

Global Bifurcations Picture in NLS and its Dynamical Aspects

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The Nonlinear Schrödinger Equation

$$i\partial_t u(t, x) = (-\Delta_x + V(x))u - |u|^{2p}u \quad (1)$$

where:

- $u : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{C}; \quad 0 < p < \frac{2}{n-2};$
- $V : \mathbb{R}^n \rightarrow \mathbb{R}, \quad \lim_{|x| \rightarrow \infty} |V(x)| = 0, \quad V \in L^q(\mathbb{R}^n) + L^r(\mathbb{R}^n),$
 $\max\{1, n/2\} < q \leq r \leq \infty.$

Applications: Nonlinear Optics, Water Waves, Quantum Physics
in particular Bose-Einstein Condensates.

Hamiltonian Formulation

The general Hamiltonian system:

$$\partial_t u = JD\mathcal{E}(u)$$

becomes equivalent to the nonlinear Schrödinger equation under the definitions:

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^n} V |u|^2 dx - \frac{1}{2p+2} \int_{\mathbb{R}^n} |u|^{2p+2} dx,$$

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\mathcal{E} : H^1(\mathbb{R}^n, \mathbb{C}) \mapsto \mathbb{R}, \quad J : H^{-1}(\mathbb{R}^n, \mathbb{C}) \mapsto H^{-1}(\mathbb{R}^n, \mathbb{C}).$$

The two conserved quantities are the energy \mathcal{E} and the mass (charge): $\mathcal{N} : H^1(\mathbb{R}^n, \mathbb{C}) \mapsto \mathbb{R}$.

$$\mathcal{N}(u) = \frac{1}{2} \int_{\mathbb{R}^n} |u|^2 dx.$$

Nonlinear Bound-States

are solutions of the form:

$$u(t, x) = e^{iEt} \psi_E(x), \quad \psi_E \in H^1(\mathbb{R}^n).$$

Hence ψ_E satisfies in the weak sense:

$$F(\psi_E, E) = (-\Delta + V + E)\psi_E - |\psi_E|^{2p}\psi_E = 0. \quad (2)$$

In applications the nonlinear bound-states are the most important solutions of the full, time-dependent NLS equation (1). Moreover:

Asymptotic Completeness Conjecture: Any solution of (1) eventually converges to a superposition of nonlinear bound-states and a radiative part which disperses to infinity.

Note: If (ψ_E, E) is a solution of (2) then so are $(e^{i\theta}\psi_E, E)$ with $0 \leq \theta < 2\pi$.

The Mathematical Framework: Find zeroes of the map:

$$F : H^2(\mathbb{R}^n, \mathbb{C}) \times \mathbb{R} \mapsto L^2(\mathbb{R}^n, \mathbb{C}), \quad F(\psi, E) = (-\Delta + V + E)\psi - |\psi|^{2p}\psi,$$

which is equivariant under the action of $O(2)$, i.e.:

$$F(e^{i\theta}\psi, E) = e^{i\theta}F(\psi, E), \quad F(\bar{\psi}, E) = \overline{F(\psi, E)},$$

and is Fréchet differentiable over the *real* Banach spaces:

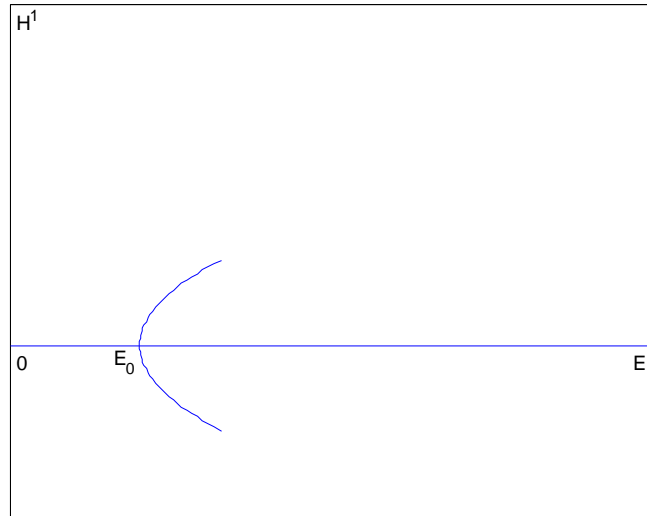
$$H^2(\mathbb{R}^n, \mathbb{C}) \cong H^2(\mathbb{R}^n, \mathbb{R}) \times H^2(\mathbb{R}^n, \mathbb{R}) \hookrightarrow L^2(\mathbb{R}^n, \mathbb{R}) \times L^2(\mathbb{R}^n, \mathbb{R}) \cong L^2(\mathbb{R}^n, \mathbb{C}).$$

For ψ real valued (hence $F(\psi, E)$ real valued) we have:

$$D_\psi F(\psi, E)[u + iv] = \begin{bmatrix} L_+(\psi, E) & 0 \\ 0 & L_-(\psi, E) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

where

$$\begin{aligned} L_+(\psi, E)[u] &= (-\Delta + V + E)u - (2p + 1)|\psi|^{2p}u \\ L_-(\psi, E)[v] &= (-\Delta + V + E)v - |\psi|^{2p}v \end{aligned}$$



- $(\psi = 0, E)$, $E \in \mathbb{R}$ is a solution, and $E \notin \text{spec}(-\Delta + V) \Rightarrow D_{\psi}F(0, E)$ is an isomorphism, hence there are no other solutions around.
- $0 < E_0 \in \text{spec}(-\Delta + V) \Rightarrow D_{\psi}F(0, E_0)$ is Fredholm hence we can get non-trivial solutions via local bifurcation theory.

Use *Lyapunov-Schmidt reduction* i.e. decompose:

$$\begin{aligned} H^2 &= \ker D_\psi F(0, E_0) \oplus M \\ L^2 &= N \oplus \text{range } D_\psi F(0, E_0) \end{aligned}$$

with associated projections $P_\parallel, P_M, P_N, P_\perp$. Then

$$F(\psi, E) = 0 \Leftrightarrow \begin{cases} P_\perp F(P_\parallel \psi + P_M \psi, E) = 0 \\ P_N F(P_\parallel \psi + P_M \psi, E) = 0 \end{cases}$$

But $G : \ker D_\psi F(0, E_0) \times M \times \mathbb{R} \mapsto \text{range } D_\psi F(0, E_0)$,

$$G(\psi_1, \psi_2, E) = P_\perp F(\psi_1 + \psi_2, E)$$

has a zero at $(0, 0, E_0)$ and $D_{\psi_2} G(0, 0, E_0)$ is an isomorphism, hence for each (ψ_1, E) near $(0, E_0)$ there is a unique ψ_2 solving the first equation, call it $h(\psi_1, E)$. The second equation becomes

$$\tilde{F}(P_\parallel \psi, E) = P_N F(P_\parallel \psi + h(P_\parallel \psi, E), E) = 0$$

where $\tilde{F} : \ker D_\psi F(0, E_0) \mapsto N$ is a finite dimensional map.

Moreover, if $P_{\parallel}, P_M, P_N, P_{\perp}$ commute with the symmetries of F then both h and \tilde{F} inherit the symmetries. In the case $-E_0$ is the lowest e-value of $-\Delta + V$:

$$\ker D_{\psi}F(0, E_0) = \text{span} \{\phi_0, i\phi_0\}, \quad M = \{\phi_0, i\phi_0\}^{\perp} \cap H^2,$$

$$N = \text{span} \{\phi_0, i\phi_0\}, \quad \text{range } D_{\psi}F(0, E_0) = \{\phi_0, i\phi_0\}^{\perp},$$

with associated (orthogonal in L^2) projections commuting with rotations and complex conjugation. Consequently, the reduced finite dimensional equation is of the form:

$$\tilde{F}(a, E) = 0 \in \mathbb{C}, \quad a \in \mathbb{C}, \quad E \in \mathbb{R}$$

with symmetries:

$$\tilde{F}(e^{i\theta}a, E) = e^{i\theta}\tilde{F}(a, E), \quad \tilde{F}(\bar{a}, E) = \overline{\tilde{F}(a, E)},$$

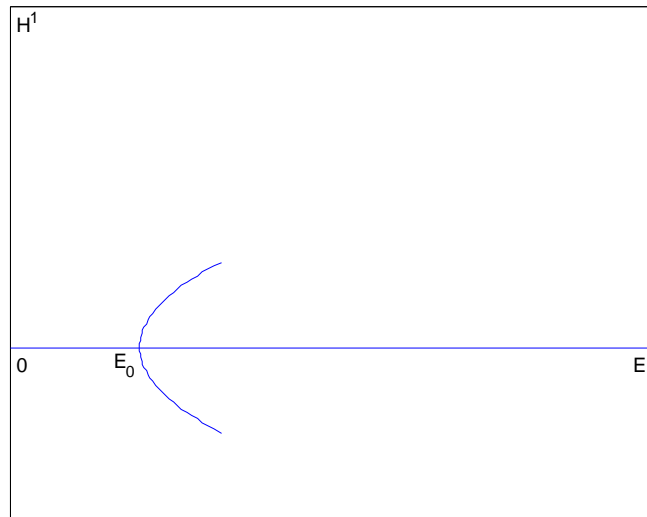
hence the solutions are generated by rotating the solutions of

$$\tilde{F}(a, E) = 0 \in \mathbb{R}, \quad a \in \mathbb{R}, \quad E \in \mathbb{R}.$$

$\tilde{F} : \mathbb{R}^2 \mapsto \mathbb{R}$ has a double zero at $(0, E_0)$, i.e. both the function and its gradient are zero, while the Hessian is nondegenerate and indefinite, leading to a quadratic normal form (Morse Lemma). Consequently,

$$\tilde{F}(a, E) = 0 \in \mathbb{R}, \quad a \in \mathbb{R}, \quad E \in \mathbb{R}$$

exhibits the pitchfork bifurcation pictured:

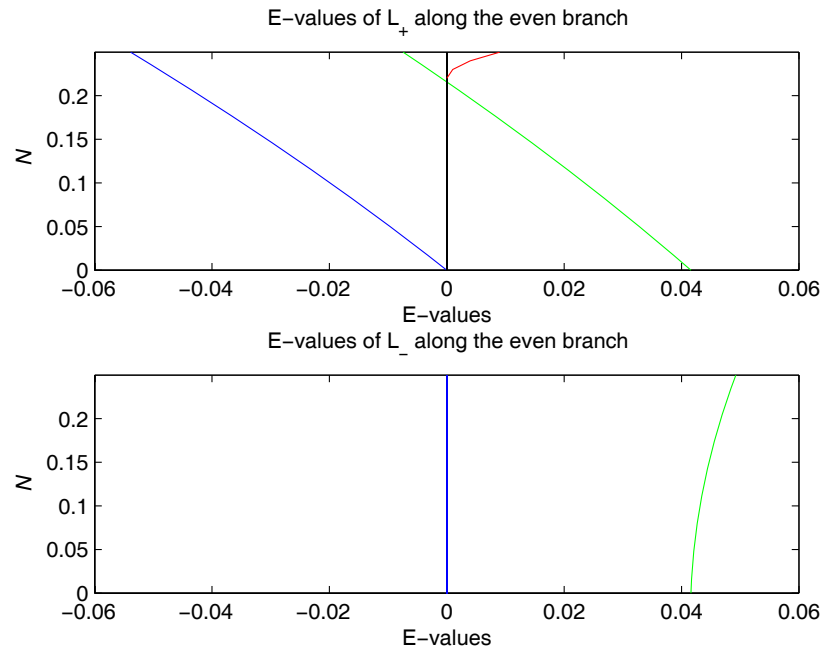


The spectrum of the linearization along the nontrivial branch:

$$D_{\psi}F(\psi_E, E) = \text{diag}\{L_+(\psi_E, E), L_-(\psi_E, E)\}$$

$$L_+(\psi, E)[v] = (-\Delta + V + E)v - (2p + 1)|\psi|^{2p}u$$

$$L_-(\psi, E)[v] = (-\Delta + V + E)v - |\psi|^{2p}v$$



How far can the branch be extended?

unique local continuation (modulo rotations) when $\ker L_+ = \{0\}$
 \Rightarrow existence of a unique maximal branch (ψ_E, E) , $E \in [E_0, E_*)$:

- $E_* < \infty$ and $\limsup_{E \nearrow E_*} \|\psi_E\|_{H^1} = +\infty$ or
- $E_* < \infty$ and $\limsup_{E \nearrow E_*} \|\psi_E\|_{H^1} < \infty$ or
- $E_* = +\infty$.

Theorem 1. *For $0 < E_* < \infty$ there are no C^1 curves $E \mapsto \psi_E$ of zeroes of F , defined on intervals $[E_1, E_*)$ or $(E_*, E_1]$, such that*

$$\limsup_{E \rightarrow E_*} \|\psi_E\|_{H^1} = \infty$$

Sketch of Proof: Consider such a curve. From the identities:

$$\begin{aligned}\mathcal{E}(E) &= \|\nabla \psi_E\|^2 + \int_{\mathbb{R}^n} V(x) |\psi_E(x)|^2 dx - \frac{1}{p+1} \|\psi_E\|_{L^{2p+2}}^{2p+2} \\ \partial_E \mathcal{E} &= -E \partial_E \|\psi_E\|_{L^2}^2, \\ \mathcal{E}(E) &= \frac{p}{p+1} \|\psi_E\|_{L^{2p+2}}^{2p+2} - E \|\psi_E\|_{L^2}^2\end{aligned}$$

we deduce:

$$\partial_E \|\psi_E\|_{L^{2p+2}}^{2p+2} = \frac{p+1}{p} \|\psi_E\|_{L^2}^2$$

Assuming $\liminf_{E \rightarrow E_*} \frac{\|\psi_E\|_{L^{2p+2}}^{2p+2}}{\|\psi_E\|_{L^2}^2} > 0$ we get for some $\epsilon > 0$ on a small interval $[E_2, E_*)$:

$$\partial_E \|\psi_E\|_{L^{2p+2}}^{2p+2} \leq \frac{p+1}{\epsilon p} \|\psi_E\|_{L^{2p+2}}^{2p+2}$$

i.e. $\|\psi_E\|_{L^{2p+2}}$ is uniformly bounded which given the blow up of the L^2 norm contradicts the strict positivity of the above limit. Hence there exists

$$E_n \rightarrow E_* \text{ such that } \frac{\|\psi_{E_n}\|_{L^{2p+2}}^{2p+2}}{\|\psi_{E_n}\|_{L^2}^2} \rightarrow 0.$$

Then $u_n = \frac{\psi_{E_n}}{\|\psi_{E_n}\|_{L^2}}$ satisfies

$$\|\nabla u_n\|_{L^2}^2 + \int_{\mathbb{R}^n} V(x)|u_n|^2 dx \rightarrow -E_* \text{ and } \|u_n\|_{L^{2p+2}} \rightarrow 0.$$

Combined they give the contradiction $\|\nabla u_n\|_{L^2}^2 \rightarrow -E_* < 0$.

Theorem 2. compactness: *If $E_* < \infty$ and*

$$\limsup_{E \nearrow E_*} \|\psi_E\|_{H^1} < \infty$$

then for all $E_n \nearrow E_$ there exists a subsequence E_{n_k} and $\psi_{E_*} \in H^1$ such that*

- $\lim_{k \rightarrow \infty} \|\psi_{E_{n_k}} - \psi_{E_*}\|_{H^1} = 0$ and
- $F(\psi_{E_*}, E_*) = 0$.

Remark 1. *Consequently, local bifurcation theory applies at (ψ_{E_*}, E_*) and gives the local manifold structure of the zeroes of F .*

Sketch of Proof: Consider $E_n \nearrow E_*$ then $\|\psi_{E_n}\|_{H^1}$ is bounded. Pass to a subsequence to get $\psi_{E_n} \xrightarrow{H^1} \psi_{E_*}$ and *concentration compactness*:

- (compactness mod translations) $\psi_{E_n}(\cdot - y_n) \xrightarrow{L^p} \psi_{E_*}$, $2 \leq p < 2n/(n-2)$ or
- (vanishing) $\psi_{E_n} \xrightarrow{L^q} 0$, $2 < q < 2n/(n-2)$ or
- (splitting) $\psi_{E_n} = u_n + v_n + w_n$ with u_n in first case, $\|v_n\|_{L^2} \not\rightarrow 0$, the distance between the support of u_n and v_n goes to infinity, and $w_n \xrightarrow{L^q} 0$, $2 \leq q < 2n/(n-2)$.

Vanishing cannot occur: $\psi_{E_n} \xrightarrow{L^{2p+2}} 0$ implies both

$$|\psi_{E_n}|^{2p}\psi_{E_n} \xrightarrow{H^{-1}} 0 \quad \text{and} \quad V\psi_{E_n} \xrightarrow{H^{-1}} 0.$$

Consequently:

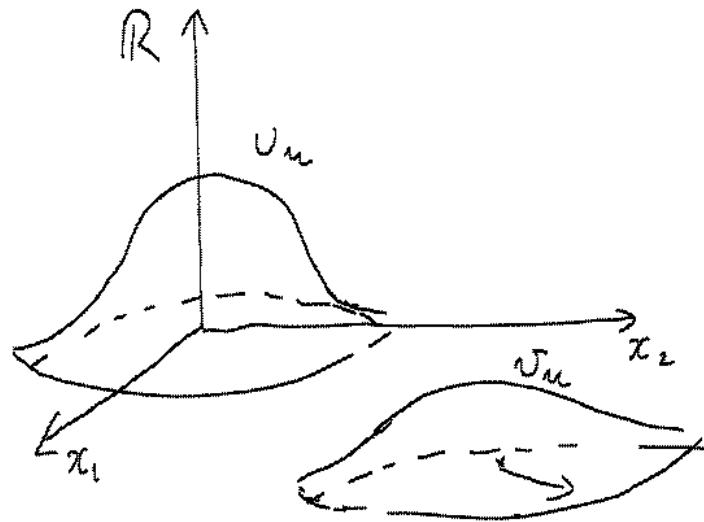
$$\psi_{E_n} = (-\Delta + E_n)^{-1}[-V\psi_{E_n} + |\psi_{E_n}|^{2p}\psi_{E_n}] \xrightarrow{H^1} 0$$

which contradicts uniqueness of trivial solution near $(0, E_*)$.

Splitting cannot occur: because we get at least 2 negative e-values for L_+ .

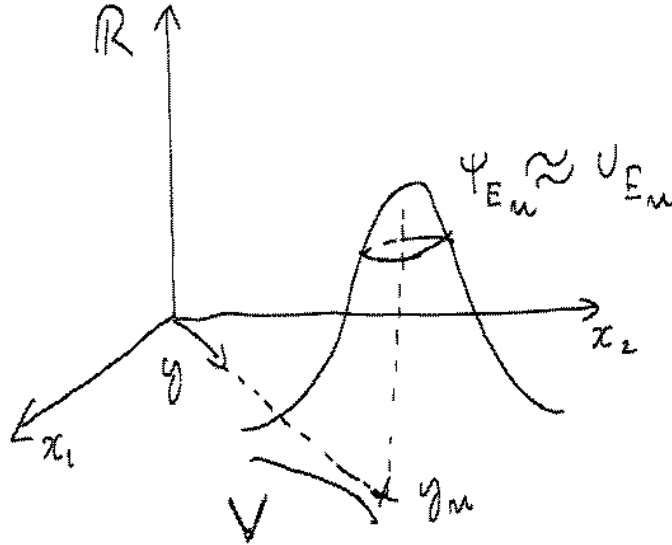
Indeed, at least one, say v_n , must drift to infinity (support condition). Then its profile converges to the (positive) solution of the problem without potential u_{E_*} . The latter has one negative e-value in the linearization. Same happens to u_n if it drifts and (by separation) each contributes one negative e-value to L_+ .

If u_n does not drift then it converges to a solution of the eq and again the linearization around u_n has one negative e-value.



All in all we must be in case 1: compactness modulo translations.
We still need to show:

Case 1: $|y_n| \not\rightarrow \infty$

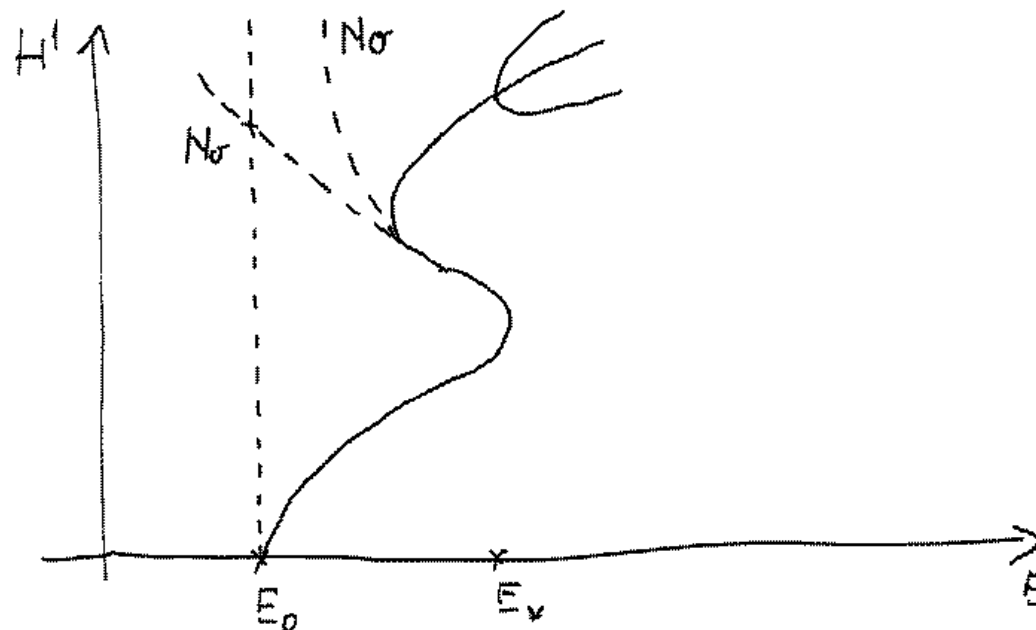


$$\langle \partial_y u_E, V u_E + V w_E + h.o.t. \rangle = 0$$

where $w_E \perp \ker L_+(u_E, E)$. Leads to contradiction if for large r :

$$|\partial_r V| > C r^{-\delta} \quad |\partial_r V| > C |\partial_{\theta_i} V|, \text{ and } \lim_{r \rightarrow \infty} V^2 / \partial_r V = 0$$

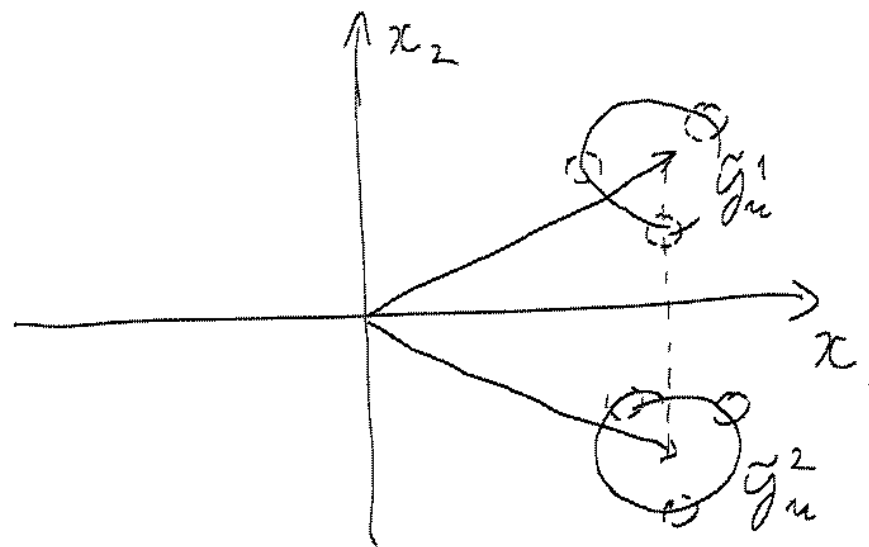
The theorem is now proven and we can go past the bifurcation point:



No crossing of the $E = E_0$ hyperplane because:

$$\langle (-\Delta + V)\psi_E, \psi_E \rangle = -E\|\psi_E\|_{L^2}^2 + \|\psi_E\|_{L^{2p+2}}^{2p+2} < -E_0\|\psi_E\|_{L^2}^2.$$

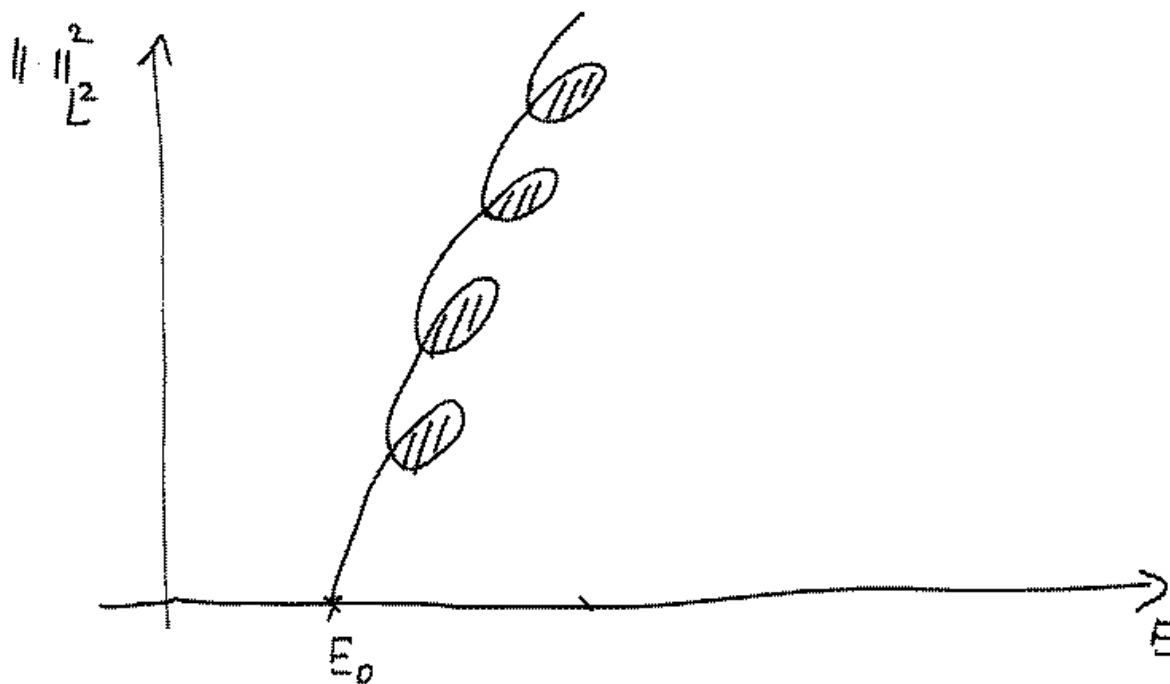
Second turning point needs new compactness argument to prevent splitting: requires grouping + configuration analysis to identify a nonzero leading term.



Compare $|\tilde{g}_n^1|$, $|\tilde{g}_n^2|$, $|\tilde{g}_n^1 - \tilde{g}_n^2|$

Now the branch MUST end up at $E = \infty$.

Except for this particular case in which the $\|\psi_E\|_{L^{2p+2}}$ cannot be controlled:



Theorem 3. *If $E \mapsto \psi_E$ is a C^1 curve of ground-states on the interval $E \in [E_1, \infty)$, and $x \cdot \nabla V \in L^\infty$ then*

$$\lim_{E \rightarrow \infty} \frac{\|\psi_E\|_{L^{2p+2}}^{2p+2}}{E^{1-n/2+1/p}} = b, \quad 0 < b < \infty$$

$$\lim_{E \rightarrow \infty} \frac{\|\psi_E\|_{L^2}^2}{E^{-n/2+1/p}} = \frac{(2-n)p+2}{2(p+1)} b$$

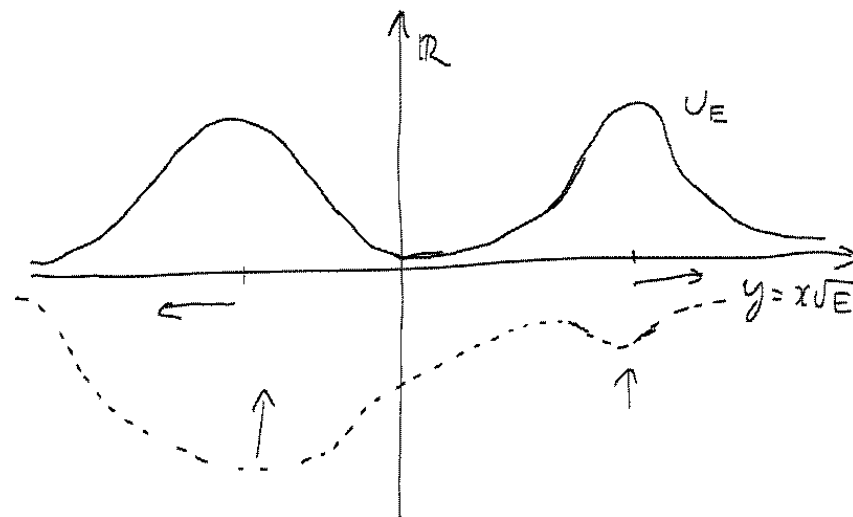
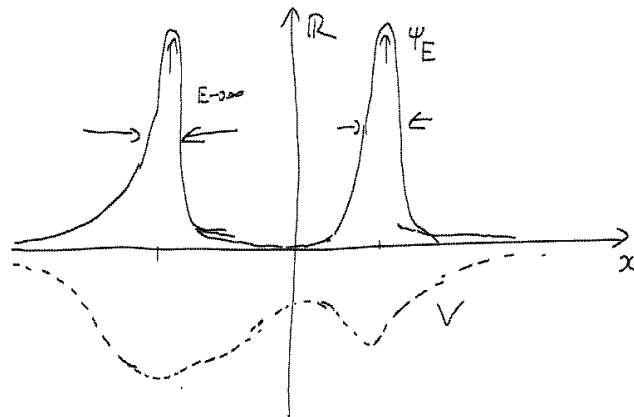
$$\lim_{E \rightarrow \infty} \frac{\|\nabla \psi_E\|_{L^2}^2}{E^{1-n/2+1/p}} = \frac{np}{2(p+1)} b$$

$$u_E(x) = E^{-1/(2p)} \psi_E(x/\sqrt{E}) \xrightarrow{H^1} u_\infty(x) = \sum_k u(x - x_k \sqrt{E}),$$

where u is the unique positive solution of

$$-\Delta u + u - |u|^{2p} u = 0 \tag{3}$$

and $\{x_k\}$ is a (nonempty) subset of critical points of the potential V .



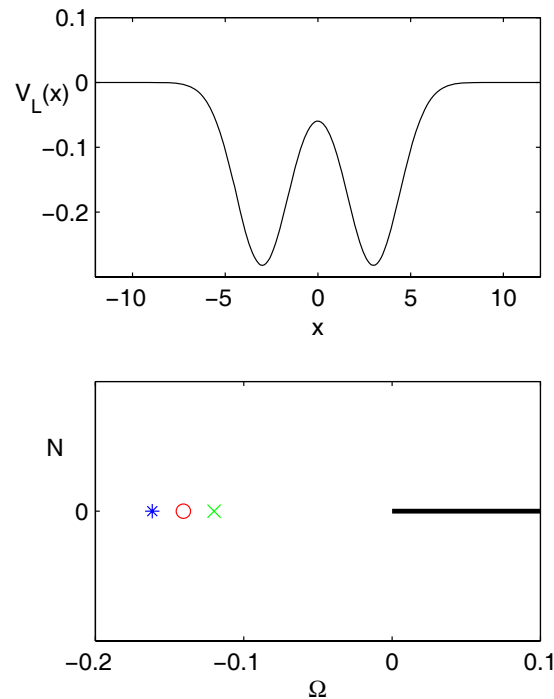
The behavior of the norms as $E \rightarrow \infty$ was anticipated by Rose-Weinstein '88.

Theorem 4. *If the potential V has only non-degenerate critical points then from each possible u_∞ defined as above bifurcates (via the above re-scaling) exactly one curve of zeroes of F . The number of negative e -values of L_+ can be calculated along each of these curves.*

The theorem is a much stronger version (includes uniqueness) of the results previously obtained in the semi-classical limit, see Oh '90 and Floer & Weinstein '86.

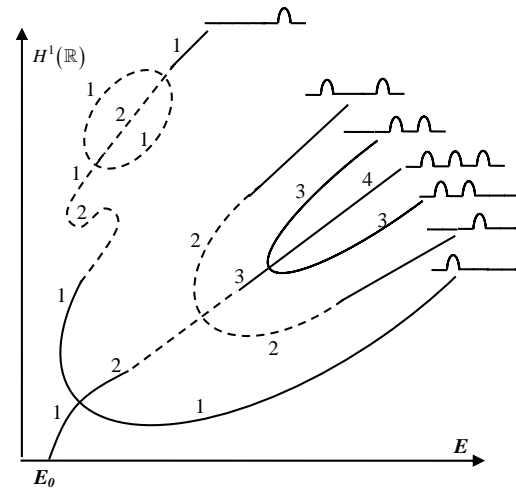
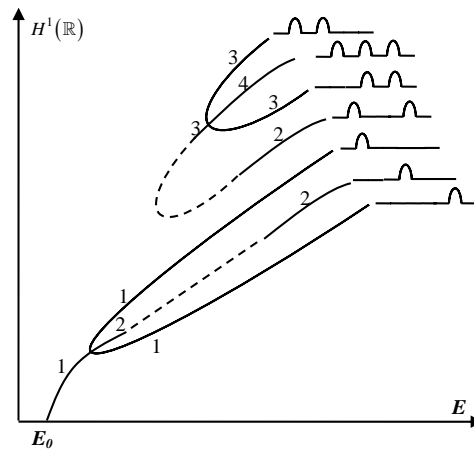
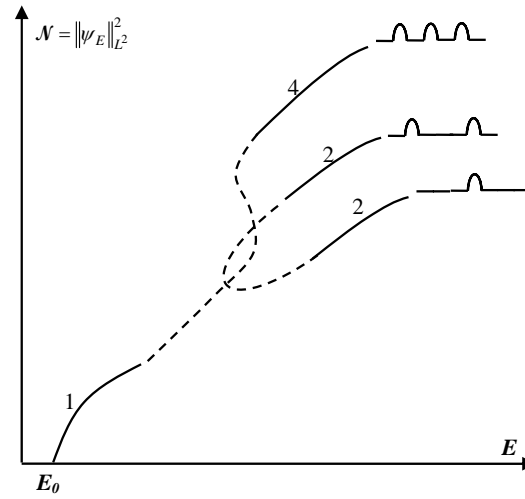
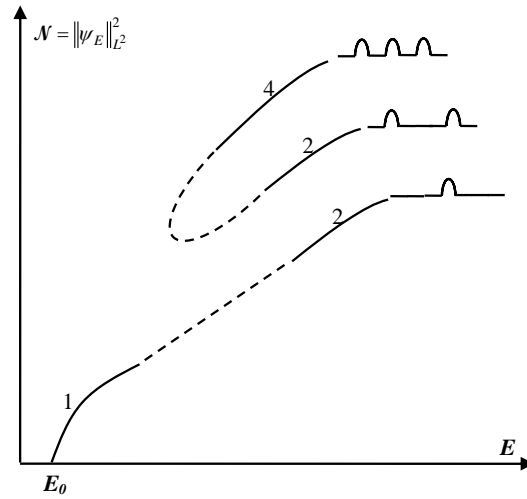
Still need to cover the case of more than one profile approaching the same maxima, see Yan-Noussair '00, Yan-Dancer '01.

An Example: $x \in \mathbb{R}^1$, symmetric double well potential



with $xV'(x) \in L^\infty(\mathbb{R})$, and p integer. Partial results appeared in Kirr-Kevrekidis-Pelinovsky '11.

All ground state manifolds



Conclusions:

1. The results have direct applications to rough ripplings (maybe even rough waves), and vortices in BEC.
2. We are on the verge of understanding the correlation between critical points of the potential and the bifurcations along the ground-state (and excited-state) manifolds. The missing link is a classification of possible bifurcations in higher dimensions when a multiple eigenvalue crosses zero.
3. Once all bound-state manifolds have been identified one can approach the asymptotic completeness conjecture in NLS by starting with the dynamics near the bifurcation points (very hard).

4. The technique is rather general for Hamiltonian PDE's, relying on energy estimates, analysis of the linearized operator, concentration compactness and properties of the limiting equation (as the parameter approaches a certain limit). Applications to rotating BEC's are underway. Other equations...

Thank you!