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Higher Order Asymptotics of Fast Diffusion with Dynamical Systems methods

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jt wk with H Koch, R. McCann

(FD) $\quad \rho_\tau(\tau, y) = \frac{1}{m}\Delta \rho^m \quad$ in $\mathbb{R}^n$

Rescaled: $x = (1+2p\tau)^{-\beta} y$

$t = \frac{1}{2p}\ln(1+2p\tau)$

with $p = \frac{2}{1-m} - n$ (moment parameter)

$\beta = 1/(2-n(1-m))$

(RFD) $\quad u_t = \frac{1}{m}\Delta u^m + \frac{2}{1-m}\operatorname{div}(\vec{x}\rho u)$

fixed pt $\quad u_B = (B+|x|^2)^{-\frac{1}{1-m}}$

Notation for critical values of m: $\quad m = 1 - \frac{2}{p+n} =: m_p$

$m > m_p \iff$ Barenblatt has p^{th} moments

Goal: local Convergence rates & higher term asymptotics for $u(t,\cdot) \to u_B$

Assuming: Center of mass of $u(0,\cdot)$ in the origin (provided $m > m_1$, else CM irrelevant)

We Measure convergence in relative L^∞ norm $\left\|\frac{u-u_B}{u_B}\right\|_{L^\infty}$

One key ingredient: Comparison argument (Vázquez)

$\frac{1}{c}u_B \le u_0 \le c\, u_B \Rightarrow \frac{1}{C}u_B \le u(t,\cdot) \le C\, u_B$

First result of this approach: (not new) (for centered data)

$$\left\|\frac{u(t)-u_B}{u_B}\right\|_{L^\infty} = O(e^{-2pt}) = O\left(\frac{1}{\tau}\right)$$

in the entire parameter range $m_0 < m < 1$

Except: $m = m_2 \to$ either only $O\left(\frac{1}{\tau^{1-\varepsilon}}\right)$ prd

$\to$ or throw in stronger hypothesis $\left\|\frac{u_0-u_B}{u_B}\right\|_{L^2(M)} < \infty$

$M = (\mathbb{R}^n, g)$ st. $d\mathrm{vol}_g = u_B^m\, d^n x$

$m < m_2$ Kim McCann 2006

$m \geq m_n$ McCann Slepčev 2006 $O(1/\tau^{1-\varepsilon})$

$m_c < m < 1$ Carrillo Vázquez 2003 (rot' symm data)

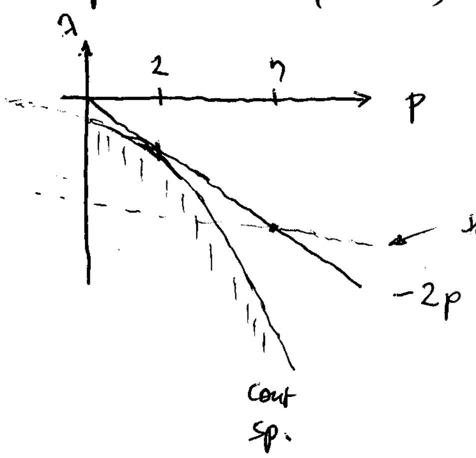
-"- Bonforte Dolbeault Grillo Vázquez 2010

-"- Blanchet -"- 2009 rate?
[& $m \leq m_0$]

Dolbeault Toscani 2005 → ~~abo~~ improved conv. rate by time shift

$O(\frac{1}{\tau})$ is the rate by which a time translated BB converges to the original BB → should get better rate.

Spectral Theory (D., McCann 2005 ARMA) for linearization in weighted $W^{1,2}$ (equiv'ly weighted L^2)



Redo Spectral theory in C^α spaces to handle Nonlinearity

With explicit eigenfcts & resolvents : routine

Crucial "lemma" : $\operatorname{Re} \operatorname{spec} H \leq -\lambda \Rightarrow \| e^{tH} \| \leq C e^{-\lambda t}$

true for sa. op's in Hilbert space

not true in general (even in finite dim.)

→ best you could hope for: $C_\varepsilon e^{(-\lambda+\varepsilon)t}$

need also sectoriality / analytic semigroup hypothesis (unproblematic here)

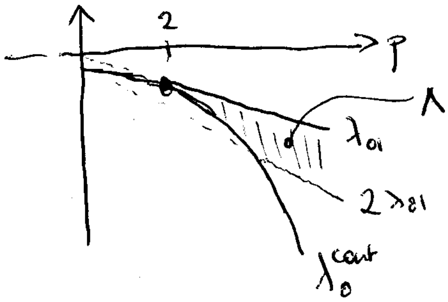
Blended approach: $\underbrace{\text{class PDE / analysis}}$ $\oplus$ fctl analysis

this will kill the ε again

Thm 2: Let $m > m_2 = 1 - \frac{2}{n+2}$ $(p > 2)$

Fix $\Lambda \in [\lambda_0^{cont}, \lambda_{01}] = [-(\frac{p}{2}+1)^2, -2p]$

Also assume $2\lambda_{01} < \Lambda$



If u solves RFD, has CM in origin, and $\| \frac{u_0}{u_B} \|_{L^\infty} < \infty$, then

$\exists$ polynomial $u_{\ell k}$ st.

$$\left\| \frac{\left(\frac{u(t,x)}{u_B(x)} - 1\right) - \sum_{\Lambda < \lambda_{\ell k} < 0} \frac{u_{\ell k}(x)}{B+|x|^2} e^{\lambda_{\ell k} t}}{(B+|x|^2)^{\frac{1}{4}(p-2-\sqrt{(p+2)^2+4\Lambda})}} \right\|_{C^\alpha(M)} = O(e^{\Lambda t})$$

$\lambda_{\ell k}$ eigenval's of linearization, $\frac{u_{\ell k}}{B+|x|^2}$ eigenfunctions

Q: No lin. comb. of eigenvalues? $e^{(\lambda_{..}+\lambda_{..})t}$ terms?

A: In principle yes, but in case $2\lambda_{01} < \Lambda$, they all go in $O(e^{\Lambda t})$ term

("Lazy" shortcut, but I don't believe it's an ess'l limitation of the method)

Note: $\Lambda \to \lambda_{01}$: old result, no weight : Λ smaller, better rate but in more permissive space

Issue : ¿uniform? parabolicity;

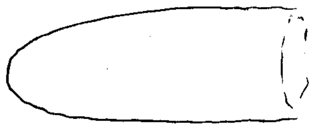
Cigar mfd $\mathcal{M} = (\mathbb{R}^n, g)$

$$dl_{\mathcal{M}}^2 = (1+|x|^2)^{-1}\, dl_{eucl}^2 \qquad (\text{w/o loss } B=1)$$

geod. distance from origin

$$s = \operatorname{arsinh}|x|$$

$\Delta_{\mathcal{M}}$ has same princ' part as lin' of RFD about BB.



$$dl_{\mathcal{M}}^2 = ds^2 + \tanh^2 s\, dl_{Sph}^2$$

~~Sketch~~:
Key feature: <u>local</u> distortion (in each coord chart) wrt "flat" is <u>uniformly</u> bdd over all charts

This restores uniformity of local parabolic Schauder estimates,

Schauder estimates targeted at quasilinear case, purely in C^α:

for HE: $(\partial_t - \Delta)v = \partial_i\partial_j f^{ij} + \partial_i b^i + c$

$$\|v\|_{C^\alpha \mathbb{R}^n_T} \le C\left(\|v_0\|_{C^\alpha \mathbb{R}^n} + \|f\|_{C^\alpha \mathbb{R}^n_T} + T^{1/2}\|b\|_{C^\alpha \mathbb{R}^n_T} + T^{1-\frac{\alpha}{2}}\|c\|_{C^\alpha \mathbb{R}^n_T}\right)$$

trading small coeff T vs ~~higher~~ lower order deriv's

Also need these to estimate regularization

$$\|v\|^*_{C^\alpha} \le C\left(\|v_0\|_\infty + \|f\|^*_{C^\alpha} + T^{1/2}\|b\|^*_{C^\alpha} + T^{1-\frac{\alpha}{2}}\|c\|^*_{C^\alpha}\right)$$

$$\|u\|^*_{C^\alpha} := \max\left\{ \sup_t t^{\alpha/2}\|u\|_{C^\alpha([t,T]\times\mathbb{R}^n)},\ \|u\|_{L^\infty}\right\}$$

<u>Goal:</u> Need a space in which the NL flow is <u>differentiable</u>

Adjusting Schauder est's to this setting, we get for the linear case

$$u_t - \partial_i\partial_j (a^{ij} u) - \partial_i (b^i u) - cu = f + \partial_i g^i$$

$$\|u\|_{C^\alpha M_T} \leq C \left(\|u_0\|_{C^\alpha M_{\cancel{T}}} + \|f\|_{L^\infty} + \|g\|_{L^\infty} \right)$$

$$\|u\|^*_{C^\alpha M_T} \leq C \left(\|u_0\|_{L^\infty} + \|f\|^{*}_{C^\alpha} + \|g\|^*_{C^\alpha} \right)$$

NL eqn

$$u_t = \partial_i\partial_j \left(f^{ij}(x,u) \right) + \partial_i \left(b^i(x,u) \right) + c(x,u)$$

local existence (by BFPT) <u>and</u> smooth dependence of sol'n on data

<u>Warning:</u> At this stage, local existence time may depend on C^α norm of initial data rather than only L^∞ norm

<u>But:</u> Vázquez comparison principle controls L^∞ norm long term, DG-N-M controls C^α in terms of L^∞.

This allows to repeat argument to get global existence (no news so far) <u>with</u> smooth dependence on data in C^α norms.

Time step iteration: $\left(\text{for } w = \frac{u}{u_B} - 1 \right)$

$$(*) \qquad w_{j+1} = S w_j + \underbrace{G(w_j)}_{\text{lin op on } C^\alpha \text{ with small norm controlled by } \|w_j\|_{L^\infty}} w_j$$

Get $\|w_j\| \leq C \, \|S\|^j \, \|w_0\|$

Weighted C^α-norms:

Cannot expect smooth dynamics in norm that permits growth at ∞ b/c that would allow u to run into singularity 0^m.

However (*) still works with $\|G(w_j)\|$ in <u>weighted</u> C^α controlled by unweighted $\|w_j\|_{L^\infty}$

Focus on spectral theory

We'll use a whole scale of weights:

$$C^{\alpha}_{\eta} := \{ (\cosh s)^{\eta} f \mid f \in C^{\alpha} \}$$

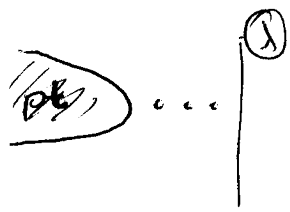
(Remember: s geodesic distance from origin

$\cosh s = \sqrt{1+|x|^2}$, so these weights are powers of BB)

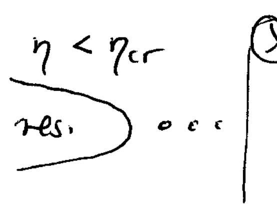
$\eta_{cr} = \frac{p}{2} - 1$

Spectrum in $C^{\alpha}_{\eta_{cr}}$ is "basically" the same as in $W^{1,2}_{u_B}$ or $L^2_{u_B^m}$
(ARMA)

$\eta > \eta_{cr}$:



$\eta < \eta_{cr}$



parabola is essential spectrum

ess spectral abscissa grows monotonically with $|\eta - \eta_{cr}|$

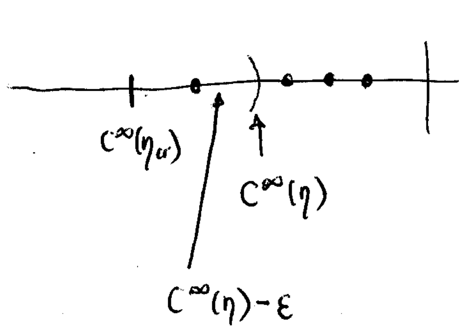
This feature will help us get the ε out of the semigroup estimates

[Do it for $\eta < \eta_{cr}$ here. The $\eta > \eta_{cr}$ is similar but more technical]

$$\tilde{v}_0 = (\cosh s)^{\eta - \eta_{cr}} \, v_0$$

lies in a "better decay" space than $v_0 \in C^{\alpha}_{\eta_{cr}} \approx L^2(\mathcal{M})$

as far as cut decay is concerned



$$\|\tilde{v}\|_{C^{\alpha}} \leq c \exp\left[(C^{\infty}(\eta) - \varepsilon) t \right]$$

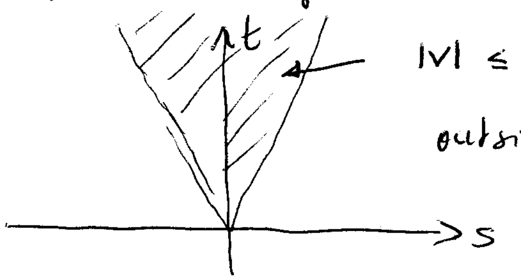
no η

Obviously, by "conservation of difficulty",
we have not controlled the stronger C^α_η norm

But we have the implication

$$|\tilde{v}(t,x)| \le C \exp[(c^*(\eta)-\varepsilon)t]$$

$$|v(t,x)| \le C \exp[(c^*(\eta)-\varepsilon)t] \, (\cosh s)^{\eta_{cr}-\eta}$$



$|v| \le C \exp[c^*(\eta) t]$

outside wedge : fct'l analysis falls short

(M)

<u>But</u> : Max principle propagates L^∞ est on ∂(wedge) into exterior region

<u>Dual case:</u> $\eta > \eta_{cr}$

max principle propagates est, outside

Full paper available at arxiv.org
(and submitted, pending referee report)