# Equidistribution of rational functions having a superattracting periodic point

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**3.5** min / **1** page

# A holomorphic family of rational functions, and its marked critical point

A holomorphic mapping $f : \Lambda \times \mathbb{P}^1 \to \mathbb{P}^1$, where $\Lambda$ is a connected complex manifold, is a *holomorphic family of rational functions of degree* $d \in \mathbb{N} \cup \{0\}$ over $\Lambda$ if for every $\lambda \in \Lambda$,

$$f_\lambda : \mathbb{P}^1 \to \mathbb{P}^1$$

defined by $f_\lambda(z) = f(\lambda, z) \in \mathbb{C}(z)$ is a rational function on $\mathbb{P}^1$ of degree $d$.
For every $\lambda \in \Lambda$ and every $n \in \mathbb{N}$, set $f_\lambda^n = f_\lambda \circ f_\lambda \circ \cdots \circ f_\lambda$ ($n$ times).

A holomorphic function $c : \Lambda \to \mathbb{P}^1$ is a *marked critical point* of the family $f$ if for every $\lambda \in \Lambda$, $c(\lambda)$ is a critical point of $f_\lambda$, i.e.,

$$f_\lambda'(c(\lambda)) = 0.$$

(In the following, we always assume $d > 1$.)

### Example

The unicritical (monic and centered) polynomial family (UPF) of degree $d$

$$P_\lambda(z) = z^d + \lambda \quad (\lambda \in \mathbb{C})$$

is regarded as a holomorphic family of rational functions over $\Lambda = \mathbb{C}$. Moreover, the constant mapping

$$c_0 \equiv 0 \quad \text{and} \quad c_1 \equiv \infty$$

from $\mathbb{C}$ to $\mathbb{P}^1$ are (all the) $2d - 2$ marked critical points of this family $P$ (taking into account their multiplicities).

# The activity locus of $c$, the bifurcation locus of $f$

Let $f, c$ as the above. For every $n \in \mathbb{N}$, define $F_n = F_n^{(f,c)} : \Lambda \to \mathbb{P}^1$ so that

$$F_n(\lambda) := f_\lambda^n(c(\lambda))$$

(so $F_0 \equiv c$). The *activity locus* of $c$ is

$$A_c := \{\lambda \in \Lambda : \{F_n : n \in \mathbb{N}\} \text{ is not equiconti. at } \lambda\},$$

where $\mathbb{P}^1$ is equipped with the chordal metric $[z, w]$.

Important Rem: Taking a finite and possibly ramified covering of $\Lambda$ if necessary, we can assume WLOG that *all* $2d - 2$ *critical points of* $f$ *are marked* in that there are $2d - 2$ marked critical points

$$c_1, \ldots, c_{2d-2} : \Lambda \to \mathbb{P}^1$$

of the family $f$ in that for every $\lambda \in \Lambda$, $c_1(\lambda), \ldots, c_{2d-2}(\lambda)$ are all the critical points of $f_\lambda$ taking into account their multiplicities.

The *bifurcation locus* of the family $f$ is

$$B_f := \Lambda \setminus (J\text{-stability locus of the family } f),$$

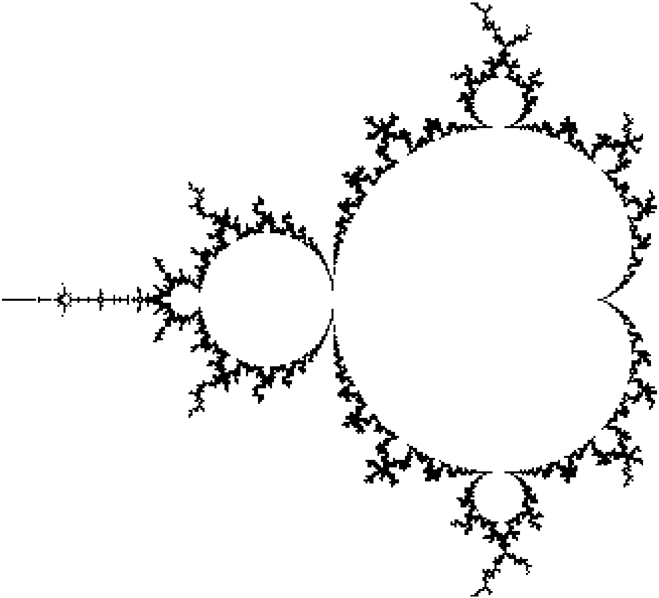
and if all $2d-2$ critical points of $f$ are marked in the above sense, then

$$B_f = \bigcup_{j=1}^{2d-2} A_{c_j}$$

(Mañé–Sad–Sullivan, Lyubich 1984).

**Example**

For the unicritical polynomial family $P_\lambda(z) = z^d + \lambda$, $F_n^{(P,c_0)}(\lambda) \in \mathbb{C}[\lambda]$ of degree $d^{n-1}$, $F_n^{(P,c_1)}(\lambda) \equiv \infty$, $A_{c_0}$ is an "interesting set", $A_{c_1} = \emptyset$, and $B_P = A_{c_0} \cup A_{c_1} = A_{c_1}$.



# The activity current of $c$, the bifurcation current of $f$

Let $\omega$ be the Fubini-Study area element on $\mathbb{P}^1$ s.t. $\omega(\mathbb{P}^1) = 1$. The limit

$$T_c := \lim_{n\to\infty} \frac{F_n^*\omega}{d^n} \quad \text{as } (1,1)\text{-currents on } \Lambda$$

exists, is called the *activity current* of $c$, and satisfies

$$\operatorname{supp} T_c = A_c.$$

The function $\lambda \mapsto L(f_\lambda) := \int_{\mathbb{P}^1} \log|f'_\lambda| d\mu_{f_\lambda}$ is psh on $\Lambda$, and the *bifurcation current* of the family $f$ is

$$T_f := dd^c_\lambda L(f_\lambda) = \sum_{j=1}^{2d-2} T_{c_j} \quad \text{as currents on } \Lambda$$

(DeMarco 2001, Dujardin–Favre 2008, where in the second equality, all $2d-2$ critical points of $f$ are marked), so $\operatorname{supp} T_f = \bigcup_{j=1}^{2d-2} \operatorname{supp} T_{c_j} = B_f$.

## (Pre)periodicity divisors of $c$ on $\Lambda$

Let $f, c, F_n$: as above. For every $n \in \mathbb{N}$ and every $k \in \mathbb{N}_{<n} \cup \{0\}$, set

$$\mathrm{PrePer}_c(n,k) := [F_n = F_k] \quad \text{on } \Lambda$$

(as a divisor or a $(1,1)$-current on $\Lambda$); moreover, for every $n \in \mathbb{N}$, set

$$\mathrm{Per}_c(n) := \mathrm{PrePer}_c(n,0), \quad \mathrm{Per}_c^*(n) := \sum_{m \in \mathbb{N} : m | n} \mu\left(\frac{n}{m}\right) \mathrm{Per}_c(m),$$

which are regarded as $(1,1)$-currents on $\Lambda$ and satisfy

$$\mathrm{supp}\,\mathrm{Per}_c(n) = \{\lambda \in \Lambda : c(\lambda) \in \mathrm{Fix}(f_\lambda^n)\} \quad \text{and}$$

$$\mathrm{supp}\,\mathrm{Per}_c^*(n) = \left\{\lambda \in \Lambda : c(\lambda) \in \mathrm{Fix}(f_\lambda^n) \setminus \bigcup_{m<n : m|n} \mathrm{Fix}(f_\lambda^m) =: \mathrm{Fix}^*(f_\lambda^n)\right\}.$$

# Main result

## Theorem (Ok 2014)

*Let* $f : \Lambda \times \mathbb{P}^1 \to \mathbb{P}^1$*: a holo. family of rat. functions of degree* $d > 1$ *and* $c : \Lambda \to \mathbb{P}^1$*: m.c.p. of* $f$*. Then*

$$\lim_{n\to\infty} \frac{\mathrm{Per}_c(n)}{d^n + 1} = T_c \quad \text{and} \quad \lim_{n\to\infty} \frac{\mathrm{Per}^*_c(n)}{d^n + 1} = T_c$$

*as currents on* $\Lambda$*.*

This is an approximation result on the activity current $T_c$ by the divisors $\mathrm{Per}_c(n)$ and $\mathrm{Per}^*_c(n)$.

Let us explain what is new, and an application of this Theorem to an approximation of the bifurcation current $T_f$ of the family $f$.

# What is new: a partial improvement of Dujardin–Favre's theorem

### Theorem (Dujardin–Favre 2008)

*Let $f : \Lambda \times \mathbb{P}^1 \to \mathbb{P}^1$: a holo. family of rat. functions of degree $d > 1$ and $c : \Lambda \to \mathbb{P}^1$: m.c.p. of $f$.*
*Assume (H): for every $\lambda \in \Lambda$, there is an immersed holomorphic curve $\gamma$ from some Riemann surface to $\Lambda$ s.t. for every $\lambda$ in an unbounded component of $\gamma \setminus A_c$ in $\gamma$, the critical orbit $(f_\lambda^n(c(\lambda)))$ tends to some (super)attracting cycle of $f_\lambda$.*
*Then for every $(k(n))$ in $\mathbb{N} \cup \{0\}$ satisfying $k(n) < n$ for every $n \in \mathbb{N}$,*

$$\lim_{n\to\infty} \frac{\mathrm{PrePer}_c(n, k(n))}{d^n + 1} = T_c \quad \text{as currents on } \Lambda.$$

So, it turns out that *the (seemingly technical) assumption (H) can be removed in the (most elementary) case that $k(n) \equiv 0$.*

# Motivating application: an improvement of Bassanelli–Berteloot's approximation of $T_f$

**Theorem (Bassanelli–Berteloot 2011)**

*Let $f : \Lambda \times \mathbb{P}^1 \to \mathbb{P}^1$: a holo. family of rat. functions of degree $d > 1$. Then (without any such assumption as (H),)*

$$\lim_{n\to\infty} \frac{\mathrm{Per}^*_f(n,0)}{d^n+1} = T_f \quad \text{as currents on } \Lambda.$$

*Here, for every $w \in \mathbb{C}$, $\mathrm{Per}^*_f(n,w) := [p^*_{f,n}(\lambda,w)]$ on $\Lambda$ is the divisor defined by the zeros of the multiplier polynomial $p^*_{f,n} : \lambda \times \mathbb{C} \to \mathbb{C}$.*

Indeed, *locally*, $\mathrm{Per}^*_f(n,0) = \sum_{j=1}^{2d-2} \mathrm{Per}^*_{c_j}(n)$ on $\Lambda$ for every $n \in \mathbb{N}$ *large enough*. Hence, our approximation of $T_c$ by $(\mathrm{Per}^*_c(n))$ with DeMarco's equality $T_f = \sum_{j=1}^{2d-2} T_{c_j}$ immediately yields the above Bassanelli–Berteloot's approximation of $T_f$ by $(\mathrm{Per}^*_f(n,0))$.

# (Outline of) the proof of the main result

**Lemma (a reduction due to Dujardin–Favre)**

*Let* $f : \Lambda \times \mathbb{P}^1 \to \mathbb{P}^1$*: a holo. family of rat. functions of degree* $d > 1$ *and* $c : \Lambda \to \mathbb{P}^1$*: m.c.p. of* $f$*. If (recalling* $F_n(\lambda) = f_\lambda^n(c(\lambda))$*,)*

$$\lim_{n\to\infty} \frac{\log[F_n, c]}{d^n + 1} = 0 \quad \text{in } L^1_{\mathrm{loc}}(\Lambda),$$

*then* $\lim_{n\to\infty} \mathrm{Per}_c(n)/(d^n + 1) = T_c$ *as currents on* $\Lambda$*.*

The convergence of $((\log[F_n, c])/(d^n + 1))$ can be shown *pointwisely* on $\Lambda$ using *Przytycki's lemma* (when $c(\lambda) \in J(f_\lambda)$) and the *classification of cyclic Fatou components* of $f_\lambda$ (when $c(\lambda) \in F(f_\lambda)$).

A *compactness principle* for loc. unif. upper bdd family of (p)sh functions on a domain in $\mathbb{R}^n$ automatically upgrades this pointwise convergence of $((\log[F_n, c])/(d^n + 1))$ to its $L^1_{\mathrm{loc}}$ convergence.

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