Functional inequalities and asymptotics of solutions to weighted porous media equations

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joint work with Matteo Muratori - Maria Michaela Porzio

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- Poincaré-type, or gap, inequalities:

$$\|f\|_2 \leq C\,\mathcal{E}(f), \quad \|f - \bar{f}\|_2 \leq C\,\mathcal{E}(f)$$

where $\bar{f}$ is the mean of f w.r.t. the measure m, provided it is finite. This is a spectral information on L.

- Logarithmic Sobolev inequalities. Suppose that $m(X) = 1$. The inequality

$$\int_X f^2 \log\left(\frac{f}{\|f\|_2}\right) \mathrm{d}m \le C\,\mathcal{E}(f), \quad \forall f \in D(\mathcal{E})$$

is stronger than the Poincaré inequality $\|f - \bar{f}\|_2 \le C\,\mathcal{E}(f)$. A typical example is the case in which $\mathrm{d}m = ce^{-|x|^2/2}\mathrm{d}x$.

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  is stronger than the Poincaré inequality $\|f - \bar{f}\|_2 \le C\,\mathcal{E}(f)$. A typical example is the case in which $\mathrm{d}m = ce^{-|x|^2/2}\mathrm{d}x$.
- The Nash-type inequality

$$\|f\|_2^{2+\vartheta} \le C\,\mathcal{E}(f)\|f\|_1^{\vartheta}, \quad \forall f \in D(\mathcal{E}) \cap \mathrm{L}^1.$$

  It turns out that such Nash inequality is equivalent to a Sobolev-type inequality if ϑ satisfy a suitable bound.

- The Sobolev inequality ($\mu > 2$)

$$\|f\|^2_{2\mu/(\mu-2)} \leq C\,\mathcal{E}(f), \quad \forall f \in D(\mathcal{E}).$$

It can be shown that the last two inequalities are equivalent if one puts $\vartheta = 4/\mu$. They are also equivalent to *families* of logarithmic Sobolev inequalities.

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Similar problems have been studied for a while also in the nonlinear setting. To make an example of the kind of new phenomena which can arise, consider a p-energy functional:

$$\mathcal{E}_p(u) = \frac{1}{p}\int_N |\nabla u|_x^p e^V \, \mathrm{d}m.$$

Here $N \subset M$, M is a manifold, m is the Riemannian measure, V a function, and Dirichlet boundary conditions are assumed in a suitable sense. Consider the nonlinear semigroup associated to the subgradient of $\mathcal{E}_p$, provided the functional is convex and l.s.c..

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THEOREM (G., JDE 2010). Consider the p-Laplacian type evolution ($p > 2$) associated to $\mathcal{E}_p$ and let $u(t)$ be a solution to such a flow. Then the functional inequality (in an appropriate space taking into account the Dirichlet b.c.)

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is equivalent to the bound ($1 \leq q_0 \leq \varrho < \infty$):

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The last bound need not hold if $\varrho = \infty$.

Weighted porous media equations-The Poincaré case

Motivated by such result, we started the study of classes of *two-weights* porous media equations, whose models are formally ($\Omega \subseteq \mathbb{R}^n, m > 1$):

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Here ρ_ν, ρ_μ are measurable weight in $L^\infty_{\rm loc}$ and s.t. $\rho_\nu^{-1}, \rho_\mu^{-1}$ are in $L^\infty_{\rm loc}$ (singularities or degeneracies are only “at infinity"). We set $d\nu = \varrho_\nu\, dx$, $d\mu = \varrho_\mu\, dx$.

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I start with the definition of solution (Dirichlet case), see also Dolbeault, Gentil, Guillin, Wang, Pot. Anal. 2008. Here $v \in V_0(\Omega;\mu)$ if $v \in W^{1,1}_{loc}(\Omega)$, $\nabla v \in L^2(\Omega;\mu)$ and there exists a sequence $\{\varphi_n\} \subset C_c^\infty(\Omega)$ such that $\|\nabla v - \nabla \varphi_n\|_{2;\mu} \to 0$.

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Definition. A function $u \in L^1((0,T);L^1_{loc}(\Omega;\nu))$ with $u^m(t) \in V_0(\Omega;\mu)$ for a.a. t, $\nabla(u^m) \in L^1((0,T);[L^2(\Omega;\mu)]^N)$ is a weak solution of (2) with initial datum $u_0 \in L^1_{loc}(\Omega;\nu)$ if it satisfies:

$$\int_0^T\int_\Omega u(\mathbf{x},t)\eta_t(\mathbf{x},t)\,d\nu\,dt = -\int_\Omega u_0(\mathbf{x})\eta(\mathbf{x},0)\,d\nu + \int_0^T\int_\Omega \nabla(u^m)(\mathbf{x},t)\cdot\nabla\eta(\mathbf{x},t)\,d\mu dt$$

$\forall \eta \in C^1(\Omega\times[0,T])$: $\operatorname{supp}\eta(\cdot,t) \Subset \Omega$, $\eta(\mathbf{x},T) = 0 \quad \forall \mathbf{x} \in \Omega$, $\forall t \in [0,T]$.

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Proposition. There exists *at most* one weak solution satisfying: $u^m \in L^{\frac{m+1}{m}}((0,T), V_0^{\frac{m+1}{m}}(\Omega;\nu,\mu)), \nabla(u^m) \in L^2((0,T);[L^2(\Omega;\mu)]^N)$ $\forall T>0$. Here $V_0^p(\Omega;\nu,\mu)$ is the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|\varphi\|_{p,2;\nu,\mu} = \|\varphi\|_{p;\nu} + \|\nabla\varphi\|_{2;\mu}$.

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Proposition. Let $\Omega \subseteq \mathbb{R}^N$ be a domain, and let ρ_ν, ρ_μ be two sufficiently regular weights such that $\rho_\nu, \rho_\mu, \rho_\nu^{-1}, \rho_\mu^{-1} \in L^\infty_{loc}(\Omega)$. If $u_0 \in L^1(\Omega;\nu) \cap L^r(\Omega;\nu)$, with $r \geq m+1$, then there exists a unique weak energy solution u. Moreover, given two energy solutions u, v one has

$$\int_\Omega (u(\mathbf{x},T) - v(\mathbf{x},T))_+ \,\mathrm{d}\nu \leq \int_\Omega (u_0(\mathbf{x}) - v_0(\mathbf{x}))_+ \,\mathrm{d}\nu.$$

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Theorem. If the inequality

$$\|v\|_{2;\nu} \leq W_P \left(\|\nabla v\|_{2;\mu} + \|v\|_{1;\nu} \right) \forall v \in W^{1,2}(\Omega; \nu, \mu)$$

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holds, then the solution $u(t)$ corresponding to $u_0 \in L^{q_0}(\Omega; \nu)$ satisfies the bound

$$\|u(t)\|_{\varrho;\nu} \leq K_2 \left(t^{-\frac{\varrho - q_0}{\varrho(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{q_0}{\varrho}} + \|u_0\|_{q_0;\nu} \right) \quad \text{for a.e. } t > 0$$

for all $q_0 \in (1, \infty) \cap [m-1, \infty)$, $\varrho > q_0$..

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$$\|u(t)\|_{m+1;\nu} \le K \left(t^{-\frac{m+1-q_0}{(m+1)(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{q_0}{m+1}} + \|u_0\|_{q_0;\nu} \right) \quad \text{for a.e. } t > 0,$$

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then there exists a constant $B > 0$ such that the functional inequality

$$\|v\|_{2;\nu} \le B \left(\|\nabla v\|_{2;\mu} + \|v\|_{\frac{q_0}{m};\nu} \right) \quad \forall v \in W^{1,2}(\Omega;\nu,\mu)$$

holds as well.

Corollary. With the above notations, the bound

$$\|u(t)\|_{\varrho;\nu} \le K\left(t^{-\frac{\varrho-m}{\varrho(m-1)}}\|u_0\|_{m;\nu}^{\frac{m}{\varrho}} + \|u_0\|_{m;\nu}\right) \quad \text{for a.e. } t>0$$

for solutions to the weighted PME, for a given $\varrho \ge m+1$, is equivalent to the validity of the functional inequality

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The use of the Poincaré inequality also allow to establish asymptotic results for t large. Decay rates (polinomial) for suitable entropies are given in Dolbeault, Nazaret, Savaré (Comm. Math. Sci. 2008). Following an old idea of Alikakos and Rostamian, Indiana Univ. Math. J. 1981 the key Lemma consists in showing that, for Φ *approximately* a power and ξ with $\overline{\xi} = 0$ and $\Phi(\xi) \in W^{1,2}(\Omega;\nu,\mu)$ the inequality

$$\|\Phi(\xi)\|_{2;\nu} \le C_\Phi \|\nabla \Phi(\xi)\|_{2;\mu}$$

holds. The existing proof uses compactness, which we have not, but one can proceed otherwise.

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If instead $\overline{u_0} \neq 0$, we have, for an unknown constant C:

$$\|u(t) - \overline{u}\|_{\varrho;\nu} \le e^{-C|\overline{u}|^{m-1} t} \, \|u_0 - \overline{u}\|_{\varrho;\nu} \quad \forall \, t > 0.$$

Theorem. Let $q_0 \in [1,\infty)$, $u_0 \in L^{q_0}(\Omega;\nu)$ and $\overline{u_0} = 0$. If the inequality

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holds, then the solution u with initial datum u_0 satisfies the following absolute bound:

$$\|u(t)\|_{\varrho;\nu} \le Q_2 \, t^{-\frac{1}{m-1}} \quad \text{for a.e. } t > 0\,,$$

for any $\varrho \in [1,\infty)$. Moreover

$$\|u(t)\|_{\varrho;\nu} \le Q_1 \, t^{-\frac{\varrho - q_0}{\varrho(m-1)}} \, \|u_0\|_{q_0;\nu}^{\frac{q_0}{\varrho}} \quad \text{for a.e. } t > 0.$$

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Convergence to the mean value in general is not uniform even if the datum is bounded, but in that case it is locally uniform.

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- *The Euclidean space* $\mathbb{R}^N$:
  - $((1 + |\mathbf{x}|^2)^{\alpha-1}, (1 + |\mathbf{x}|^2)^{\alpha})$ for $\alpha < 1 - \frac{N}{2}$;
  - $(e^{-a|\mathbf{x}|^2}, e^{-a|\mathbf{x}|^2})$ for $a > 0$.

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THEOREM . Let $\nu(\Omega) < \infty$ and let inequality (4) hold true for some $\sigma > 1$. Then for the solution u corresponding to an initial datum $u_0 \in L^{q_0}(\Omega;\nu)$ with $q_0 \in [1,\infty)$ the following estimate holds $\forall t > 0$:

$$\|u(t)\|_\infty \le K \left(t^{-\frac{\sigma}{(\sigma-1)q_0+\sigma(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{(\sigma-1)q_0}{(\sigma-1)q_0+\sigma(m-1)}} + \|u_0\|_{q_0;\nu} \right) ,$$

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The bound is sharp in the non-weighted case (Barenblatt solutions...). In the weighted case, there are explicit weights for which (4) holds for a best possible σ and such that the power of time is attained for special solutions (whose initial trace is singular “at infinity").

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Also a converse result holds true

THEOREM . Let $\nu(\Omega) < \infty$ and suppose that there exist a constant $K > 0$ and $\sigma > 1$, $q_0 \in [m, m+1)$ such that, for all $u_0 \in L^{q_0}(\Omega;\nu)$, the solution u corresponding to the initial datum u_0 satisfies $\forall t > 0$:

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Then there exists a constant $B > 0$ such that the functional inequality

$$\|v\|_{2\sigma;\nu} \le B \left(\|\nabla v\|_{2;\mu} + \|v\|_{\frac{q_0}{m};\nu} \right) \quad \forall v \in W^{1,2}(\Omega; \nu, \mu) \tag{6}$$

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holds as well. In particular, the validity of (5) for some, hence all, $q_0 \ge m$ (and, a posteriori, hence for all $q_0 \ge 1$) and of (6) for $q_0 = m$, are equivalent.

As for the long-time behaviour, the bounds for zero mean data are qualitatively similar to one proved in the Poincaré case, but bounds hold in L^∞: e.g. one can prove that, for such data and any $t \geq 0$ ($\overline{u_0} = 0$):

$$\|u(t)\|_\infty \leq Q_1\, t^{-\frac{\sigma}{(\sigma-1)q_0+\sigma(m-1)}} \frac{1}{\left(Q_2 t + \|u_0\|_{q_0;\nu}^{1-m}\right)^{\frac{(\sigma-1)q_0}{(m-1)[(\sigma-1)q_0+\sigma(m-1)]}}},$$

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The situation is more interesting for non-zero mean solutions. In fact, the only existing result in the weighted, Neumann-type case is due to Kamin and Rosenau (CPAM 1982, 1-dim). They prove, for some explicit weights, *local* uniform convergence to the mean value. As we saw in the Poincaré case, data with compact support may generate solutions which are compactly supported for all times.

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Examples (weighted Sobolev inequalities)

- *Intervals*:
  - (x^α, x^β) on $(0,b)$ $(b>0)$: $\beta>1$, $\alpha>\beta-2$ and $\sigma\in\left(1,\frac{\alpha+1}{\beta-1}\right]$;
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- *Bounded Lipschitz domains* ($N > 1$):
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  - $((1+|\mathbf{x}|)^\alpha, (1+|\mathbf{x}|)^\beta)$: $\beta \geq 2-N$, $\alpha < -N$ and $\sigma \in \left(1, \frac{N}{N-2}\right]$ OR $\beta < 2-N$, $\alpha < \beta - 2$ and $\sigma \in \left(1, \min\left(\frac{N}{N-2}, \frac{\alpha+N}{\beta+N-2}\right)\right]$;
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THANK YOU FOR YOUR ATTENTION!