

A soup of potential estimates, with elliptic and parabolic ingredients

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July 3, 2012

Recent Trends in Nonlinear Diffusion - Scuola Normale Superiore

Part 1: Ellipticity

- Consider the model case

$$-\Delta u = \mu \quad \text{in } \mathbb{R}^n$$

The classical potential estimates

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- We have

$$u(x) = \int G(x, y) \mu(y) dy$$

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$$u(x) = \int G(x, y) \mu(y) dy$$

where

$$G(x, y) \approx \begin{cases} |x - y|^{2-n} & \text{se } n > 2 \\ -\log |x - y| & \text{se } n = 2 \end{cases}$$

- Previous formula gives

$$|u(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-2}} = I_2(|\mu|)(x)$$

Estimates via Riesz potentials

- Previous formula gives

$$|u(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-2}} = I_2(|\mu|)(x)$$

- while, after differentiation, we obtain

$$|Du(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}} = I_1(|\mu|)(x)$$

What happens in the nonlinear case?

- **For instance for nonlinear equations with linear growth**

$$-\operatorname{div} a(Du) = \mu$$

that is equations well posed in $W^{1,2}$ (p -growth and $p = 2$)

- **And degenerate ones like**

$$-\operatorname{div} (|Du|^{p-2} Du) = \mu$$

- In bounded domains one uses

$$I_{\beta}^{\mu}(x, R) := \int_0^R \frac{\mu(B(x, \varrho))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho} \quad \beta \in (0, n]$$

since

$$\begin{aligned} I_{\beta}^{\mu}(x, R) &\lesssim \int_{B_R(x)} \frac{d\mu(y)}{|x-y|^{n-\beta}} \\ &= I_{\beta}(\mu \llcorner B(x, R))(x) \\ &\leq I_{\beta}(\mu)(x) \end{aligned}$$

for non-negative measures

- We consider equations

$$-\operatorname{div} a(Du) = \mu$$

- under the assumptions

$$\begin{cases} |a(z)| + |\partial a(z)||z| \leq L|z|^{p-1} \\ \nu^{-1}|z|^{p-2}|\lambda|^2 \leq \langle \partial a(x, z)\lambda, \lambda \rangle \end{cases}$$

with

$$p \geq 2$$

this last bound is assumed in order to keep the exposition brief

- **The nonlinear Wolff potential is defined by**

$$\mathbf{W}_{\beta,p}^{\mu}(x, R) := \int_0^R \left(\frac{|\mu|(B(x, \varrho))}{\varrho^{n-\beta p}} \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \quad \beta \in (0, n/p]$$

which for $p = 2$ reduces to the usual Riesz potential

$$\mathbf{I}_{\beta}^{\mu}(x, R) := \int_0^R \frac{\mu(B(x, \varrho))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho} \quad \beta \in (0, n]$$

- **The nonlinear Wolff potential** plays in nonlinear potential theory the same role the Riesz potential plays in the linear one

A fundamental estimate

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Another approach to this result has been given by Trudinger & Wang (Amer. J. Math. 02)

Integral estimates follow via Wolff inequalities

- We have

$$\mu \in L^q \implies \mathbf{W}_{\beta,p}^\mu \in L^{\frac{nq(p-1)}{n-qp\beta}} \quad q \in (1, n)$$

with related explicit estimates, also in Marcinkiewicz spaces

- Such a property allows to reduce the study of integrability of solutions to that of nonlinear potentials
- The key is the following inequality:

$$\int_0^\infty \left(\frac{|\mu|(B(x, \varrho))}{\varrho^{n-\beta p}} \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \lesssim I_\beta \left\{ [I_\beta(|\mu|)]^{\frac{1}{p-1}} \right\} (x)$$

the last quantity is called Havin-Maz'ja potential

The potential gradient estimate for $p = 2$

Theorem (Min., JEMS 2011)

$$|D_\xi u(x)| \lesssim \mathbf{I}_1^{|\mu|}(x, R) + \int_{B(x, R)} |D_\xi u| dy$$

holds for almost every point x and $\xi \in \{1, \dots, n\}$

A general potential gradient estimate

Theorem (Duzaar & Min., Amer. J. Math. 2011)

$$|Du(x)| \lesssim \mathbf{W}_{1/p,p}^\mu(x, R) + \int_{B(x,R)} |Du| dy$$

holds for almost every point x

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holds for almost every point x

This means

$$|Du(x)| \leq c \int_0^R \left(\frac{|\mu|(B(x, \varrho))}{\varrho^{n-1}} \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} + c \int_{B(x,R)} |Du| dx$$

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$$|Du(x)| \lesssim \mathbf{W}_{1/p,p}^\mu(x, R) + \int_{B(x,R)} |Du| dy$$

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- **Since**

$$\mu \in L^q \implies \mathbf{W}_{1/p,p}^\mu \in L^{\frac{nq(p-1)}{n-q}} \quad q \in (1, n)$$

therefore, for instance

$$\mu \in L^q \implies Du \in L_{\text{loc}}^{\frac{nq(p-1)}{n-q}}(\Omega) \quad q \in (1, n)$$

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- More in general, estimates in all rearrangement invariant spaces follow, recovering all those already known for the model case $\text{div}(|Du|^{p-2}Du) = \mu$, and fixing **open borderline cases**

- **The two potential estimates are**

$$|u(x)| \lesssim \mathbf{W}_{1,p}^\mu(x, R) + c \int_{B(x,R)} |u| dy$$

and

$$|Du(x)| \leq c \mathbf{W}_{1/p,p}^\mu(x, R) + \int_{B(x,R)} |Du| dy$$

- **They basically provide size estimates on u and Du**
- **The aim is now to provide estimates on the oscillations of solutions and/or alternatively, on intermediate derivatives**

- **The following definition is due to DeVore & Sharpley (Mem. AMS, 1982)**
- Let $\alpha \in (0, 1]$, $q \geq 1$, and let $\Omega \subset \mathbb{R}^n$ be a bounded open subset. A measurable function v , finite a.e. in Ω , belongs to the Calderón space $C_q^\alpha(\Omega)$ if and only if there exists a nonnegative function $m \in L^q(\Omega)$ such that

$$|v(x) - v(y)| \leq [m(x) + m(y)]|x - y|^\alpha$$

holds for almost every couple $(x, y) \in \Omega \times \Omega$.

- In other words

$$m(x) \approx \partial^\alpha v(x)$$

- Indeed DeVore & Sharpley take

$$M_{\#}^\alpha v(x) = \sup_{B(x,\varrho)} \varrho^{-\alpha} \int_{B(x,\varrho)} |v(y) - (v)_{B(x,\varrho)}| dy$$

- For $\alpha \in (0, 1)$ and $q > 1$ we have

$$W^{\alpha,q} \subset C^{\alpha,q} \subset W^{\alpha-\varepsilon,q}$$

therefore such spaces, although not being of interpolation type, are just another way to say “fractional differentiability”

A “universal potential estimate”

Theorem (Kuusi & Min., JFA 2012)

The estimate

$$\begin{aligned} & |u(x) - u(y)| \\ & \leq c \left[\mathbf{W}_{1 - \frac{\alpha(p-1)}{p}, p}^\mu(x, R) + \mathbf{W}_{1 - \frac{\alpha(p-1)}{p}, p}^\mu(y, R) \right] |x - y|^\alpha \\ & \quad + c \int_{B_R} |u| d\xi \cdot \left(\frac{|x - y|}{R} \right)^\alpha \end{aligned}$$

holds uniformly in $\alpha \in [0, 1]$, whenever $x, y \in B_{R/4}$

- The cases $\alpha = 0$ and $\alpha = 1$ give back the two known potential estimates as endpoint cases

The homogeneous case

- The estimate tells that

$$“\partial^\alpha u(x) \lesssim \mathbf{W}_{1-\frac{\alpha(p-1)}{p}, p}^\mu(x, R)”$$

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- The case $\mu = 0$ reduces to the classical estimate

$$|u(x) - u(y)| \leq \int_{B_R} |u| d\xi \cdot \left(\frac{|x - y|}{R} \right)^\alpha$$

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$$|u(x) - u(y)| \leq \int_{B_R} |u| d\xi \cdot \left(\frac{|x - y|}{R} \right)^\alpha$$

- In the case $p = 2$ we have

$$\begin{aligned} |u(x) - u(y)| &\leq c \left[\mathbf{I}_{2-\alpha}^{|\mu|}(x, R) + \mathbf{I}_{2-\alpha}^{|\mu|}(y, R) \right] |x - y|^\alpha \\ &\quad + c \int_{B_R} |u| d\xi \cdot \left(\frac{|x - y|}{R} \right)^\alpha \end{aligned}$$

which in the classical case $-\Delta u = \mu$ can be derived directly from the standard representation formula via potentials

The second universal potential estimate

Theorem (Kuusi & Min., JFA 2012)

The estimate

$$\begin{aligned} & |Du(x) - Du(y)| \\ & \leq c \left[\mathbf{W}_{1 - \frac{(1+\alpha)(p-1)}{p}, p}^\mu(x, R) + \mathbf{W}_{1 - \frac{(1+\alpha)(p-1)}{p}, p}^\mu(y, R) \right] |x - y|^\alpha \\ & \quad + c \int_{B_R} |Du| d\xi \cdot \left(\frac{|x - y|}{R} \right)^\alpha \end{aligned}$$

holds whenever $\alpha < \alpha_M$, whenever $x, y \in B_{R/4}$

- The case $\alpha = 0$ gives back the gradient potential estimate

Part 2: Parabolicity

- **The model case is here given by**

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \mu,$$

more in general we consider

$$u_t - \operatorname{div} a(Du) = \mu.$$

- A basic reference for existence and a priori estimates is the work of Boccado, Dall'Aglio, Galloüet and Orsina, *J. Funct. Anal.*, 1997

Problematic aspects

- For basic scaling reasons the previous potential estimates do not hold in the case $p \neq 2$
- For the case $p = 2$ it holds

Theorem (Duzaar & Min., Amer. J. Math. 2011)

$$|Du(x, t)| \lesssim \mathbf{I}_1^{|\mu|}(x, t; r) + \int_{Q_r(x, t)} |Du| dz$$

holds for almost every point (x, t) and $\xi \in \{1, \dots, n\}$

- Here $\mathbf{I}_1^{|\mu|}(x, t; r)$ denotes the parabolic Riesz potential

$$\mathbf{I}_\beta^{|\mu|}(x, t; r) := \int_0^r \frac{|\mu|(Q_\varrho(x, t))}{\varrho^{N-\beta}} \frac{d\varrho}{\varrho}, \quad \beta < N := n + 2$$

and

$$Q_\varrho(x, t) = B(x, \varrho) \times (t - \varrho^2, t)$$

is a standard parabolic cylinder

- **The case $p \neq 2$ is a very different story**

Recall the elliptic estimate

Theorem (Duzaar & Min., Amer. J. Math. 2011)

$$|Du(x)| \lesssim \mathbf{W}_{1/p,p}^\mu(x, R) + \int_{B(x,R)} |Du| dy$$

holds for almost every point x

Recall the elliptic estimate

Theorem (Duzaar & Min., Amer. J. Math. 2011)

$$|Du(x)| \lesssim \mathbf{W}_{1/p,p}^\mu(x, R) + \int_{B(x,R)} |Du| dy$$

holds for almost every point x

This means

$$|Du(x)| \lesssim \int_0^R \left(\frac{|\mu|(B(x, \varrho))}{\varrho^{n-1}} \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} + \int_{B(x,R)} |Du| dx$$

Another way to say the same thing

if

$$c \int_{B(x,R)} |Du| dy + c \mathbf{W}_{1/p,p}^\mu(x, R) \leq \lambda$$

then

$$|Du(x)| \leq \lambda$$

The intrinsic geometry of DiBenedetto

- **The basic analysis is the following: consider intrinsic cylinders**

$$Q_\rho^\lambda(x, t) = B(x, \rho) \times (t - \lambda^{2-p}\rho^2, t)$$

where it happens that

$$|Du| \approx \lambda \quad \text{in } Q_\rho^\lambda(x, t)$$

then the equation behaves as

$$u_t - \lambda^{p-2} \Delta u = 0$$

that is, scaling back in the same cylinder, as the heat equation

- **On intrinsic cylinders estimates “ellipticize”; in particular, they become homogeneous**

- The effect of intrinsic geometry

Theorem (DiBenedetto & Friedman, Crelle J. 85)

There exists a universal constant $c \geq 1$ such that

$$c \left(\int_{Q_r^\lambda(x,t)} |Du|^{p-1} dz \right)^{1/(p-1)} \leq \lambda$$

then

$$|Du(x, t)| \leq \lambda$$

- Define the intrinsic Wolff potential such that

$$\mathbf{W}_\lambda^\mu(x, t; r) := \int_0^r \left[\frac{|\mu|(Q_\varrho^\lambda(x, t))}{\lambda^{2-p}\varrho^{N-1}} \right]^{1/(p-1)} \frac{d\varrho}{\varrho}$$

- Note that

$$\mathbf{W}_\lambda^\mu(x, t; r) = \mathbf{I}_1^{|\mu|}(x, t; r) \quad \text{when } p = 2$$

and

$$\mathbf{W}_\lambda^\mu(x, t; r) = \mathbf{W}_{1/p, p}^\mu(x, r) \quad \text{when } \mu \text{ is time independent}$$

this is the elliptic case

The parabolic Wolff gradient bound

Theorem (Kuusi & Min.)

There exists a universal constant $c \geq 1$ such that

$$c\mathbf{W}_\lambda^\mu(x, t; r) + c \left(\int_{Q_r^\lambda(x, t)} |Du|^{p-1} dz \right)^{1/p-1} \leq \lambda$$

then

$$|Du(x, t)| \leq \lambda$$

- **When $\mu \equiv 0$ this reduces to the sup estimate of DiBenedetto & Friedman (Crelles J. 84)**

- Consider the equation

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \delta,$$

where δ denotes the Dirac unit mass charging the origin

- The so called Barenblatt (fundamental solution) is

$$\mathcal{B}_p(x, t) = \begin{cases} t^{-\frac{n}{\theta}} \left(c_b - \theta^{\frac{1}{1-p}} \left(\frac{p-2}{p} \right) \left(\frac{|x|}{t^{1/\theta}} \right)^{\frac{p}{p-1}} \right)_+^{\frac{p-1}{p-2}} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

for $\theta = n(p-2) + p$ and a suitable constant c_b such that

$$\int_{\mathbb{R}^n} \mathcal{B}_p(x, t) dx = 1 \quad \forall t > 0$$

- A direct computation shows the following upper optimal upper bound

$$|DB_p(x, t)| \leq ct^{-(n+1)/\theta}$$

- The intrinsic estimate above **exactly reproduces this upper bound**
- This decay estimate is indeed reproduced for all those solutions **that are initially compactly supported**

- The previous bound always implies a priori estimates on standard parabolic cylinders

Theorem (Kuusi & Min.)

$$|Du(x, t)| \lesssim \left[\int_0^r \left(\frac{|\mu|(Q_\varrho(x, t))}{\varrho^{N-1}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho} \right]^{p-1} + \int_{Q_r(x, t)} (|Du| + s + 1)^{p-1} dz$$

holds for every standard parabolic cylinder Q_r

Theorem (Kuusi & Min.)

$$|Du(x, t)| \lesssim [\mathbf{W}_1^\mu(x, t; r)]^{p-1} + \int_{Q_r(x, t)} (|Du| + s + 1)^{p-1} dx dt$$

holds for every standard parabolic cylinder Q_r

- The scaling deficit exponent $p - 1$ appears

Occurrence of deficit scaling exponents

- For solutions to

$$u_t - \Delta_p u = \operatorname{div}(|F|^{p-2}F)$$

Theorem (Acerbi & Min., Duke Math. J. 2007)

$$\left(\int_{Q_r} |Du|^q dz \right)^{\frac{1}{q}} \lesssim \left[\left(\int_{Q_{2r}} |Du|^p dz \right)^{\frac{1}{p}} + \left(\int_{Q_{2r}} |F|^q dz \right)^{\frac{1}{q}} + 1 \right]^{\frac{p}{2}}$$

for every $q \geq p$

- The scaling deficit exponent $p/2$ appears

Theorem (Kuusi & Min.)

It μ is time independent than

$$|Du(x, t)| \lesssim \mathbf{W}_{1/p, p}^\mu(x, r) + \int_{Q_r} (|Du| + s + 1)^{p-1} dx dt$$

holds for every standard parabolic cylinder Q_r

Theorem (Kuusi & Min., General regularity estimate)

$$A, B, q \geq 1 \quad \text{and} \quad \varepsilon \in (0, 1).$$

Then there exists a constant $\delta_\varepsilon \in (0, 1/2)$

$$\frac{\lambda}{B} \leq \sup_{Q_{\delta_\varepsilon r}^\lambda} \|Dw\| \leq \sup_{Q_r^\lambda} \|Dw\| \leq A\lambda$$

holds, then

$$E_q(Dw, \delta_\varepsilon Q_r^\lambda) \leq \varepsilon E_q(Dw, Q_r^\lambda)$$

holds, where

$$E_q(Dw, Q_\varrho^\lambda) := \left(\int_{Q_\varrho^\lambda} |Dw - (Dw)_{Q_\varrho^\lambda}|^q dx dt \right)^{1/q}$$

Theorem (Kuusi & Min.)

Assume that

$$\lim_{r \rightarrow 0} \sup_{(x,t) \in \Omega_T} \mathbf{W}_1^\mu(x, t; r) = 0$$

holds, then Du is continuous in Q_T

Gradient continuity via potentials

Theorem (Kuusi & Min.)

Assume that

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Previous assumption reads as

$$\lim_{r \rightarrow 0} \sup_{(x,t) \in \Omega_T} \int_0^r \left(\frac{|\mu|(Q_\varrho(x, t))}{\varrho^{N-1}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho} = 0$$

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Previous assumption reads as

$$\lim_{r \rightarrow 0} \sup_{(x,t) \in \Omega_T} \int_0^r \left(\frac{|\mu|(Q_\varrho(x, t))}{\varrho^{N-1}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho} = 0$$

i.e. the convergence is uniform

Theorem (Kuusi & Min., General regularity estimate)

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Then there exists a constant $\delta_\varepsilon \in (0, 1/2)$

$$\frac{\lambda}{B} \leq \sup_{Q_{\delta_\varepsilon r}^\lambda} \|Dw\| \leq \sup_{Q_r^\lambda} \|Dw\| \leq A\lambda$$

holds, then

$$E_q(Dw, \delta_\varepsilon Q_r^\lambda) \leq \varepsilon E_q(Dw, Q_r^\lambda)$$

with

$$\delta_\gamma = \frac{1}{c(A)} \left(\frac{\varepsilon}{B} \right)^{1/\alpha},$$

Theorem (Kuusi & Min.)

Assume that

$$|\mu|(Q_\varrho) \lesssim \varrho^{N-1+\delta}$$

holds, then there exists α , depending on δ , such that

$$Du \in C^{0,\alpha} \quad \text{locally in } Q_T$$

Thanks, with a work of Serena Nono



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A soup of potential estimates, with elliptic and parabolic ingredients