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RATIONAL CURVES ON ALGEBRAIC VARIETIES

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ABSTRACT. We describe some characterization of uniruled varieties, we define special families of rationla curves, we introduce the tangent map and we use it to characterize some special varieties. Finally we apply the above technique to the case of complex symplectic (projective) manifold.

1. Hilbert Scheme

Let X and Y be a normal projective schemes (of finite type over k algebraically closed field).

We denote with Hilb(X) the Hilbert scheme of proper subschemes of X; with Hom(X,Y) the open subscheme of Hilb$(X \times Y)$ of morphisms from X to Y (the construction of the schemes is due to Grothendieck and Mumford).

Theorem 1.1. *Let $f : X \longrightarrow Y$ be a morphism. Assume that X is without embedded points and that Y has no embedded points contained in $f(X)$ and the image of every irreducible component of X intersect the smooth locus of Y. Then*

- *The tangent space of $Hom(X,Y)$ at $[f]$ is naturally isomorphic to*
$$Hom_X(f^*{\Omega_Y}^1, \mathcal{O}_X).$$
- *The dimension of every irreducible component of $Hom(X,Y)$ at $[f]$ is at least*
$$dim Hom_X(f^*{\Omega_Y}^1, \mathcal{O}_X) - dim {Ext_X}^1(f^*{\Omega_Y}^1, \mathcal{O}_X).$$

Let $f : C \longrightarrow X$ be a morphism from a proper curve to a scheme and L a line bundle on X. We use the following notation to denote the intersection number of C and L:

$$C \cdot L := deg_C f^* L$$

In the special case of the Hilbert scheme of curves, thank to Riemann Roch theorem, we have the following spectacular result.

Theorem 1.2. *Let C be a proper algebraic curve without embedded points and $f : C \longrightarrow Y$ a morphism to a smooth variety Y of pure dimension n. Then*

$$dim_{[f]} Hom(C,Y) \geq -K_Y \cdot C + n\chi(\mathcal{O}_C).$$

Moreover equality holds if $H^1(C, f^*T_Y) = 0$.

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Proof. If F is a locally free sheaf on a scheme Z, then $\mathrm{Ext}^i_X(F, \mathcal{O}_Z) = H^i(Z, F^*)$. Therefore we have $dim_{[f]}\mathrm{Hom}(C,Y) \geq dim\mathrm{Hom}_C(f^*\Omega_Y{}^1, \mathcal{O}_C) - dim\mathrm{Ext}_C{}^1(f^*\Omega_Y{}^1, \mathcal{O}_C) = h^0(C, f^*T_Y) - h^1(C, f^*T_Y) = \chi(C, f^*T_Y) = degf^*T_Y + n\chi(\mathcal{O}_C) = -K_Y{\cdot}C + n\chi(\mathcal{O}_C)$.

2. Rational Curves

Definition 2.1. Let $\mathrm{Hom}_{bir}(\mathbb{P}^1, X) \subset \mathrm{Hom}(\mathbb{P}^1, X)$ be the open suscheme corresponding to those morphisms $f : \mathbb{P}^1 \longrightarrow X$ which are birational onto their image. f is birational onto its image iff f is an immersion at its generic point. This is an open condition. By the Lüroth theorem, every morphism $g : \mathbb{P}^1 \longrightarrow X$ is either constant or can be written as

$$\mathbb{P}^1 \xrightarrow{h} \mathbb{P}^1 \xrightarrow{f} X$$

where f is birational onto its image. Thus, at least set theoretically, $\mathrm{Hom}_{bir}(\mathbb{P}^1, X)$ contains all information about $\mathrm{Hom}(\mathbb{P}^1, X)$.
There is a universal morphism $F : Hom_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1 \longrightarrow X$ defined by $F(f,p) = f(p)$.

It is clear that the scheme $\mathrm{Hom}_{bir}(\mathbb{P}^1, X)$ is still too large for many purpose. If $f : \mathbb{P}^1 \longrightarrow X$ is any morphism and $h \in Aut(\mathbb{P}^1)$, then$f \circ h$ is counted as a different morphism. The group $Aut(\mathbb{P}^1)$ acts on $\mathrm{Hom}_{bir}(\mathbb{P}^1, X)$ an it is the quotient that "really parametrizes" morphisms of $\mathbb{P}^1$ into X. It can be proved that the quotient in the sense of Mumford (Mori-Mumford-Fogarty) exists; it will be called $RatCurves(X)$. Taking the normalizations we obtain a morphism

$$\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \longrightarrow RatCurves^n(X)$$

which has the structure of a $Aut(\mathbb{P}^1)$-bundle.
We obtain therefore a diagram as follows

(2.0.1)
$$\begin{array}{ccccc} \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1 & \xrightarrow{U} & Univ(X) & \xrightarrow{i} & X \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \\ \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) & \xrightarrow{u} & RatCurves^n(X) & & \end{array}$$

U and u have the structure of principal $Aut(\mathbb{P}^1)$-bundle; π is a $\mathbb{P}^1$-bundle. The composition $i \circ U$ is the universal morphism F in Definition 2.1 and the restriction of i to any fiber of π is generically injective, i.e. birational onto its image.
$RatCurves^n(X)$ is called the *space of rational curve on* X.

Given a point $x \in X$, one can similarly find a scheme $\mathrm{Hom}(\mathbb{P}^1, X, [0:1] \to x)$ whose geometric points correspond to generically injective morphisms from $\mathbb{P}^1$ to X which map the point $[0:1]$ to x. Also one can construct the quotient, in the sense of Mumford, by the group of automorphism of $\mathbb{P}^1$ which fixes the point $[0:1]$: this will be denoted by $RatCurves^n(x, X)$ and called the *space of rational curves through* x.

We obtain the diagram:
(2.0.2)

$$\begin{array}{ccccc} \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, [0:1] \to x) \times \mathbb{P}^1 & \xrightarrow{U} & Univ(x, X) & \xrightarrow{i_x} & X \\ \downarrow & & \downarrow \pi & & \\ \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, [0:1] \to x) & \xrightarrow{u} & RatCurves^n(x, X) & & \end{array}$$

2.1. **Existence of Rational Curves.** The following is a fundamental result of S. Mori ([Mo82]).

Theorem 2.2. *Let X be a smooth projective variety over an algebraically closed field (of any characteristic), C a smooth, projective and irreducible curve and $f : C \longrightarrow X$ a morphism. Assume that*

$$-K_X{\cdot}C > 0.$$

Then for every $x \in f(C)$ there is a rational curve $D_x \subset X$ containing x (and such for any nef $\mathbb{R}$-divisor L :

$$L{\cdot}D_x \le 2dimX(\frac{L{\cdot}C}{-K_X{\cdot}C}) \qquad and \qquad -K_X{\cdot}D_x \le dimX + 1.)$$

Idea of Proof. If C has genus 0, then we are done. Let $g := g(C) > 0$ and $n = dimX$. We have seen that

$$dim_{[f]}\mathrm{Hom}(C, Y) \ge -K_Y{\cdot}C + n(1 - g).$$

Take $x = f(0) \in f(C)$; since n conditions are required to fix the image of the basepoint 0 under f, morphisms f of C into X sending 0 to x have a deformation space of dimension

$$\ge -K_Y{\cdot}C + n(1 - g) - n = -K_Y{\cdot}C - ng.$$

Thus whenever this quantity is positive there must be a non-trivial one-parameter family of deformations of the map f keeping the image of 0 fixed.
In particular, we can find a nonsingular (affine) curve D and a morphism $g : C \times D \to X$, thought of as a nonconstant family of maps, *all sending* 0 *to the same point* x. We can argue here that D cannot be complete, otherwise the family would have to be constant (rigidity Lemma or Bend an Break I, see [KM98] Lemma 1.6). So let $D \subset \overline{D}$ be a completion where $\overline{D}$ is a nonsingular projective curve. Let $G : C \times \overline{D} \dashrightarrow X$ be the rational map. Blow up a finite number of points to resolve the undefined points to get $Y \to C \times \overline{D}$ whose composition given by $\pi : Y \to X$ is an honest morphism. Let $E \subset Y$ be the exceptional curve of the last blow up needed. Since it was actually needed, it can't be collapsed to a point, and hence $\pi(E)$ is our desired curve.

If $char(k) = p > 0$ it is easy to show that

$$-K_Y{\cdot}C - ng > 0.$$

In fact choose r large enough so that $-p^r(C_0{\cdot}K_X) \ge ng + 1$. Define $q = p^r$. Let $F : C \longrightarrow C$ be the q-th power, k-linear Frobenius map and denote $f : C \to X$ the composition. We only changed the structure sheaf and not the topological space, so C still has genus g. In this way we prove the existence of a rational curve through

x for almost all $p > 0$; then it holds also for any algebraically closed field k, by a general principle of Algebra.

Definition 2.3. A normal proper variety is called uniruled if it is covered by rational curves. The above Theorem proves that Fano manifolds are uniruled.

The following Theorem of Y. Miyaoka, which generalizes the Mori's result, is the most powerful uniruledness criteria.

Theorem 2.4. *Let X be a smooth and proper variety over $\mathbb{C}$. Then X is uniruled if and only if there is a quotient sheaf ${\Omega_X}^1 \longrightarrow F$ and a family of curves $\{C_t\}$ covering an open subset of X such that $F_{|C_t}$ is locally free and $deg(F_{|C_t}) < 0$ for every t.*

Remark 2.5. An irreducible component $\mathcal{V}$ of $RatCurves^n(X)$ is called a family of rational curve. Note that X is uniruled if and only if there exists a family of rational curve $\mathcal{V}$ such that $i : Univ(X) \longrightarrow X$ is dominant. (This follows from the fact that the irreducible components of $RatCurves^n(X)$ are numerable; which in turn follows from the fact that families of a given degree, with respect to a very ample line bundle, are finite, depending on the Hilbert polynomial). In this case we call $\mathcal{V}$ a unruling for X.

2.2. **Families of rational curves.** Let X be a smooth projective variety of dimension n and B a finite set of points. A rational curve $f : \mathbb{P}^1 \longrightarrow X$ is called free (over B) if $f^*TX \otimes I_B$ is generated by its global sections and $H^1(\mathbb{P}^1, f^*T(X) \otimes I_B) = 0$. By Grothendieck' s Theorem any vector bundle over $\mathbb{P}^1$ splits as a direct sum of line bundles, so free means $f^*TX \otimes I_B = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$, with $a_i \geq 0$ for every i (for more see [K095] II.3).
Free rational curves have a nice deformation theory because $H^1(\mathbb{P}^1, f^*T(X)) = 0$: this cohomology group contains the obstruction to realizing deformations of $f(C)$ from its infinitesimal deformations in $H^0(\mathbb{P}^1, f^*T(X))$.

Let $F : \mathbb{P}^1 \times Hom^n_{bir}(\mathbb{P}^1, X, B) \to X$ be the universal map in Definition 2.1, with $B = \{x\}$ or $\emptyset$.
Let $V \subset \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$, be an irreducible component and $[f] \in V$ a general element; let V_x be the set of elements in V passing through $x \in X$. We denote $Locus(V) := F(\mathbb{P}^1 \times V)$, where F is the universal morphism in Definition 2.1.. If V_x is the subfamily in $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, 0 \to x)$ then $Locus(V, 0 \to x) := F(\mathbb{P}^1 \times V_x)$, with $x \in X$.
The following follows from the first two Theorems of this note (1.1 and 1.2). Assume $char(k) = 0$ and $B = \emptyset$ (respectively $B = \{x\}$).
If $a_i \geq -1$ (resp. $a_i \geq 0$), then V (resp. V_x) is generically smooth.
Moreover

$$dimLocus(V) = rk(dF) \text{ at a generic point } x \in X = \sharp\{i : a_i \geq 0\},$$

and $dimV = dimX + \Sigma a_i$.
Similarly, for general $x \in X$, $dimLocus(V_x = \sharp\{i : a_i \geq 1\}$.
The above computations and observations can be used to prove the following.

Proposition 2.6. *(Assume $char(k) = 0$). X is uniruled if and only through a general point $x \in X$ there is a free rational curve.*

Theorem 2.7. *Let $\mathcal{V}$ be an irreducible component of $RatCurves^n(X)$. Denote by $\mathcal{V}^{free} \subset \mathcal{V}$ the parameter space of members of $\mathcal{V}$ that are free. Then $\mathcal{V}$ is a uniruling if and only if $\mathcal{V}^{free}$ is nonempty. In this case, $\mathcal{V}^{free}$ is a Zariski open subset of the smooth locus of $\mathcal{V}$.*

Given a uniruling $\mathcal{V}$ on X and a point $x \in X$, let $\mathcal{V}_x$ be the normalization of the subvariety of $\mathcal{V}$ parametrizing members of $\mathcal{V}$ passing through x. Since by the above Theorem non-free rational curves do not cover X, for general point $x \in X$, the structure of $\mathcal{V}_x$ is particularly nice ([K095] II.3.11):

Theorem 2.8. *For a uniruling $\mathcal{V}$ on a projective manifold X and a general point $x \in X$, all members of $\mathcal{V}_x$ belongs to $\mathcal{V}^{free}$. Furthermore, the variety $\mathcal{V}_x$ is a finite union of smooth quasi-projective varieties of dimension $deg_{K_X^{-1}}(\mathcal{V}) - 2$.*

Definition 2.9. A family of rational curve $\mathcal{V}$ on a projective manifold X is locally unsplit or unbreakable if $\mathcal{V}_x$ is projective for a general $x \in X$. Members of an unbreakable uniruling on X will be called minimal rational curves on X. (In [K095] IV.2.1 the definition is given for all points $x \in X$, not only general, and it is called unsplit family).

Unbreakable unirulings exist on any uniruled projective manifold. To see this, we need the following notion.

Definition 2.10. Let L be an ample line bundle on a projective manifold X. A uniruling $\mathcal{V}$ is a minimal with respect to L, if $deg_L(\mathcal{V})$ is minimal among all unirulings of X. A uniruling is a minimal uniruling if it is minimal with respect to some ample line bundle.

Proposition 2.11. *Minimal unirulings exist on any uniruled projective manifold. A minimal uniruling is unbreakable. In particular there exist unbreakable unirulings on any uniruled projective manifold.*

The geometric idea behind the unbreakability of minimal unirulings. Suppose for a uniruling $\mathcal{V}$, which is minimal with respect to an ample line bundle L, the variety $\mathcal{V}_x$ is not projective for a general point $x \in X$. Then the members of $\mathcal{V}_x$ degenerate to reducible curves all components of which are rational curves of smaller L-degree than the members of $\mathcal{V}$ and some components of which pass through x. Collecting those components passing through x as x varies over the general points of X gives rise to another uniruling $\mathcal{V}'$ satisfying $deg_L(\mathcal{V}') < deg_L(\mathcal{V})$, a contradiction to the minimality of $deg_L(\mathcal{V})$. This argument gives an intuitive picture behind the definition of an unbreakable uniruling: if a uniruling is not unbreakable, its members can be broken into members of another uniruling. More figuratively speaking, if a uniruling is not unbreakable, it can be broken into a smaller uniruling.
Mori shows that a weaker version of this property continues to hold for unbreakable unirulings:

Theorem 2.12. *Let $\mathcal{V}$ be an unbreakable family. Then for a general point $x \in X$ and any other point $y \in X$, there does not exist a positive-dimensional family of members of $\mathcal{V}$ that pass through both x and y.*
(This is the definition of generically unsplit family in [K095] *I.V.2.1. The Theorem is* [K095] *I.V.Proposition 2.3.)*

The theorem is proved again by a bend-and-break argument. Geometrically, it says that any 1-dimensional family of rational curves which share two distinct points

in common must degenerate into a reducible curve. This is the most important geometric property of an unbreakable uniruling.
Equivalently, let $V = u^{-1}(\mathcal{V})$ and $\Pi : V \to X \times X$ be the map $[f] \to (f(0), f(\infty))$, we have that the fiber of Π over the generic point of $Im\Pi$ has dimension at most one.

Proposition 2.13. *Let $V \subset Hom^n_{bir}(\mathbb{P}^1, X)$ such that $u^{-1}(u(V) = V$ (we say that V is closed under $Aut\mathbb{P}^1$)). If $\mathcal{V} := (u(V)$ is generically unsplit (i.e. satisfies the assumption of Theorem 2.12) and $x \in X$ is a general point in $locus(V)$, then*

$$dimV = dimLocus(V) + dimLocus(V, 0 \to x) + 1.$$

Proof. The proof is immediate from the semi continuity of the fiber dimension (see [K095] IV.2.5) □

Combining this Theorem with 1.2 we obtain the following result.

Corollary 2.14. *Let $\mathcal{V}$ be a generically unsplit family and let $V := u^{-1}(\mathcal{V})$. Then*

- $dimX + deg_{-K}V \leq dimLocus(V) + dimLocus(V, 0 \to x) + 1$
- $dimX + deg_{-K}V \leq 2dimLocus(V) + 1 \leq 2dimX + 1$
- $deg_{-K}V \leq dimLocus(V, 0 \to x) + 1 \leq dimX + 1$

An infinitesimal version of Theorem 2.12 is important for us. A key notion here is the following

Definition 2.15. A rational curve $C \subset X$ is unbending if under the normalization $v_C : \mathbb{P}^1 \longrightarrow C \subset X$, the vector bundle ${v_C}^*T(X)$ has the form

$${v_C}^*T(X) = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}$$

for some integer p satisfying $0 \leq p \leq n-1$, where $n = dimX$.
(This is the definition of Minimal free morphism in [K095] I.V.2.8.)

Theorem 2.16. *A general member of an unbreakable uniruling is unbending.*

Proof. (See [K095] 2.9, 2.10) Sketch of proof: Let $f] \in V \subset \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ a general element of an irreducible component V which is an unbreakable uniruling (i.e. $V = u^{-1}\mathcal{V}$ with $\mathcal{V}$ an unbreakable uniruling); let $f^*TX = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$. By assumption $a - i \geq 0$ for every i. Then

$$dimX + \Sigma a_i = dimV = dimLocus(V) + dimLocus(V, 0 \to x) + 1 = dimX + \sharp\{i : a_i \geq 1\} + 1.$$

Therefore $\Sigma a_i = \sharp\{i : a_i \geq 0\} + 1$, that is at most one of the a_i is at least two.

Remark 2.17. If $f : \mathbb{P}^1 \to C \subset X$ is an unbending member of $\mathcal{V}_x$ the differential $Tf : T(\mathbb{P}^1) \longrightarrow f^*T(X)$ is an isomorphism of $T(\mathbb{P}^1)$ and the unique $\mathcal{O}(2)$ summand. Therefore Tf_p is non zero at every $p \in \mathbb{P}^1$. Recall that a curve is *immersed* if its normalization has rank one at every point; therefore an unbending member is immersed.

3. Tangent Map

Let X be a smooth projective variety and $\mathcal{V} \subset RatCurves^n(X)$, a closed irreducible component; fix a point $x \in X$ and consider $\mathcal{V}_x$. Let t be a local coordinate around $0 \in \mathbb{P}^1$; take the derivative map $\Phi_x : \mathcal{V}_x \to P(T_xX) = P((f^*TX)_0)$ which is defined at $[f] \in \mathcal{V}_x$, if f is an immersion at 0, by $\Phi_x([f]) = [(Tf)_0(\partial/\partial t)]$, c.f. [Mori79,

pp.602-603]. In the formula $Tf : T\mathbb{P}^1 \to f^*TX$ is the tangent map and T_xX is identified naturally, via f^*, with $(f^*TX)_0$.
By P we denote the "natural projectivisation" (that is vector spaces modulo homotheties) in opposition to "Grothendieck projectivisation" (that is projective spectrum of the symmetric algebra of a vector space) which we denote by $\mathbb{P}$.

Definition 3.1. The rational map $\Phi_x : \mathcal{V}_x - - - - > P(T_xX)$ is called the tangent map.
It sends a member of $\mathcal{V}_x$ that is smooth (or an immersion at 0) to its tangent direction.

Theorem 3.2. *If $f : \mathbb{P}^1 \to C \subset X$ is an unbending member of $\mathcal{V}_x$, the tangent map above defined can be extended to $[f]$, even when C is singular at x, because the differential $Tf : T(\mathbb{P}^1) \longrightarrow f^*T(X)$ is injective. Moreover Φ_x : is immersive at $[f] \in \mathcal{V}_x$.*

In particular, Theorem 2.16 implies that for an unbreakable uniruling $\mathcal{V}$ and a general point $x \in X$, the tangent map Φ_x is generically finite over its image.
Although not all members of $\mathcal{V}_x$ are as nice as we would wish them to be, they are still considerably well behaved. Kebekus has carried out an in-depth analysis of singularities of members of $\mathcal{V}_x$. Among other things, he ([Keb02] [Theorem 3.3] has shown

Theorem 3.3. *For an unbreakable uniruling $\mathcal{V}$ and a general point $x \in X$, members of $\mathcal{V}_x$ which are singular are a finite number. Moreover the singular ones are immersed at the point corresponding to x.*

That is, given a morphism $v_C : \mathbb{P}^1 \longrightarrow C \subset X$ with a point $0 \in \mathbb{P}^1$ satisfying $v_C(0) = x$, the Theorem says that $(dv_C)_0 : T_0(\mathbb{P}^1) \longrightarrow T_x(X)$ is injective.
Using this, Kebekus has shown the following important result ([Keb02-2] [Theorem 3.4].

Theorem 3.4. *For an unbreakable uniruling $\mathcal{V}$ and a general point $x \in X$ (as in Theorem 3.3), the tangent morphism $\Phi_x : \mathcal{V}_x \longrightarrow P(T_xX)$ can be defined by assigning to each member C of $\mathcal{V}_x$ its tangent direction $\mathbb{P}((dv_C)(T_0(\mathbb{P}^1)) \in \mathbb{P}(T_x(X))$. This morphism Φ_x is finite over its image.*

Proof. Let i_x be the map in 2.0.2; by 3.3 the preimage $i_x^{-1}(x)$ contains a section, which we call $\sigma_\infty \simeq \mathcal{V}_x$, and at most a finite number of further points.
Since all curves are immersed at x, the tangent morphism of i_x gives a nowhere vanishing morphism of vector bundles,

$$Ti_x : T_{U_x|\mathcal{V}_x|\sigma_\infty} \to i_x^*(T_{X|x}).$$

The tangent morphism is then given by the projectivization of the above map. Assuming that Φ_x is not finite, then we can find a curve $C \subset \mathcal{V}_x$ such that N_{σ_∞, U_x} is trivial along C. But σ_∞ can be contracted and the normal bundle must be negative.

The next result was proved in a special case (when the tangent map is surjective) by Kebekus, in general it has been proved by Hwang and Mok [HM04].

Theorem 3.5. *For an unbreakable uniruling $\mathcal{V}$ and a general point $x \in X$ (as in Theorem 3.3), the tangent morphism $\Phi_x : \mathcal{V}_x \longrightarrow P(T_xX)$ is birational (i.e. generically injective) over its image.*

Definition 3.6. We define $S_x \subset P(T_xX)$ as the closure of the image of the map Φ_x and we call it *tangent cone of curves from V at the point x.*

J.-M.Hwang and N. Mok call this variety of minimal rational tangents. The name tangent cone follows from the fact that S_x is (at least around $[f]$) the tangent cone to Locus(V_x). Indeed, let $\pi : \widehat{X_x} \to X$ be the blow-up of X at x with the exceptional divisor $E_x = P(T_xX)$. Consider $\hat{f} : \mathbb{P}^1 \to \widehat{C} \subset \widehat{X_x}$, the lift-up of f; it exists for the universal property of the blow-up.
Then $\hat{f}^*(T\widehat{X_x}) = \mathcal{O}(2) \oplus \mathcal{O}^{\oplus(d)} \oplus \mathcal{O}(-1)^{\oplus(n-d-1)}$. Thus Hom$(\mathbb{P}^1, \widehat{X_x})$ is smooth at $[\hat{f}]$ and of dimension $d+3$. Moreover, by [K095], II.3.4, the evaluation morphism $\hat{F} : \mathrm{Hom}(\mathbb{P}^1, \widehat{X_x}) \times \mathbb{P}^1 \to \widehat{X_x}$ is an immersion along $[\hat{f}] \times \mathbb{P}^1$ and moreover, by definition, $\hat{F}([\hat{f}], 0) = \Phi_x([f])$.
On the other hand if we take an irreducible component $\hat{V}$ of Hom$(\mathbb{P}^1, \widehat{X_x})$ which contains $[\hat{f}]$ then Locus$(\hat{V}_x)$ outside of E_x coincides with a component of Locus(V_x). Thus around $\Phi_x([f])$ we get $S_x = E_x \cap \widetilde{\mathrm{Locus}(V_x)}$, with $\widetilde{\mathrm{Locus}(V_x)}$ denoting the strict transform of Locus(V_x), so S_x is the tangent cone to Locus(V_x).

For our purposes we need the following observation which follows from the above discussion.

Lemma 3.7. *The projectivised tangent space of the tangent cone S_x at $\Phi_x([f])$ is equal to $P((f^*TX)^+_0) \subset P((f^*TX)_0) = P(T_xX)$.*

Proof. By [Ko] II.3.4 the tangent space to Locus(V_x) at $f(p)$ for $p \neq 0$ is the image of the evaluation of sections of the twisted pull-back of TX which is $\mathrm{Im}(T\hat{F})_p = (f^*TX)^+_p \subset (f^*TX)_p = T_{f(p)}X$. Thus passing with p to 0 we get the result.

4. Characterization of $\mathbb{P}^n$

The following is the celebrated Theorem of Mori of 1979 ([Mo79].

Theorem 4.1. *Let X be a complex projective manifold of dimension $n \geq 3$. Assume that TX is ample. Then X is isomorphic to the projective space.*

The next Theorem was first proved by Cho-Miyaoka-Shepherd Barron; subsequently Kebekus gave a shorter proof in [Keb02-2]. Mori's Theorem follows nowdays immediately from it.

Theorem 4.2. *Let X be a complex projective manifold of dimension $n \geq 3$. Assume that for every curve $C \subset X$ we have $-K_X \cdot C \geq n+1$. Then X is isomorphic to the projective space.*

Proof. Take an unbreakable uniruling $\mathcal{V}$. By 2.8, and our assumption, for a general point $x \in X$ we have that $\mathcal{V}_x$ is smooth and $dim(\mathcal{V}_x) = (n-1)$.
By 3.4 and 3.5 and Zariski Main Theorem (birational morphism into a normal scheme has connected fibers), we have that $\mathcal{V}_x \simeq \sigma_\infty \simeq \mathbb{P}^{n-1}$. Let $\tilde{i}_x : \mathcal{V}_x \to \tilde{X} = Bl_xX$ be the lift up of i_x; since Ti_x has rank one along σ_∞, then $T\tilde{i}_x$ has maximal rank along σ_∞, in particular $N_{\sigma_\infty, U_x} \simeq N_{E/X} = O_{\mathbb{P}^{n-1}}(-1)$.
We conclude via an argument of Mori, see 3.2 in [Keb02-2].

The next generalization of 4.1 is due to Andreatta and Wisniewski [AW01].

Theorem 4.3. *Let X be a complex projective manifold of dimension $n \geq 3$. Assume that there exist a subsheaf $E \subset TX$ which is an ample vector bundle. Then X is isomorphic to the projective space.*

Proof. By the assumption that there exist a subsheaf $E \subset TX$ which is an ample vector bundle we can apply the Theorem 2.4 and therefore X is uniruled. Take an unbreakable uniruling $\mathcal{V}$.
For a general $f \in \mathcal{V}$ we have $f^*TX = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus d} \oplus \mathcal{O}^{\oplus(n-d-1)}$ where $d = deg(f^*(-K_X)) - 2$ (see 2.16).

Lemma 4.4. *For any $f \in \mathcal{V}$ the pull-back f^*E is isomorphic either to $\mathcal{O}(1)^{\oplus r}$ or to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r-1)}$. In particular the family of curves parametrized by V is unsplit.*

Proof. For a general $f \in \mathcal{V}$ the pull-back f^*E is an ample subbundle of $f^*TX = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(d)} \oplus \mathcal{O}^{\oplus(n-d-1)}$ and thus it is as in the lemma. Since E is ample this is true also for all $f \in V$. Since $deg(f^*E) = r$ or $deg(f^*E) = r + 1$ and $r > 1$, and for any ample bundle $\mathcal{E}$ over a rational curve we have $deg(\mathcal{E}) \geq rank(\mathcal{E})$, it follows that no curve from V can be split into a sum of two or more rational curves, hence V is unsplit. □

The family $\mathcal{V}$ defines a relation of *rational connectedness with respect to* $\mathcal{V}$, which we shall call rc$\mathcal{V}$ relation for short, in the following way: $x_1,\ x_2 \in X$ are in the rc$\mathcal{V}$ relation if there exists a chain of rational curves parametrized by morphisms from $\mathcal{V}$ which joins x_1 and x_2. The rc$\mathcal{V}$ relation is an equivalence relation and its equivalence classes can be parametrized generically by an algebraic set. More precisely, we have the following result due to Campana and, independently, to Kollár-Miyaoka-Mori.

Theorem 4.5. *(see* [K095]*, IV.4.16). There exist an open subset $X^0 \subset X$ and a proper surjective morphism with connected fibers $\varphi^0 : X^0 \to Z^0$ onto a normal variety, such that the fibers of φ^0 are equivalence classes of the rc$\mathcal{V}$ relation.*

We shall call the morphism φ^0 an rc$\mathcal{V}$ fibration. If Z_0 is just a point then we will call X a rationally connected manifold with the respect to the family $\mathcal{V}$, in short an rc$\mathcal{V}$ manifold.

Lemma 4.6. *Let X be a manifold which is rationally connected with the respect to a unsplit family $\mathcal{V}$. Then $\rho(X) := dimN_1(X) = 1$ and X is a Fano manifold.*

Also in this case the proof is a sort of an (easy) bend and break lemma.

We shall analyze X using the notions of rc$\mathcal{V}$ relation and rc$\mathcal{V}$ fibration. The following is a key observation.

Lemma 4.7. *Let X, E and $\mathcal{V}$ be as above and moreover assume that $\varphi^0 : X^0 \to Z^0$ is an rc$\mathcal{V}$ fibration. Then E is tangent to a general fiber of φ^0. That is, if X_g is a general fiber of φ^0, then the injection $E_{|X_g} \to TX_{|X_g}$ factors via $E_{|X_g} \hookrightarrow TX_g$.*

Proof. Choose a general X_g (in particular smooth) and let moreover $x \in X_g$ and $f \in \mathcal{V}_x$ be general as well. Then $Locus(\mathcal{V}_x) \subset X_g$. By [K095] II.3.4 the tangent space to $Locus(\mathcal{V}_x)$ at $f(p)$ is the image of the evaluation of sections of the twisted pull-back of TX, which is $= (f^*TX)^+_p$, therefore $(f^*TX)^+_p \subset (f^*TX_g)_p$ for every $p \in \mathbb{P}^1 \setminus \{0\}$. This implies that $E_{|X_g} \to TX_{|X_g}$ factors to $E_{|X_g} \to TX_g$ generically and since the map $TX_g \to TX_{|X_g}$ has cokernel which is torsion free (it is the normal sheaf which is locally free) this yields $E_{|X_g} \hookrightarrow TX_g$, a sheaf injection.

Proposition 4.8. *The general fiber of φ^0, X_g, is $\mathbb{P}^k$ and $E_{|X_g} = \mathcal{O}(1)^{\oplus r}$ or $E_{|X_g} = TX_g$.*

Proof. By abuse during the proof we denote the general fiber with $X := X_g$. We consider here only the case when for $f \in \mathcal{V}$ the pull-back f^*E is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r-1)}$. In particular $f^*E \subset (f^*TX)^+$
Comparing the splitting type of f^*E and f^*TX we see that the tangent map $Tf : T\mathbb{P}^1 \to f^*TX$ factors to a vector bundle (nowhere degenerate) injection $T\mathbb{P}^1 \to f^*E$. (In other words, we have surjective morphism $(f^*E)^* \to \Omega_{\mathbb{P}^1} \simeq \mathcal{O}(-2)$.
The vector bundle (nowhere degenerate) injection $T\mathbb{P}^1 \to f^*E$ implies $(f^*TX)^+ \hookrightarrow f^*E$. In fact, choose a general f which is an immersion at $0 \to x$. Then $\Phi_x([f]) \in P(E_x) = P((f^*E)_0) \subset P(T_xX) = P((f^*TX)_0)$ and the same holds for morphisms in a neighborhood of $[f]$ in V_x. Thus around $\Phi_x([f])$ the tangent cone S_x is contained in $P(E_x) = P((f^*E)_0)$, so is its tangent space $P((f^*TX)_0^+)$ (see 3.7).
Thereforet $f^*E = (f^*TX)^+$ and thus $\deg(f^*E) = \deg(f^*(-K_X))$. Since $\rho(X) = 1$ it follows that $\det(E) = -K_X$.
The embedding $E \hookrightarrow TX$ gives rise to a non-trivial morphism $\det(E) \to \Lambda^r TX$ and thus to a non-zero section of $\Lambda^r TX \otimes K_X$. We use dualities to have the equalities:

$$h^0(X, \Lambda^r TX \otimes K_X) = h^n(X, \Omega^r_X) = h^r(X, \Omega^n_X) = h^r(X, K_X) = h^{n-r}(X, \emptyset_X)$$

and, since X is Fano, the latter number is non-zero only if $r = n$. Thus $\Lambda^r TX \otimes (\det E)^{-1} \simeq \mathcal{O}_X$ so $E \hookrightarrow TX$ is nowhere degenerate, hence an isomorphism. We conclude by Theorem 4.1.

We conclude proving that $dimZ_0$ is zero , i.e. X is rationally connected. By contradiction if $dimZ_0 \geq 1$ in [AW01] we proved that :

Lemma 4.9. *Outside a subset of codimension ≥ 2 the morphism φ_0 is a $\mathbb{P}^k$-bundle (in the analytic topology).*

Then we take a complete curve $B \subset Z_0$ and we consider the $\mathbb{P}^k$ bundle $f^* : X_B \to B$ and the ample vector bundle $E_{|X_B}$.
We get a contradiction applying the following result, which is due to Campana and Peternell.

Lemma 4.10. *Let X be a n-dimensional projective manifold, $\varphi : X \to Y$ a $\mathbb{P}^k$ bundle $(k < n)$ of the form $X = \mathbb{P}(V)$ with a vector bundle V on Y. Then the relative tangent sheaf $T_{X/Y}$ does not contain an ample locally free subsheaf*

5. Rational Curves on Symplectic varieties

5.1. **Symplectic contractions.** A holomorphic 2-form ω on a smooth variety is called **symplectic** if it is closed and non-degenerate at every point. A **symplectic variety** is a normal variety Y whose smooth part admits a holomorphic symplectic form ω_Y such that its pull back to any resolution $\pi : X \to Y$ extends to a holomorphic 2-form ω_X on X. We call π a **symplectic resolution** if ω_X is non degenerate on X, i.e. it is a symplectic form. More generally, a map $\pi : X \to Y$ is called a **symplectic contraction** if X is a symplectic manifold, Y is normal and π is a birational projective morphism. If moreover Y is affine we will call $\pi : X \to Y$ a **local symplectic contraction** or **local symplectic resolution**. The following facts are well known.

Proposition 5.1. *Let Y be a symplectic variety and $\pi : X \to Y$ be a resolution. Then the following statement are equivalent: (i) $\pi^* K_Y = K_X$, (ii) π is symplectic, (iii) K_X is trivial, (iv) for every symplectic form on Y_{reg} its pull-back extends to a symplectic form on X.*
Moreover by the Grauert Riemeschneider Teorem $R^i\pi_*\mathcal{O}_X = 0$ for all positve i: in particular T has rational singularities. Note that Y is Gorenstein and K_Y is trivial.

Corollary 5.2. *All exceptional fibers of π are uniruled.*

Theorem 5.3. *Let $\pi : X \to Y$ be a symplectic resolution with $dimX = 2n$. Let also $f : \mathbb{P}^1 \to X$ be a non constant morphism such that $f(\mathbb{P}^1)$ is an f-exceptional curve. Then f deforms in a family of dimension at least $2n + 1$. In other words the Chow variety of X has dimension at least $2n - 2$ at the point corresponding to $f(\mathbb{P}^1)$.*

Proof. The Theorem was proved by Z. Ran [Ra95] in the case X is projective. In general it was proved by J. Wierzba [Wie03].
Roughly speaking Wierza proves first that there exists a first order symplectic deformation of X, which stays in an unobstructed deformation χ, such that all deformations of f stay in X. After showing that all the pertinent deformations are "represented" by algebraic spaces, he shows that $g : \mathbb{P}^1 \to X \subset \chi$ deform in a family of dimension (Mori)

$$dim_g Hom(\mathbb{P}^1, \chi) = \chi(\mathbb{P}^1, g^*T_\chi) \geq dim\chi - degf^*K_\chi \geq 2n + 1.$$

Since all the deformation of f stays in X then $dim\ Hom_f(\mathbb{P}^1, X) = dim\ Hom_g(\mathbb{P}^1, \chi)$ and we are done. □

It follows from this last result the following Theorem.

Theorem 5.4. *A symplectic resolution $\pi : X \to Y$ is semismall, that is for every closed subvariety $Z \subset X$ we have $2\operatorname{codim} Z \geq \operatorname{codim} \pi(Z)$. If equality holds Z then is called a maximal cycle.*

Proof. Sketch: let $F \subset X$ be a generic fiber of $Z \to \pi(Z)$, let also $d = dimZ$ and $e = dim(\pi(Z))$. We know that all exceptional fibers are uniruled (5.2); take then $\mathcal{V}$ be a generically unsplit family which covers F as in 2.12 and let $V := u^{-1}(\mathcal{V})$. Then by 2.13 we have

$$dimV = dimLocus(V) + dimLocus(V, 0 \to x) + 1 \leq 2dimF + 1 = 2d - 2e + 1.$$

Let $f : \mathbb{P}^1 \longrightarrow F$ be an rational curve in V; since $f(\mathbb{P}^1)$ gets contracted under π, all its deformation in X stay in the exceptional set and we may assume that all small deformations stay in Z. Therefore

$$dim_{[f]}\mathrm{Hom}(\mathbb{P}^1, X) = dim_{[f]}\mathrm{Hom}(\mathbb{P}^1, Z) = dim_{[f]}\mathrm{Hom}(\mathbb{P}^1, F) + e \leq 2d - e + 1.$$

By the above Theorem 5.3 we have on the other hand that $dim_{[f]}\mathrm{Hom}(\mathbb{P}^1, X) \geq 2n + 1$ and the Theorem follows. □

Corollary 5.5. *Any two symplectic resolutions $\pi_i : X_i \to Y$, where $i = 1,\ 2$, are isomorphic in codimension 1.*

Proof. By 5.4 every exceptional divisor of a symplectic resolution π_i is mapped to a codimension 2 set in Y. On the other hand, any symplectic resolution of Y is uniquely determined in codimension 2 as the resolution of the surface Du Val singularities. □

6. Local symplectic contractions in dimension 4.

6.1. **MDS structure.** In this section $\pi : X \to Y$ is a local symplectic contraction, as defined in 5.1, and $\dim X = 4$. By the semismall property (see Theorem 5.4), the fibers of π have dimension less or equal to 2. We will denote with 0 the unique (up to shrinking Y to a smaller affine set) point such that $\dim \pi^{-1}(0) = 2$. If π is divisorial then the general non trivial fiber has dimension 1.
By $N_1(X/Y)$ we denote the $\mathbb{Q}$ vector space of 1-cycles proper over Y, modulo numerical equivalence (c.f. [KM98, Ex. 2.16]). Then $N_1(X/Y)$ and $N^1(X/Y)$ are dual via the intersection pairing. Since $R^i\pi_*\mathcal{O}_X = 0$ for $i > 0$, it follows that $N^1(X/Y)$ is a finite dimensional vector space.
We start by recalling the following theorem of Wierzba-Wisniewski. It is a sort of *relative* characterization of the projective space: the hard part is to prove that the two dimensional fiber is normal, then the proof is as in Section 4.

Theorem 6.1. *Suppose that π is small (i.e. it does not contract any divisor). Then π is locally analytically isomorphic to the collapsing of the zero section in the cotangent bundle of $\mathbb{P}^2$. Therefore X admits a Mukai flop*

The above theorem, together with Matsuki's termination of 4-dimensional flops is the key ingredient in the proof of the following result.

Theorem 6.2. *Let $\pi : X \to Y$ be a 4-dimensional local symplectic contraction and let $\pi^{-1}(0)$ be its only 2-dimensional fiber. Then X is a Mori Dream Space over Y. Moreover any SQM model of X over Y is smooth and any two of them are connected by a finite sequence of Mukai flops whose centers are over $0 \in Y$. In particular, there are only finitely many non isomorphic (local) symplectic resolutions of Y,.*

7. Rational curves and differential forms

7.1. **The set-up.** Our starting point is the paper of Wierzba [Wie03] . In particular Theorem 1.3 of [Wie03] says that a general fiber of π over any component of S is a configuration of $\mathbb{P}^1$'s with dual graph being a Dynkin diagram.
Choose an irreducible component of S, call it S'. Take an irreducible curve $C \simeq \mathbb{P}^1$ in a (general) fiber over a point in $S' \setminus \{0\}$ and let D' be the irreducible component of D which contains C; note that $\pi(D') = S'$ and S' may be (and usually is) non-normal. Let $\mathcal{V}' \subset Chow(X/Y)$ be an irreducible component of the Chow scheme

of X containing C. By $\mathcal{V}$ we denote its normalization and $p : \mathcal{U} \to \mathcal{V}$ is the normalized pullback of the universal family over $\mathcal{V}'$. Finally, let $q : \mathcal{U} \to D' \subset X$ be the evaluation map. The contraction π determines a morphism $\tilde{\pi} : \mathcal{V} \to S'$, which is surjective because C was chosen in a general fiber over S'. We let $\mu : \mathcal{V} \to \widetilde{S}' \to S'$ be its Stein factorization. In particular $\widetilde{S}'$ is normal and $\nu : \widetilde{S}' \to S'$ is a finite morphism, ètale outside $\nu^{-1}(0)$, whose fibers are related to the orbits of the action of the group of automorphism of the Dynkin diagram, [Wie03, 1.3]. We will assume that μ is not an isomorphism which is equivalent to say that D' has a 2-dimensional fiber over 0. Also, since we are interested in understanding the local description of the contraction in analytic category we will assume that S' is analytically irreducible at 0 or that $\nu^{-1}(0)$ consists of single point. The exceptional locus of μ is $\mu^{-1}(\nu^{-1}(0)) = \bigcup_i V_i$ where $V_i \subset \mathcal{V}$ are irreducible curves.

(7.1.3)
$$\begin{array}{ccccc} \mathcal{U} & & \xrightarrow{\quad q \quad} & & D' \subset X \\ \downarrow{\scriptstyle p} & & & & \downarrow{\scriptstyle \pi} \\ \mathcal{V} & \xrightarrow{\mu} & \widetilde{S}' & \xrightarrow{\nu} & S' \subset Y \end{array}$$

If necessary, we can take $\mathcal{V}$ to be smooth, eventually by replacing it with its desingularization and $\mathcal{U}$ with the normalized fiber product.

7.2. **The differentials.** Let us consider the derivative map $Dq : q^*\Omega_X \to \Omega_{\mathcal{U}}$. We have another derivation map into $\Omega_{\mathcal{U}}$, namely $Dp : p^*\Omega_{\mathcal{V}} \to \Omega_{\mathcal{U}}$. It fits in the exact sequence

(7.2.4)
$$p^*\Omega_{\mathcal{V}} \longrightarrow \Omega_{\mathcal{U}} \longrightarrow \Omega_{\mathcal{U}/\mathcal{V}} \longrightarrow 0,$$

whose dual sequence is

(7.2.5)
$$0 \longrightarrow T_{\mathcal{U}/\mathcal{V}} \longrightarrow T_{\mathcal{U}} \longrightarrow p^*T_{\mathcal{V}}$$

The symplectic form on X, that is ω_X, gives an isomorphism $\omega_X : T_X \to \Omega_X$. We consider the following diagram involving morphism of sheaves over $\mathcal{U}$ appearing in the above sequences.

(7.2.6)
$$\begin{array}{ccccccc} T_{\mathcal{U}/\mathcal{V}} & \longrightarrow & T_{\mathcal{U}} & \xrightarrow{(Dp)^*} & p^*(T_{\mathcal{V}}) & \overset{p^*(\omega_{\mathcal{V}})}{\dashrightarrow} & p^*(\Omega_{\mathcal{V}}) & & \\ & & \downarrow{\scriptstyle (Dq)^*} & & & & \downarrow{\scriptstyle Dp} & & \\ & & q^*T_X & \xrightarrow[q^*(\omega_X)]{} & q^*\Omega_X & \xrightarrow{Dq} & \Omega_{\mathcal{U}} & \longrightarrow & \Omega_{\mathcal{U}/\mathcal{V}} \end{array}$$

We claim that the dotted arrow exists and it is obtained by a pull back of a two form $\omega_{\mathcal{V}}$ on $\mathcal{V}$, and it is an isomorphism outside the exceptional set of μ which is $\bigcup_i V_i$.

Indeed, the composition of arrows in the diagram which yields $T_{\mathcal{U}} \to \Omega_{\mathcal{U}}$ is given by the 2-form $Dq(\omega_X)$; it is zero on $T_{\mathcal{U}/\mathcal{V}} \subset T_{\mathcal{U}}$, because this is a torsion free sheaf and its restriction to any fiber of p outside $\bigcup_i V_i$ (any fiber of p is there a $\mathbb{P}^1$) is $\mathcal{O}(2)$ while the restriction of $\Omega_{\mathcal{U}}$ is $\mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O}$. Therefore we have that it is in fact a map $p^*T_{\mathcal{V}} \to \Omega_{\mathcal{U}}$.

By the same reason the composition $T_{\mathcal{U}} \to \Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}/\mathcal{V}}$ is zero since $T_{\mathcal{U}}$ on any fiber of p outside $\bigcup_i V_i$ is $\mathcal{O}(2) \oplus \mathcal{O} \oplus \mathcal{O}$ while $\Omega_{\mathcal{U}/\mathcal{V}}$ is $\mathcal{O}(-2)$. Thus the map $Dq(\omega_X) : p^*T_{\mathcal{V}} \to \Omega_{\mathcal{U}}$ factors through $p^*(T_{\mathcal{V}}) \to p^*(\Omega_{\mathcal{V}})$.
As a result, since it is trivial on the fiber of p, $Dq(\omega_X) = Dp(\omega_{\mathcal{V}})$, for some 2-form $\omega_{\mathcal{V}}$ on $\mathcal{V}$.
Since Dq is of maximal rank outside of $p^{-1}(\bigcup_i V_i)$ and p is just a $\mathbb{P}^1$-bundle there, it follows that $\omega_{\mathcal{V}}$ does not assume zero outside the exceptional set of μ. Hence $K_{\mathcal{V}} = \sum a_i V_i$, with $a_i \geq 0$ being the discrepancy of V_i.
We have proved the following.

Theorem 7.1. *The surface $\widetilde{S}'$ has at most Du Val (or $\mathbb{A} - \mathbb{D} - \mathbb{E}$) singularity at $\nu^{-1}(0)$ and $\mu : \mathcal{V} \to \widetilde{S}'$ is its, possibly non-minimal, resolution. In particular every V_i is a rational curve.*

We note that although the surface $\widetilde{S}'$ is the same for all the symplectic resolutions of Y, the parametric scheme for lines, which is a resolution of $\widetilde{S}'$ may be different for different SQM models, see 8.3 for an explicit example.

8. Quotient symplectic singularities

Example 8.1. Let S be a smooth surface (proper or not). Denote by $S^{(n)}$ the *symmetric product* of S, that is $S^{(n)} = S^n/\sigma_n$, where σ_n is the symmetric group of peremutations of n elements. Let also $Hilb^n(S)$ be the *Hilbert scheme* of 0-cycles of degree n. A classical result (c.f. [**?**]) says that $Hilb^n(S)$ is smooth and that $\tau : Hilb^n(S) \to S^{(n)}$ is a crepant resolution of singularities. We will call it a Hilb-Chow map.
Suppose now that $S \to S'$ is a resolution of a Du Val singularity which is of type $S' = \mathbb{C}^2/H$ with $H < SL(2, \mathbb{C})$ a finite group. Then the composition $Hilb^n(S) \to S^{(n)} \to (S')^{(n)}$ is a local symplectic contraction.
We note that $(S')^{(n)}$ is a quotient singularity with respect to the action of the wreath product $H \wr \sigma_n = (H^n) \rtimes \sigma_n$ (the group σ_n permutes factors in $H^n = H^{\times n}$).

8.1. **Preliminaries.** In this section $G < Sp(\mathbb{C}^4) =: Sp(4)$ is a finite subgroup preserving a symplectic form. We will discuss some examples in which $Y := \mathbb{C}^4/G$ admits a symplectic resolution $\pi : X \to Y$.

8.2. **Direct product resolution.** Let $H_1, H_2 < SL(2)$ be finite subgroups and consider $G := H_1 \times H_2$ acting on $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$. Let $\pi_i : S_i \to \mathbb{C}^2/H_i$ be minimal resolutions and $n_i = |H_i| - 1$ be the number of exceptional rational curves in S_i. The product morphism $\pi = \pi_1 \times \pi_2 : X := S_1 \times S_2 \to Y := \mathbb{C}^4/G$ is a symplectic resolution with the central fiber isomorphic to the product of the exceptional loci of π_i. In particular X does not admit any flop and $\mathrm{Mov}(X/Y) = \mathrm{Nef}(X/Y)$. Every component of $Chow(X/Y)$ containing an exceptional curve of π_i is isomorphic to S_j, with $i \neq j \in \{1, 2\}$.

8.3. **Elementary contraction to $\mathbb{C}^4/\sigma_3$.** Let σ_3 be a group of permutation of 3 elements; σ_3 acts on $\mathbb{C}^2$ via the standard representation. Let σ_3 acts on $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$ as the diagonal action of the standard representation. A symplectic resolution of the quotient $\mathbb{C}^4/\sigma_3$ can be obtained as a section of the Hilbert-Chow morphism $\tau : Hilb^3(\mathbb{C}^2) \to (\mathbb{C}^2)^{(3)}$ in 8.1. This is a local version of Beauville's construction.

There are three conjugacy classes in σ_3 which are related to three maximal cycles, of complex dimension 4, 3 and 2, each related to a 1-dimensional group of homology for the resolution $\pi : X \to Y = \mathbb{C}^4/\sigma_3$.
Since the normalizer of any order 2 element in σ_3 (any transposition or any reflection, if one thinks about σ_3 as the dihedral group) is trivial, by Lemma **??** it follows that the normalization of the singular locus S of Y is smooth. Hence, by **??** we can compute both the parametrizing scheme for rational curves in X and the respective universal family. That is, the parametrizing scheme $\mathcal{V}$ is just a blow-up of the normalization of S, the evaluation map $q : \mathcal{U} \to X$ drops its rank over 0 and the exceptional divisor of π, which is the image of q, is non-normal over 0.

8.4. **Wreath product.** Let $H < SL(2)$ be a finite subgroup and let $G := H^{\times 2} \rtimes \mathbb{Z}_2$ where $\mathbb{Z}_2$ interchanges the factors in the product. We write $G = H \wr \mathbb{Z}_2$. Note that $\mathbb{Z}_{n+1} \wr \mathbb{Z}_2$ has another nice presentation, namely $(\mathbb{Z}_{n+1})^{\times 2} \rtimes \mathbb{Z}_2 = D_{2n} \rtimes \mathbb{Z}_n$, where D_{2n} is the dihedral group of the regular n-gon and $\mathbb{Z}_n$ acts on it by rotations.
We consider the projective symplectic resolution described in 8.1 (with $n = 2$):

$$\pi : X := Hilb^2(S) \to S^{(2)} \to (\mathbb{C}^2/H)^{(2)} := Y$$

where $\nu : S \to \mathbb{C}^2/H$ is the minimal resolution with the exceptional set $\bigcup_i C_i$, where C_i, $i = 1, ..., k$, are (-2)-curves.
The morphism $\tau : Hilb^2(S) \to S^{(2)}$ is just a blow-up of the locus of $\mathbb{A}_1$ singularities (the image of the diagonal under $S^2 \to S^{(2)}$) with irreducible exceptional divisor E_0 which is a $\mathbb{P}^1$ bundle over S. We set $S' = \pi(E_0)$. By E_i, with $i = 1, \ldots, k$ we denote the strict transform, via τ, of the image of $C_i \times S$ under the map $S^2 \to S^{(2)}$. By e_i we denote the class of an irreducible component of a general fiber of $\pi_{|E_i}$. The image $\pi(E_i)$ for $i \geq 1$ is the surface $S'' \simeq C^2/H$. The singular locus of Y is the union $S = S' \cup S''$.
The irreducible components of $\pi^{-1}(0)$ are described in the following.

- $P_{i,i}$, for $i = 1, ..., k$. They are the strict transform of $C_i^{(2)}$ via τ. They are isomorphic to $\mathbb{P}^2$.
- $P_{i,j}$, for $i, j = 1, ..., k$ and $i < j$. They are the strict transform via τ of the image of $C_i \times C_j$ under the morphism $S^2 \to S^{(2)}$. They are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ if $C_i \cap C_j = \emptyset$ and to the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ if $C_i \cap C_j \neq \emptyset$.
- Q_i, for $i = 1, ..., k$. They are the preimage $\tau^{-1}(\Delta_{C_i})$, where Δ_{C_i} is the diagonal embedding of C_i in $S^{(2)}$. They are isomorphic to $\mathbb{P}(T_{S|C_i}) = \mathbb{P}(\mathcal{O}_{C_i}(2) \oplus \mathcal{O}_{C_i}(-2))$, i.e. to the Hirzebruch surface F_4.

Let us also describe some intersections between these components. Namely, $P_{i,i}$ intersects Q_i along a curve which is a (-4)-curve in Q_i and a conic in $P_{i,i}$. If $C_i \cap C_j = \{x_i\}$ then $P_{i,j}$ intersect $P_{i,i}$ (respectively $P_{j,j}$) along a curve which is a (-1) curve in $P_{i,j}$ and a line in $P_{i,i}$ (respectively in $P_{j,j}$). Moreover in this case $P_{i,j}$ intersect Q_i (respectively Q_j) in a curve which is a (-1) curve in $P_{i,j}$ and a fiber in Q_i (respectively Q_j).
The next lemma is straightforward, a proof of it can be found in [**?**, Lemma 4.2].

Lemma 8.2. *The strict transform of Q_i under any sequence of Mukai flops along components in $\pi^{-1}(0)$ is not isomorphic to $\mathbb{P}^2$.*

8.5. **Resolutions of $\mathbb{C}^4/(\mathbb{Z}_3 \wr \mathbb{Z}_2)$.** The Figure 1 presents a "realistic" description of configurations of components in the special fiber of symplectic resolutions of $\mathbb{C}^4/(\mathbb{Z}_3 \wr \mathbb{Z}_2)$. By abuse, the strict transforms of the components and the results of the flopping of $\mathbb{P}^2$'s are denoted by the same letters.

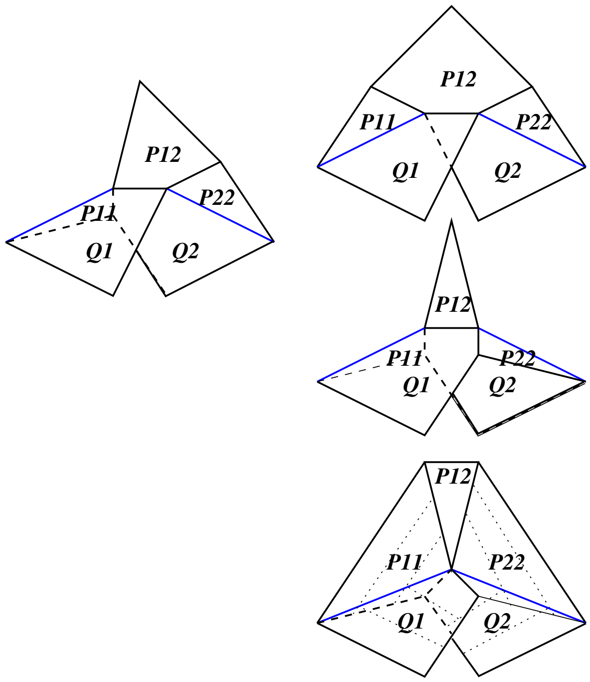

FIGURE 1. Components of the central fiber in resolutions of $\mathbb{C}^4/(\mathbb{Z}_3 \wr \mathbb{Z}_2)$

The position of these configurations in Figure 1 is consistent with the decomposition of the cone $\mathrm{Mov}(X/Y)$. In particular, the configuration at the top is associated with the Hilbert-Chow resolution. Note that the central configuration of this diagram cointains three copies of $\mathbb{P}^2$, denoted P_{ij}, which contain lines whose classes are $e_0 - e_1$, $e_0 - e_2$ and $e_1 + e_2 - e_0$.
On the other hand, the configuration in the bottom is associated with the resolution which can be factored by two different divisorial elementary contractions of classes e_1 and e_2. In fact, contracting both e_1 and e_2 is a resolution of $\mathbb{A}_2$ singularities which is a part of a resolution of Y which comes from presenting $\mathbb{Z}_3 \wr \mathbb{Z}_2$ as $D_6 \rtimes \mathbb{Z}_3$. That is, X is then obtained by first resolving the singularities of the action of $D_6 = \sigma_3$ and then by resolving the singularities of the $\mathbb{Z}_3$ action on this resolution. The rulings of the respective surfaces coming from this last blow up are indicated by dotted line segments. We will call such X a $D_6 \rtimes \mathbb{Z}_3$-resolution.
This example is convenient for understanding the contents of Theorem 7.1 and of Proposition **??**. We refer to diagram 7.1.3 and let S' and S'' be the closure of the locus of $\mathbb{A}_1$ and $\mathbb{A}_2$ singularities in $Y = \mathbb{C}^4/(\mathbb{Z}_3 \wr \mathbb{Z}_2)$. From Lemma **??** we find out that the normalization of S' as well as S'' has a singularity of type $\mathbb{A}_2$.
By $\mathcal{V}_0$ we denote the component of $Chow(X/Y)$ dominating S' and parametrizing curves equivalent to e_0, while by $\mathcal{V}_1$ and $\mathcal{V}_2$ we denote components dominating S''

parameterizing deformations of e_1 and e_2. The surfaces $\mathcal{V}_i$ may depend on the resolution and, in fact, while $\mathcal{V}_1$ and $\mathcal{V}_2$ remain unchanged, the component $\mathcal{V}_0$ will change under flops.

Lemma 8.3. *If X is the Hilbert-Chow resolution then $\mathcal{V}_0$ is the minimal resolution of $\mathbb{A}_2$ singularity. If X is the $D_6 \rtimes \mathbb{Z}_3$-resolution then $\mathcal{V}_0$ is non-minimal, with one (-1) curve in the central position of three exceptional curves.*

Proof. The first statement is immediate. To see the second one, note that we have the map of $\mathcal{V}_0$ to Chow of lines in the resolution of $\mathbb{C}^4/\sigma_3$ divided by $\mathbb{Z}_3$ action. The $\mathbb{Z}_3$-action in question is just a lift up of the original linear action on the fixed point set of rotations in $\sigma_3 = D_6$ hence $\mathcal{V}_0$ resolves 2 cubic cone singularities associated with the eigenvectors of the original action. □

One may verify that in the $D_6 \rtimes \mathbb{Z}_3$-resolution the exceptional set in $\mathcal{V}_0$ parametrizes curves consisting of three components: $\mathbb{Q}_2 \cap P_{11}$, $Q_1 \cap P_{22}$ and a line in P_{12}, whose classes are, respectively, e_2, e_1 and $e_0 - (e_1 + e_2)$.

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