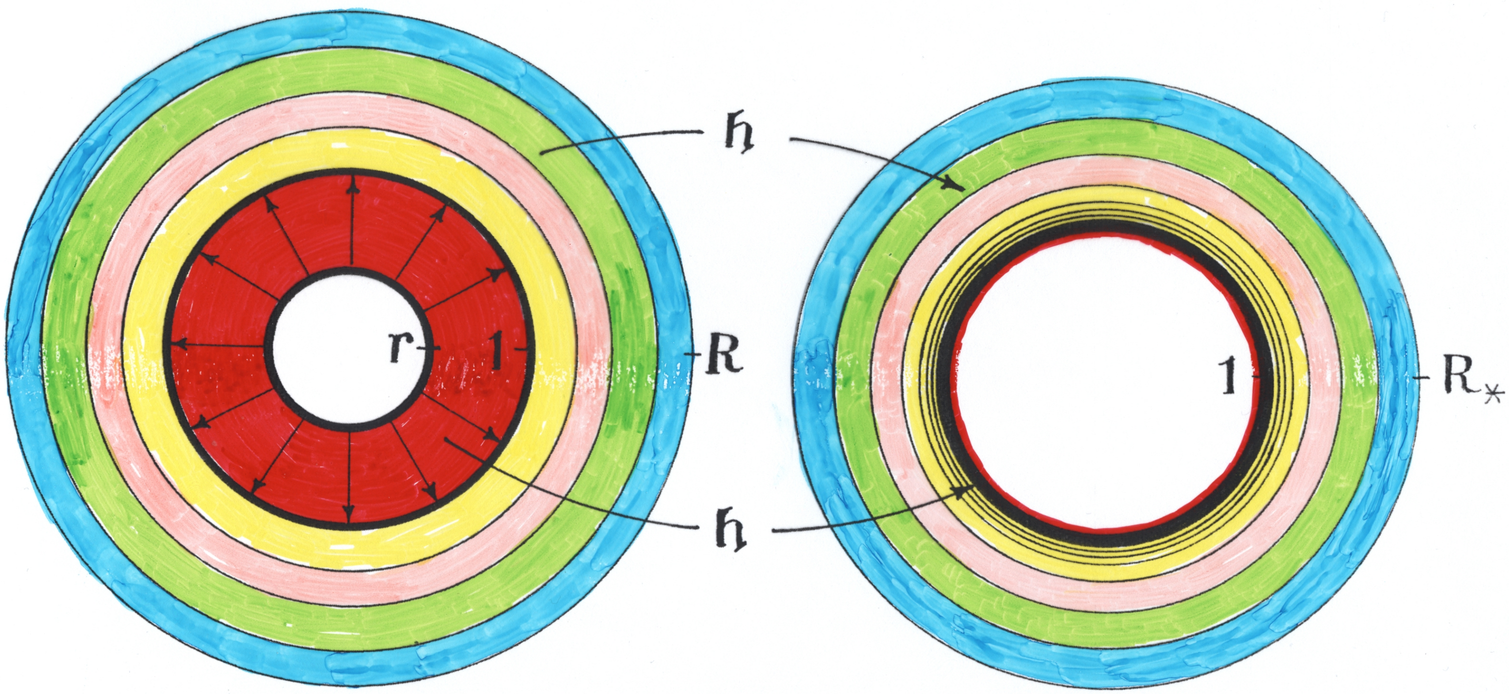
Nonharmonic Energy-Minimal Solutions

Consider two annuli $\mathbb{X} = \{z\colon\ r < |z| < R\}$ and $\mathbb{Y} = \{w\colon\ 1 < |w| < R_*\}$ where $0 < r < 1 < R < \infty$ and $R_* = \frac{1}{2}(R + R^{-1})$. The map

$$h(z) = \begin{cases} \frac{z}{|z|}\,, & \text{if} \quad r < |z| \leqslant 1\,, \quad \text{squeezing to the unit cirle} \\ \frac{1}{2}(z + \frac{1}{\bar{z}})\,, & \text{if} \quad 1 \leqslant |z| < R\,, \quad \text{the Nitsche harmonic map} \end{cases}$$

takes $\mathbb{X}$ onto $\mathbb{Y} \cup \partial_\circ \mathbb{Y}$. It satisfies the Hopf-Laplace equation

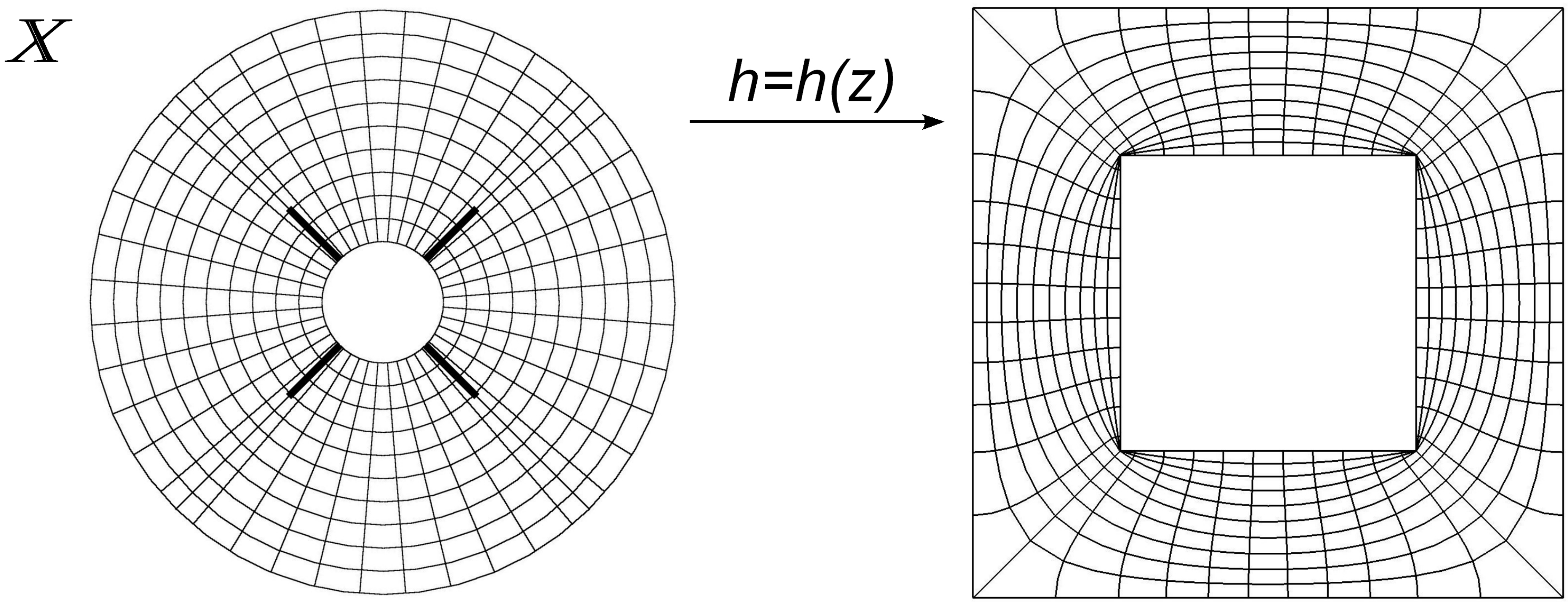
$$h_z\,\overline{h_{\bar{z}}} = \varphi(z) = \frac{-1}{4z^2}$$



$$h(z) = \begin{cases} \frac{z}{|z|}\,, & \text{if} \quad r < |z| \leqslant 1\,, \\ \frac{1}{2}\left(z + \frac{1}{\bar{z}}\right), & \text{if} \quad 1 \leqslant |z| < R\,, \end{cases}$$

hammering into the unit circle

the Nitsche harmonic map



The energy minimal map is locally Lipschitz in the entire annulus $\mathbb{X}$, including cuts that are mapped into corners of the square hole (concave part of $\partial\mathbb{Y}$)

Lipschitz Regularity for the Hopf-Laplace Equation

THEOREM
Every $\mathscr{W}^{1,2}_{\text{loc}}(\mathbb{X})$ -solution to the Hopf-Laplace equation with nonnegative Jacobian is locally Lipschitz but not necessarily $\mathscr{C}^1$-smooth.

This is a corollary from *Cristina, Kovalev, Onninen, T.I.* arXiv:1011.5934. For more general Lipschitz continuity results for solutions to inner variational equations see (*Kovalev, Onninen, T.I.* arXiv:1109.0720), to appear in Duke Mathematical Journal.

Failure of $\mathscr{C}^1$ -Regularity

We use the polar coordinates for z in the closed unit disk $\overline{\mathbb{D}}$, $z = \rho e^{i\theta}$, $0 \leqslant \rho \leqslant 1$ and $0 \leqslant \theta < 2\pi$. Define a continuous map $h \colon \overline{\mathbb{D}} \to \mathbb{C}$

$$h(\rho e^{i\theta}) = 2\rho \left[\sqrt{\rho} \sin(3/2\, \theta) + i \sin \theta\right] = z - \bar{z} - i \left[z^{3/2} - \bar{z}^{3/2}\right].$$

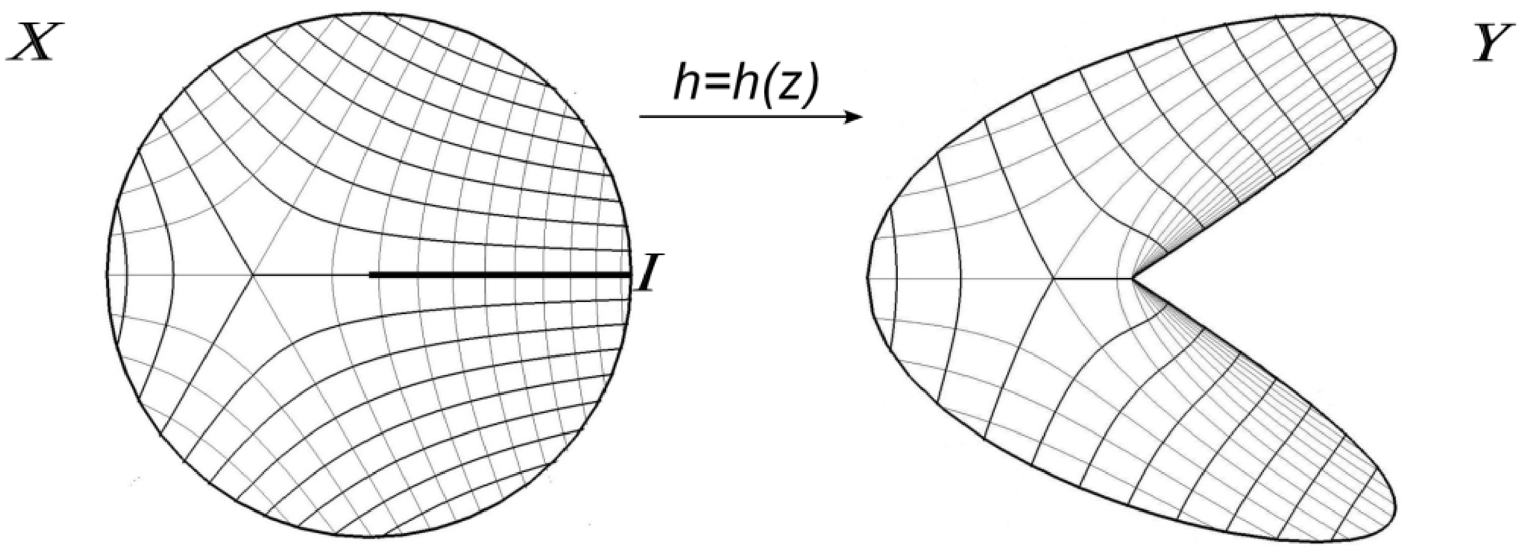
This mapping is Lipschitz continuous, since it has bounded derivatives

$$h_z = 1 - 3/2\, i \sqrt{z}, \qquad h_{\bar{z}} = -1 + 3/2\, i \sqrt{\bar{z}}.$$

Moreover, its Hopf differential is holomorphic, $h_z \overline{h_{\bar{z}}} = -1/4\, (4 + 9z)$. Thus h solves the Hopf-Laplace equation $\frac{\partial}{\partial \bar{z}} \left(h_z \overline{h_{\bar{z}}}\right) = 0$.

However h fails to be $\mathscr{C}^1$-smooth in any neighborhood of the ray $\mathbf{I} = \{z \colon \operatorname{Im} z = 0 \text{ and } 0 \leqslant \operatorname{Re} z \leqslant 1\}$.

Topologically, h is a harmonic diffeomorphism of $\mathbb{D} \setminus \mathbf{I}$ onto the butterfly domain $\mathbb{Y} \subset \mathbb{C}$. The Figure shows the grid of horizontal and vertical trajectories in $\mathbb{X}$ as well as their images in $\mathbb{Y}$.



The radius $\mathbf{I}$ is squeezed into the origin. Observe that the origin is a point in $\partial\mathbb{Y}$ where $\mathbb{Y}$ fails to be convex.

The Hopf Product $h_z \overline{h_{\bar z}} \in \mathscr{C}^\alpha(\mathbb{X})$

THEOREM. Let $h \in \mathscr{W}^{1,2}_{\text{loc}}(\mathbb{X})$ be a mapping with nonnegative Jacobian. Suppose that the Hopf product $h_z \overline{h_{\bar z}}$ is bounded and Hölder continuous. Then h is locally Lipschitz.

Hölder continuity of $\phi = h_z \overline{h_{\bar z}}$ cannot be relaxed to continuity.

EXAMPLE. Let $h(z) = z \log\log|z|^{-2}$, for $|z| < 1/2$. This mapping is an orientation preserving homeomorphism which belongs to $\mathscr{W}^{1,p}$ for all $p < \infty$. We compute

$$h_z = \log\log\frac{1}{|z|^2} - \log^{-1}\frac{1}{|z|^2} \qquad \text{and} \qquad h_{\bar z} = \frac{z}{\bar z}\log^{-1}\frac{1}{|z|^2}.$$

Clearly, $\phi = h_z\overline{h_{\bar z}}$ is continuous. However, h is not Lipschitz.

The Inner-Variational Equations

Let us consider the energy integral for mappings $h : \mathbb{X} \to \mathbb{C}$

$$\mathscr{E}[h] = \iint_{\mathbb{X}} \mathbf{E}(z, h, h_z, h_{\bar z})\, \mathrm{d}x_1\, \mathrm{d}x_2 \;, \qquad z = x_1 + ix_2$$

where $\mathbf{E} = \mathbf{E}(z, w, \xi, \zeta)$ is a given *stored-energy function.*

Given any test function $\eta \in \mathscr{C}_0^\infty(\mathbb{X})$ and a complex parameter t, small enough so that the map $z \mapsto z + t\,\eta(z)$ represents a diffeomorphism of $\mathbb{X}$ onto itself, consider the inner variation $h^t(z) = h(z + t\eta)$ and its energy

$$\mathscr{E}[h^t] = \iint_{\mathbb{X}} \mathbf{E}(z, h^t, h^t_z, h^t_{\bar z})\, \mathrm{d}x_1\, \mathrm{d}x_2 \;, \qquad z = x_1 + ix_2$$

First we make a substitution $w = z + t\,\eta(z)$ and then differentiate to obtain an integral form of the equilibrium equation $\frac{\partial}{\partial t}\big|_{t=0}\mathscr{E}[h^t] = 0$. We eliminate η through integration by parts to arrive at what is called the *inner-variational equation*

$$\frac{\partial}{\partial \bar z}\Big[h_z\,\mathbf{E}_\zeta \;+\; \overline{h_{\bar z}}\,\mathbf{E}_{\bar\xi}\Big] \;+\; \frac{\partial}{\partial z}\Big[h_z\,\mathbf{E}_\xi \;+\; \overline{h_{\bar z}}\,\mathbf{E}_{\bar\zeta} \;-\; \mathbf{E}\Big] \;+\; \mathbf{E}_z \;=0$$

Hereafter the subscripts under $\mathbf{E}$ stand for complex partial derivatives of $\mathbf{E} = \mathbf{E}(z, w, \xi, \zeta)$. The partial derivatives $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar z}$ are understood in the sense of distributions. The most basic example is the Dirichlet integrand $\mathbf{E} = |\xi|^2 + |\zeta|^2$ and the associated Hopf-Laplace equation

$$\frac{\partial}{\partial \bar z}\left(h_z\,\overline{h_{\bar z}}\,\right) = 0\;, \quad \text{for } \; h \in \mathscr{W}^{1,2}_{\mathsf{loc}}(\mathbb{X})$$

Poincaré Disk

Let the target be the Poincaré disk $\mathbb{D} = \{ w \in \mathbb{C} \colon |w| < 1 \}$ equipped with the hyperbolic metric $\mathrm{d}s = \frac{|\mathrm{d}w|}{1-|w|^2}$. The associated Dirichlet integral

$$\mathscr{E}[h] = \iint_{\mathbb{X}} \frac{|h_z|^2 + |h_{\bar z}|^2}{(1-|h|^2)^2} \, \mathrm{d}x_1 \, \mathrm{d}x_2$$

is certainly infinite for homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{D}$ in the Sobolev space $\mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{X}, \mathbb{D})$. Nonetheless, it is interesting to examine the inner-variational equation and all its solutions, not necessarily homeomorphisms.

$$\frac{\partial}{\partial \bar z} \frac{h_z \, \overline{h_{\bar z}}}{(1-|h|^2)^2} = 0 \, , \quad \text{for } \; h \in \mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{X}, \mathbb{D})$$

Weighted Dirichlet integral

$$\mathscr{E}[h] = \iint_{\mathbb{X}} \left(|h_z|^2 + |h_{\bar z}|^2 \right) \rho(z,h)\, \mathrm{d}x_1\, \mathrm{d}x_2$$

and its inner-variational equation

$$\frac{\partial}{\partial \bar z} \left[\rho(z,h)\, h_z\, \overline{h_{\bar z}} \right] \;=\; \rho_z(z,h) \left(|h_z|^2 + |h_{\bar z}|^2 \right) , \quad \text{for } h \in \mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{X}, \mathbb{D})$$

THEOREM. Suppose $\rho = \rho(z,w) \geqslant 1$ is Lipschitz continuous in the z-variable and Hölder continuous in the w-variable. If $h \in \mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{X}, \mathbb{D})$ is a solution of (??) with nonnegative Jacobian, then h is locally Lipschitz continuous.

Proof of the weighted case

We are dealing with a nonhomogeneous Cauchy-Riemann equation

$$\frac{\partial U}{\partial \bar z} = u\,, \quad \textbf{where} \quad U = \rho(z,h)\, h_z \overline{h_{\bar z}}\;, \quad u = \left(|h_z|^2 + |h_{\bar z}|^2\right) \rho_z(z,h)$$

At the beginning we only know that $U, u \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{X})$**. We shall recurrently improve integrability properties of these terms. First observe that** U**, having** $\frac{\partial}{\partial \bar z}$**-derivative in** $\mathscr{L}^1_{\mathrm{loc}}(\mathbb{X})$**, lies in** $\mathscr{L}^s_{\mathrm{loc}}(\mathbb{X})$ **for every exponent** $1 < s < 2$**. Then, in view of pointwise inequality** $|h_{\bar z}|^2 \leqslant \rho(z,h)\, |h_z||h_{\bar z}| = |U|$**, we see that** $|h_{\bar z}|^2 \in \mathscr{L}^s_{\mathrm{loc}}(\mathbb{X})$**. This implies that also** $|h_z|^2 \in \mathscr{L}^s_{\mathrm{loc}}(\mathbb{X})$**. In this way we gain higher integrability of the right hand side of (??); namely,** $u = \left(|h_z|^2 + |h_{\bar z}|^2\right) \rho_z(z,h) \in \mathscr{L}^s_{\mathrm{loc}}(\mathbb{X})$**, because** $\rho_z(z,h)$ **is bounded. Now equation (??) places**

U **in the space** $\mathscr{L}^{\frac{2s}{2-s}}_{\text{loc}}(\mathbb{X})$**. This, in view of** $|h_{\bar z}|^2 \leqslant |U|$**, yields** $|h_{\bar z}|^2 \in \mathscr{L}^{\frac{2s}{2-s}}_{\text{loc}}(\mathbb{X})$**;** $|h_z|^2 \in \mathscr{L}^{\frac{2s}{2-s}}_{\text{loc}}(\mathbb{X})$ **as well. Thus we gained even more integrability of** u**;** $u \in \mathscr{L}^p_{\text{loc}}(\mathbb{X})$**, with** $p = \frac{2s}{2-s} > 2$**. We again appeal to equation (??). This time the equation yields Hölder continuity of** U**; precisely,** $U \in \mathscr{C}^\alpha_{\text{loc}}(\mathbb{X})$ **with** $\alpha = 1 - \frac{2}{p} > 0$**. Let us write the equation as**

$$h_z \, \overline{h_{\bar z}} = \frac{\psi(z)}{\rho(z,h)}\,, \quad \textbf{where} \quad \psi \in \mathscr{C}^\alpha_{\text{loc}}(\mathbb{X})$$

We observe that h **is also locally Hölder continuous, because** $h_{\bar z} \in \mathscr{L}^{2p}_{\text{loc}}(\mathbb{X})$ **with exponent** $2p > 2$**. The conclusion is that the** ***Hopf product*** $h_z \, \overline{h_{\bar z}}$ **is a Hölder continuous function. Thus** h **is locally Lipschitz.**

The General Setting

Let $\mathcal{H} = \mathcal{H}(z,\xi)$ be a continuous function in $\mathbb{X} \times \{\xi \colon R < |\xi| \leqslant \infty\}$, where $0 \leqslant R < \infty$. there is a constant $0 \leqslant L < \infty$ such that for every $z \in \mathbb{X}$ it holds:

$$|\mathcal{H}(z,\xi_1) - \mathcal{H}(z,\xi_2)| \leqslant L \cdot \left|\frac{1}{\xi_1} - \frac{1}{\xi_2}\right| , \text{ for } R < |\xi_1| \leqslant |\xi_2| \leqslant \infty$$

$$\sup_{z\in\mathbb{X}} |\mathcal{H}(z,\xi)| \;+\; \sup_{z_1\neq z_2} \frac{|\mathcal{H}(z_1,\xi) - \mathcal{H}(z_2,\xi)|}{|z_1 - z_2|^\alpha} \leqslant M\,, \quad \text{for } z_1\,,\, z_2 \in \mathbb{X}$$

The Main Result

DEFINITION A function $h \in \mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{X})$ is said to be a solution to the *generalized Hopf-Laplace* equation $h_{\bar z} = \mathcal{H}(z, h_z)$ if it holds for almost every point $z \in \mathbb{X}$, whenever $|h_z(z)| > R$.

Note we impose no condition at the points where $|h_z(z)| \leqslant R$. *At such points the gradient of* h *is bounded,* $|h_{\bar z}| \leqslant |h_z| \leqslant R$.

THEOREM. Suppose the equation $h_{\bar z} = \mathcal{H}(z, h_z)$ satisfies the conditions (??) and (??). Then every solution $h \in \mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{X})$ with nonnegative Jacobian is locally Lipschitz continuous. Specific gradient estimates near $\partial\mathbb{X}$, are also available.

We will construct a continuous family $\{F^\lambda\}_{\lambda\in\mathbb{C}}$ of so-called "good" solutions of $F^\lambda_{\bar{z}} = \mathcal{H}(z, F^\lambda_z)$ such that all mappings $g(z) = F^\lambda - h$, with $|\lambda| \geqslant \lambda_\circ$, satisfy (point-wise) the distortion inequality $|g_{\bar{z}}| \leqslant k|g_z|$, meaning that g is quasiregular .
After that we appeal to the topology of quasiregular mappings.

The interested reader is referred to recent papers by D. Faraco, B. Kirchheim and L. Székelyhidi which also combine the theory of quasiregular mappings with topological arguments.

An Application to Nonlinear Elasticity

In nonlinear elasticity of isotropic materials one considers the energy of h of the form

$$\mathscr{E}[h] = \iint_{\mathbb{X}} W\big(z, h, |h_z|^2, |h_{\bar z}|^2\big)$$

Specifically, neo-Hookean models of elasticity deal with the integrands W which blow up as the Jacobian determinant approaches zero. To emphasize a possible neo-Hookean character of the integrand we bring to the stage the following integral,

$$\mathscr{E}_{\mathbb{X}}[h] = \iint_{\mathbb{X}} \frac{|Dh(z)|^{2p}}{J(z,h)^{\,p-1}} \, \mathrm{d}z \; = \; \iint_{\mathbb{X}} \frac{\big(|h_z|^2 + |h_{\bar z}|^2\big)^p}{(|h_z|^2 - |h_{\bar z}|^2)^{p-1}}, \quad p \geqslant 1$$

subject to homeomorphisms $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in the Sobolev space $\mathscr{W}^{1,2}(\mathbb{X})$.

THEOREM.

Let $h \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{X})$ be an inner-stationary mapping for the above energy integral $\mathscr{E}_{\mathbb{X}}[h] < \infty$. Then h is locally Lipschitz continuous. Furthermore the stored energy function $\mathbf{E}(Dh)$ is locally bounded.

This integral gains additional interest in Geometric Function Theory because the transition to the energy of the inverse mapping $f = h^{-1} \colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ results in the $\mathscr{L}^p$-norm of the distortion function.

$\mathscr{L}^p$-norm of the distortion function

$$\mathscr{E}_{\mathbb{Y}}[f] = \iint_{\mathbb{Y}} K_f(w)^p \, \mathrm{d}\, w, \quad K_f(w) = \frac{|Df(w)|^2}{J_f(w)} \geqslant 1\,, \quad J_f(w) = \frac{1}{J(z,h)}$$

We see that conformal mappings, for which $K_f \equiv 1$, are the absolute minimizers. In general, $\mathscr{L}^p$-integrability of the distortion function only guarantees that $f \in \mathscr{W}^{1,\frac{2p}{p+1}}(\mathbb{Y})$. Indeed,

$$\iint_{\mathbb{Y}} |Df|^{\frac{2p}{p+1}} = \iint_{\mathbb{Y}} K_f^{\frac{p}{p+1}} J_f^{\frac{p}{p+1}} \leqslant \left(\iint_{\mathbb{Y}} K_f^p \right)^{\frac{1}{p+1}} \left(\iint_{\mathbb{Y}} J_f \right)^{\frac{p}{p+1}}$$
$$= \|K_f\|_{\mathscr{L}^p(\mathbb{Y})}^{\frac{p}{p+1}} \cdot |\mathbb{X}|^{\frac{p}{p+1}} < \infty$$

Will explain Lipschitz regularity for Coffee

The Proof of Lipschitz Regularity in the Model Case : Hopf-Laplace Equation

THEOREM. Suppose that the Hopf product $h_z \, \overline{h_{\bar z}} = \phi(z)$ is analytic and bounded in a domain $\mathbb{X} \subset \mathbb{C}$, for some $h \in \mathscr{W}^{1,2}(\mathbb{X}) \cap \mathscr{L}^\infty(\mathbb{X})$ with nonnegative Jacobian. Then h is locally Lipschitz. Moreover, for almost every $z \in \Omega$ we have

$$|\nabla h(z)| \leqslant \frac{13 \ \operatorname{osc}_{\mathbb{X}}[h]}{\textsf{dist}(z, \partial \mathbb{X})} \ + \ 5 \, \|\phi\|^{1/2}_{\mathscr{L}^\infty(\mathbb{X})}$$

Note that $|h_{\bar z}|^2 \leqslant |h_z \, h_{\bar z}| \leqslant |\phi| \in \mathscr{L}^\infty(\mathbb{X})$. But this is only good enough to infer that $Dh \in \mathbf{BMO}_{\textsf{loc}}(\mathbb{X})$; ***the inclusion*** $h_z \in \mathscr{L}^\infty_{\textsf{loc}}(\mathbb{X})$ ***is the true challenge.***

Step 1. Good Family of Solutions

Finding good solutions to the equation $h_z \overline{h_{\bar{z}}} = \phi(z)$ in which ϕ is analytic presents no difficulty. First consider $\mathbb{X} = \mathbb{D}$ -the unit disk, and assume that ϕ is bounded and analytic in $\mathbb{D}$. Denote by $\Phi = \Phi(z)$ its anti-derivative such that $\Phi(0) = 0$. Thus $\Phi_{\bar{z}} = 0$ and $\Phi_z = \phi$. Clearly, Φ extends continuously to the closed unit disk $\mathbf{D} = \overline{\mathbb{D}}$. The mappings $F^\lambda(z) = \lambda z + f^\lambda(z)$, where $f^\lambda(z) = \overline{\lambda^{-1}\Phi(z)}$ with complex parameter $\lambda \neq 0$, solve the same equation $F_z^\lambda \overline{F_{\bar{z}}^\lambda} = \phi(z)$. Also note that $\|f^\lambda\|_\infty \leqslant |\lambda|^{-1}\|\phi\|_\infty$.

A short computation reveals that the difference $g = g^\lambda(z) = F^\lambda(z) - h(z)$ is a $\mathscr{W}^{1,2}(\mathbb{D})$-solution to a linear elliptic equation

Step . A Family of Quasiregular Mappings

$g = g^{\lambda}(z) = F^{\lambda}(z) - h(z)$

$$g_{\bar{z}}(z) = \nu(z)\, \overline{g_z(z)}\,, \quad \nu(z) = \frac{-h_{\bar{z}}(z)}{\bar{\lambda}}\,, \quad |\nu(z)| \leqslant \frac{1}{2}$$

whenever $|\lambda| \geqslant 2\,\|h_{\bar{z}}\|_{\infty}$. Now consider a continuous family of mappings $G^{\lambda}(z) = \frac{1}{\lambda}\, g^{\lambda}(z) = z + \frac{1}{\lambda}\big[f^{\lambda}(z) - h(z)\big]$. We have

$$|\, G^{\lambda}(z) - z\,| \leqslant \frac{\|\phi\|_{\infty}}{|\lambda|^2} + \frac{\|h\|_{\infty}}{|\lambda|} < \frac{1}{3}$$

provided $|\lambda| \geqslant 2\sqrt{\|\phi\|_{\infty}}$ and $|\lambda| \geqslant 13\,\|h\|_{\infty}$. This shows, in particular, that G^{λ} is a nonconstant quasiregular mapping, thus orientation-preserving, open and discrete. At this point we appeal to a Rouché type lemma.

Step 3. Rouche's Lemma

Let $G = G^\lambda(z)$ be a continuous family of mappings $G^\lambda : \mathbf{D} = \overline{\mathbb{D}} \to \mathbb{C}$ parametrized by complex numbers λ with $\varrho \leqslant |\lambda| \leqslant \infty$, such that
(i) $G^\infty(z) \equiv z$
(ii) $|G^\lambda(z) - z| < \frac{1}{3}$, for $z \in \mathbf{D}$ and $|\lambda| \geqslant \varrho$
(iii) For every $|\lambda| \geqslant \varrho$ the map $G^\lambda : \mathbb{D} \to \mathbb{C}$ is orientation preserving open and discrete.
Then, given any $z_\circ \in \frac{1}{3}\mathbf{D}$ and parameter $|\lambda| \geqslant \varrho$, the equation

$$G^\lambda(z) = G^\lambda(z_\circ) \,, \quad \text{for } z \in \mathbf{D} \,;$$

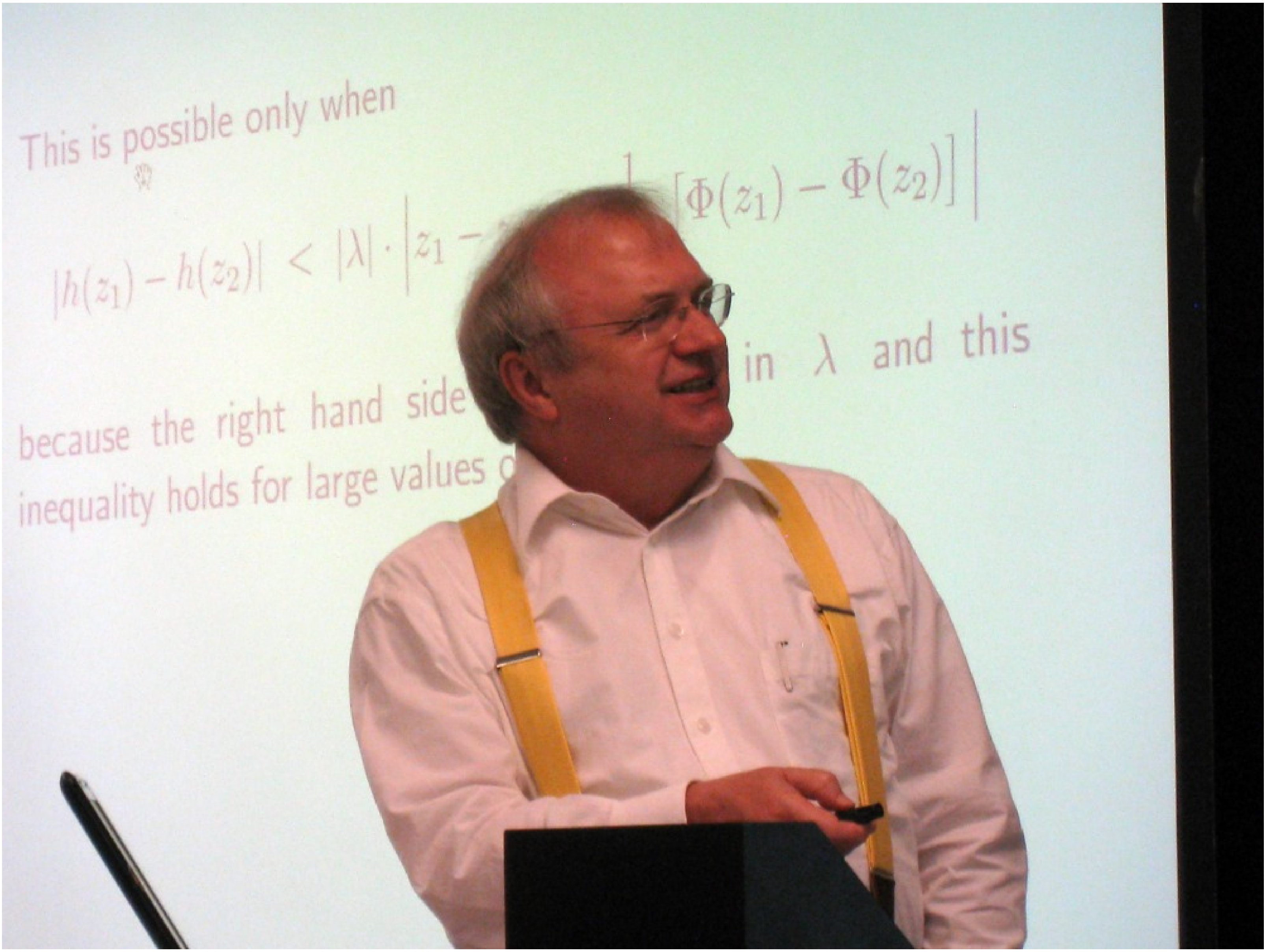
admits exactly one solution $z = z_\circ$.

Step 4. Injectivity of g^λ

We infer that the mappings $G^\lambda(z) = \frac{1}{\lambda} g^\lambda(z)$ are injective in the disk $\frac{1}{3}\mathbf{D}$. So are the mappings $g^\lambda(z) = \lambda z + \overline{\lambda^{-1}\Phi(z)} - h(z)$. This reads as follows:
$h(z_1) - h(z_2) \;\neq\; \lambda \cdot \left\{ z_1 - z_2 \;+\; \frac{1}{|\lambda|^2}\left[\Phi(z_1) - \Phi(z_2)\right] \right\}$
for $z_1 \neq z_2$ in the disk $\frac{1}{3}\mathbf{D}$. Letting λ run over a circle of radius $|\lambda|$ we conclude that
$|h(z_1) - h(z_2)| \;\neq\; |\lambda| \cdot \left| z_1 - z_2 \;+\; \frac{1}{|\lambda|^2}\left[\Phi(z_1) - \Phi(z_2)\right] \right|$
This is possible only when

$$|h(z_1) - h(z_2)| \;<\; |\lambda| \cdot \left| z_1 - z_2 \;+\; \frac{1}{|\lambda|^2}\left[\Phi(z_1) - \Phi(z_2)\right] \right|$$

because the right hand side is continuous in λ and the inequality (**??**) holds for large values of $|\lambda|$.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Step 5. Conclusion

A conclusion is immediate;

$$\|\nabla h\|_{\mathscr{L}^\infty(\frac{1}{3}\mathbf{D})} \leqslant |\lambda| + \frac{1}{|\lambda|}\|\phi\|_{\mathscr{L}^\infty(\mathbf{D})}$$

All the conditions we have encountered for the parameter λ are satisfied if we set

$$|\lambda| = \max \begin{cases} 2\,\|h_{\bar z}\|_{\mathscr{L}^\infty(\mathbf{D})} \\ 2\,\|\phi\|^{1/2}_{\mathscr{L}^\infty(\mathbf{D})} \\ 13\,\|h\|_{\mathscr{L}^\infty(\mathbf{D})} \end{cases}$$

Therefore,

$$\|\nabla h\|_{\mathscr{L}^\infty(\frac{1}{3}\mathbf{D})} \leqslant 2\,\|h_{\bar z}\|_{\mathscr{L}^\infty(\mathbf{D})} + 13\,\|h\|_{\mathscr{L}^\infty(\mathbf{D})} + 3\,\|\phi\|^{1/2}_{\mathscr{L}^\infty(\mathbf{D})}$$

General Equation

Step 1. Good Family of Solutions

We are looking for a family $\{F^\lambda\}_{|\lambda|\geqslant\lambda_\circ}$, $F^\lambda(z) = \lambda z + f^\lambda(z)$, of "good" solutions to the equation $F^\lambda_{\bar z} = \mathcal{H}(z, F^\lambda_z)$. Equivalently,

$$f^\lambda_{\bar z} = \mathcal{H}(z, \lambda + f^\lambda_z)$$

in the closed unit disk $\mathbb{X} = \mathbf{D} = \{z \colon |z| \leqslant 1\}$. The good solutions $\{f^\lambda\}_{|\lambda|\geqslant\lambda_\circ}$ are obtained by fixed point method. In fact we extend the equation to the entire complex plane $\mathbb{C}$. Then the problem reduces to a singular integral equation for the function $\omega = f^\lambda_{\bar z}$, which is found uniquely in the Besov space $\mathscr{B}^p_\alpha(\mathbb{C}) \subset \mathscr{L}^\infty(\mathbb{C})$, $p = 3/\alpha > 3$.

$$\|\,\omega\,\|_{\alpha,p} \;:=\; \|\omega\|_p \,+\, \sup_{\tau\neq 0} \frac{\|\,\omega(\cdot+\tau)-\omega(\cdot)\|_p}{|\tau|^\alpha} \;<\; \infty$$

PROPOSITION.

There is $\lambda_\circ = \lambda_\circ(\mathcal{H})$ and a family $\{f^\lambda\}_{|\lambda| \geqslant \lambda_\circ}$ of solutions in $\mathbb{D}$ such that

$$f^\lambda(0) = 0 \tag{1}$$

$$|f^\lambda(z_1) - f^\lambda(z_2)| \leqslant \lambda_\circ \cdot |z_1 - z_2| \tag{2}$$

$$|f^{\lambda_1}(z) - f^{\lambda_2}(z)| \leqslant \lambda_\circ \cdot \left| \frac{\lambda_1 - \lambda_2}{\lambda_1 \cdot \lambda_2} \right| \tag{3}$$

We have the family $\{F^\lambda\}_{|\lambda| \geqslant \lambda_\circ}$, $F^\lambda(z) = \lambda z + f^\lambda(z)$ of "good" solutions.

For sufficiently large $|\lambda|$, say $|\lambda| > \sigma$, all $g^\lambda = F^\lambda - h$ are nonconstant K -quasiregular mappings, hence open and discrete.

Step. 2 The Difference $g^\lambda = F^\lambda - h$

We apply Rouche's Lemma to the family

$$\begin{cases} G^\lambda(z) = \frac{1}{\lambda}\, g^\lambda(z) = z + \frac{1}{\lambda}\left[f^\lambda(z) \; - \; h(z)\right], \text{ for } z \in \mathbf{D} \text{ and } |\lambda| \geqslant \sigma \\ G^\infty(z) \equiv z \end{cases}$$

to conclude that $G^\lambda(z_1) \neq G^\lambda(z_2)$, whenever z_1 and z_2 are distinct points in $\frac{1}{3}\mathbf{D}$ and $|\lambda| \geqslant \sigma$. This reads as follows

COROLLARY. For all complex parameters λ with $|\lambda| \geqslant \sigma$ the mappings $g^\lambda(z) = \lambda z + f^\lambda(z) - h(z)$ are injective in the disk $\frac{1}{3}\mathbf{D}$; that is, for $z_1 \neq z_2$ in $\frac{1}{3}\mathbf{D}$

$$h(z_1) - h(z_2) \;\neq\; \lambda(z_1 - z_2) + f^\lambda(z_1) - f^\lambda(z_2)$$

Step. 3 A Lipschitz Bound

We shall infer from this, using topological degree arguments, the following inequality

LEMMA. For every circle $\mathbb{T}_\rho = \{\lambda\colon |\lambda| = \rho\}$ with $\rho \geqslant \sigma$ there exists $\lambda \in \mathbb{T}_\rho$ such that

$$|h(z_1) - h(z_2)| \leqslant |\lambda(z_1 - z_2) + f^\lambda(z_1) - f^\lambda(z_2)|$$

We invoke this inequality with $\rho = \sigma$ to conclude with the desired Lipschitz bound

$$|h(z_1) - h(z_2)| \leqslant t|a| \leqslant \sigma\, |z_1 - z_2| + |f^{\,\rho\, e^{i\theta}}(z_1) - f^{\,\rho\, e^{i\theta}}(z_2)| \leqslant (\sigma + \lambda_\circ)|z_1 - z_2|$$

Proof of the Lemma (optional)

This inequality certainly holds for large values of ρ. To simplify writing we denote $a = h(z_1) - h(z_2)$ and assume, as we may, that $a \neq 0$. We shall consider a family of mappings $\Phi_\rho^a : \mathbb{T} \to \mathbb{T}$, with parameter $\rho \geqslant \sigma$, given by

$$\Phi_\rho^a(e^{i\theta}) = \frac{F(\rho\, e^{i\theta}) - a}{|F(\rho\, e^{i\theta}) - a|}, \quad \text{where } F(\lambda) = \lambda \cdot (z_1 - z_2) + f^\lambda(z_1) - f^\lambda(z_2)$$

By virtue of the inequalities (**??**), each such mapping has well defined degree, denoted by $\deg \Phi_\rho^a$, also known as winding number. Letting the parameter ρ vary we obtain an integer-valued continuous function in ρ, thus constant. We identify this constant by letting $\rho \to \infty$. The mappings converge uniformly to $\Phi_\infty^a : \mathbb{T} \to \mathbb{T}$, where $\Phi_\infty^a(e^{i\theta}) := \frac{z_1 - z_2}{|z_1 - z_2|} \cdot e^{i\theta}$. The degree of this limit map is equal to 1. Hence we conclude that

$$\deg \Phi_\rho^a = 1\,, \quad \text{for all parameters} \quad \rho \geqslant \sigma$$

We now fix $\rho \geqslant \sigma$ and move the point $a \neq 0$ to ∞ along the straight half-line $\{ta\colon t \geqslant 1\}$, to observe that for some $t \geqslant 1$ the point ta lies in $F(\mathbb{T}_\rho)$. For if not, we would have well defined degree of the mappings $\Phi_\rho^{ta} : \mathbb{T} \to \mathbb{T}$, given by

$$\Phi_\rho^{ta}(e^{i\theta}) = \frac{F(\rho e^{i\theta}) - ta}{|F(\rho e^{i\theta}) - ta|}$$

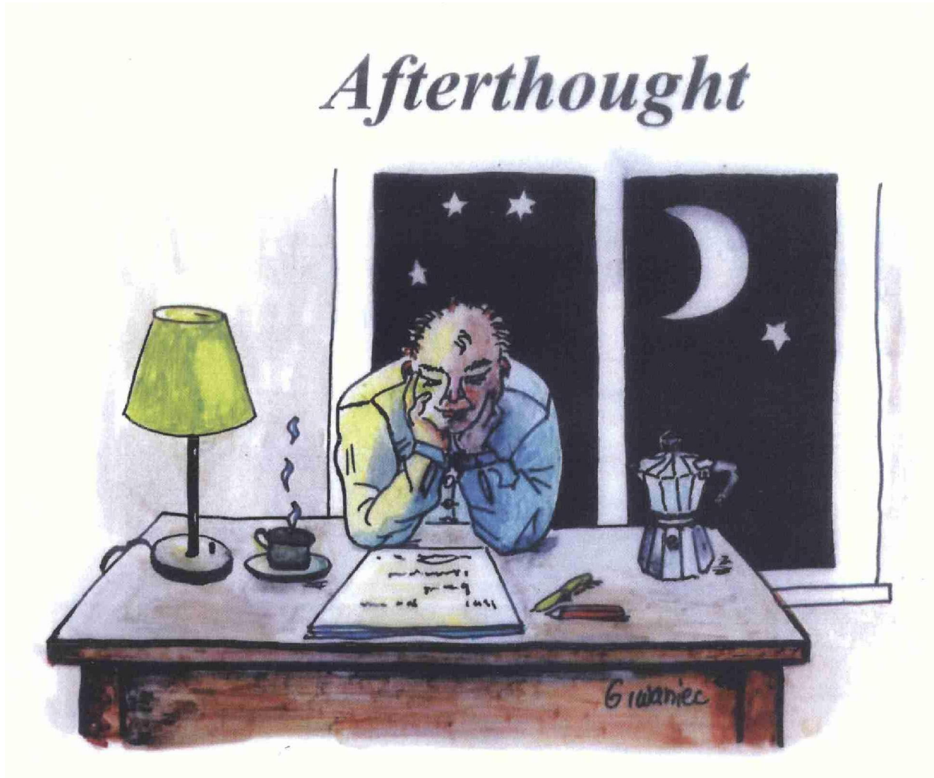
By virtue of continuity with respect to the parameter t we would have

$$\deg \Phi_\rho^{ta} = \deg \Phi_\rho^a = 1\,, \quad \text{for all } t \geqslant 1$$

On the other hand letting $t \to \infty$ the mappings $\Phi_\rho^{ta} : \mathbb{T} \to \mathbb{T}$ converge uniformly to a constant map $\Phi_\rho^\infty = \frac{a}{|a|}$, whose degree is zero, in contradiction with the case $t = 1$. Thus $ta \in F(\mathbb{T}_\rho)$, for some $t \geqslant 1$, meaning that

$$ta = \lambda \cdot (z_1 - z_2) + f^\lambda(z_1) - f^\lambda(z_2) \, , \quad \text{for some} \quad \lambda \in \mathbb{T}_\rho$$

which yields the desired inequality.



If a variational equation admits nice family of solutions then, most likely, other solutions are also nice.

$*$ - Motto in Geometric Theory of Variational PDEs