Lagrange spectra of Veech surfaces

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The Diophantine approximation point of view

The classical theorem of Dirichlet says that for all real numbers α there are infinitely many integers p and $q \neq 0$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

The results of Markoff and Hurwitz moreover, state that for *all* reals α the previous inequality can be improved to $1/\sqrt{5}q^2$ and that the result is sharp, as one can show choosing $\alpha = \frac{1+\sqrt{5}}{2}$. This leads to the following natural question:

Given a *fixed* real number α what is the best constant $C > 0$ such that the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Cq^2},$$

holds for infinitely many integers p and $q \neq 0$?

This leads to the classical definition of the *Lagrange spectrum* $\mathcal{L}$:

$$\mathcal{L} = \left\{ L(\alpha) = \limsup_{p,q\to\infty} \frac{1}{q|q\alpha - p|}, \alpha \in \mathbb{R} \right\} \subset \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}.$$

Since its introduction, this set has been intensively studied and generalised to a variety of contexts. For example, it is known that:

- It is a closed subset of $\overline{\mathbb{R}}$ and quadratic irrationals are dense in $\mathcal{L}$;
- It is discreet from $\sqrt{5}$ to 3, that is the first accumulation point;
- It is a Cantor-like set of increasing Hausdorff dimension from 3 up to $\mu \approx 4.527$;
- Contains the whole interval $[\mu, +\infty]$. This portion is called *Hall ray*.

The classic proof of the Hall ray

The starting point is a formula for $L(\alpha)$ due to Perron. Let $\alpha = a_0 + [a_1, a_2, \dots]$. Then

$$L(\alpha) = \limsup_{n\to\infty} \left([a_{n-1}, \dots, a_0] + a_n + [a_{n+1}, a_{n+2}, \dots]\right).$$

In particular, this shows that $L(\alpha)$ is finite iff α is badly approximable.

Fix $n \in \mathbb{N}$ and define the following Cantor set in $[0, 1]$

$$F(n) = \{x = [a_1, a_2, \dots] \in [0, 1] : a_i \leq n, \forall i \in \mathbb{N}\}$$

Then, the crucial result by Hall is that, if $n \geq 4$ then $F(n) + F(n)$ is an *interval* of length greater than 1.

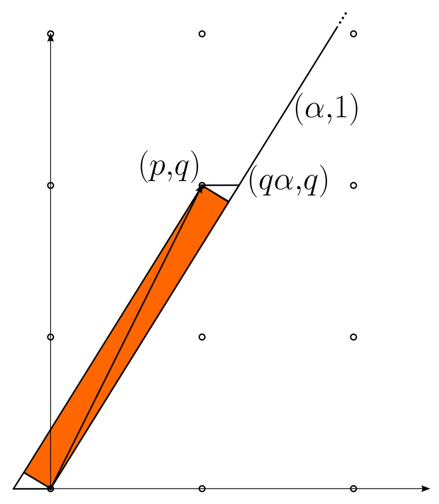
Take an $x > 6$, we can write it as $x = k + [b_1, b_2, \dots] + [c_1, c_2, \dots]$, where $k \geq 5$ and the rest is in $F(4) + F(4)$. Call

$$\alpha = [b_1, k, c_1, b_2, b_1, k, c_1, c_2, b_3, b_2, b_1, k, c_1, c_2, c_3, \dots]$$

Then, for this α, $L(\alpha) = x$, thanks to Perron's formula.

The Lagrange Spectrum has been generalised to many different contexts. We will be interested in the one defined by Hubert, Marchese and Ulcigrai in [5] for *translation surfaces*. We are now going to recast the Diophantine approximation definition of $\mathcal{L}$ into more geometric terms.

The geometric interpretation and translation surfaces



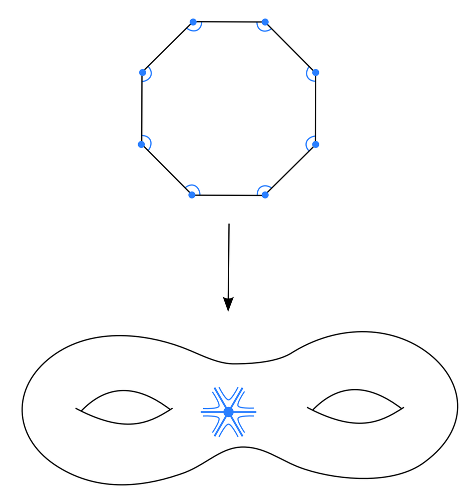
Fixing $\alpha \in \mathbb{R}$, we can interpret the quantity $q|q\alpha - p|$ as the *area* of the parallelogram draw in the picture. In turn, this is asymptotic to the area of the orange rectangle that has (p, q) as a diagonal, two sides in direction $(\alpha, 1)$ and two in the perpendicular one. Remark that, if we look only at $(p, q) = (p_n, q_n)$, for some $n \in \mathbb{N}$, where $\frac{p_n}{q_n}$ is the n-th convergent of α, then the rectangle does not contain any point of the lattice $\mathbb{Z}^2$ in its interior.

The torus is an example of a (Veech) translation surface. A translation surface is a surface obtained by glueing a finite number of polygons in the plane, identifying pairs of isometric parallel sides by translations.

Given a direction θ and a saddle connection v, that is a straight line connecting two singularities without any singularity in its interior, we define

$$\mathrm{Area}(v, \theta) = |\Re(e^{i\theta}v)| \cdot |\Im(e^{i\theta}v)|$$

to be the area of the embedded rectangle on the surface that has v as a diagonal, two sides in direction θ and two in the orthogonal one.



Lagrange spectrum of Veech surfaces

Veech translation surfaces are surfaces with many symmetries. An example is given by the surfaces obtained by glueing opposite sides of a regular $2n$-gon. The Lagrage spectrum of a Veech surface S is defined as $\mathcal{L}(S) = \{L_S(\theta), 0 \leq \theta < 2\pi\}$, where

$$L_S(\theta) = \limsup_{|\Im(e^{i\theta}v)|\to\infty} \frac{1}{\mathrm{Area}(v, \theta)}.$$

Theorem (A, Marchese, Ulcigrai, [1])

Let S be a Veech translation surface and let $\mathcal{L}(S)$ be its Lagrange spectrum. Then $\mathcal{L}(S)$ contains a *Hall ray*, that is there exists $r = r(s) > 0$ such that

$$[r(S), +\infty] \subset \mathcal{L}(S).$$

Strategy of the proof

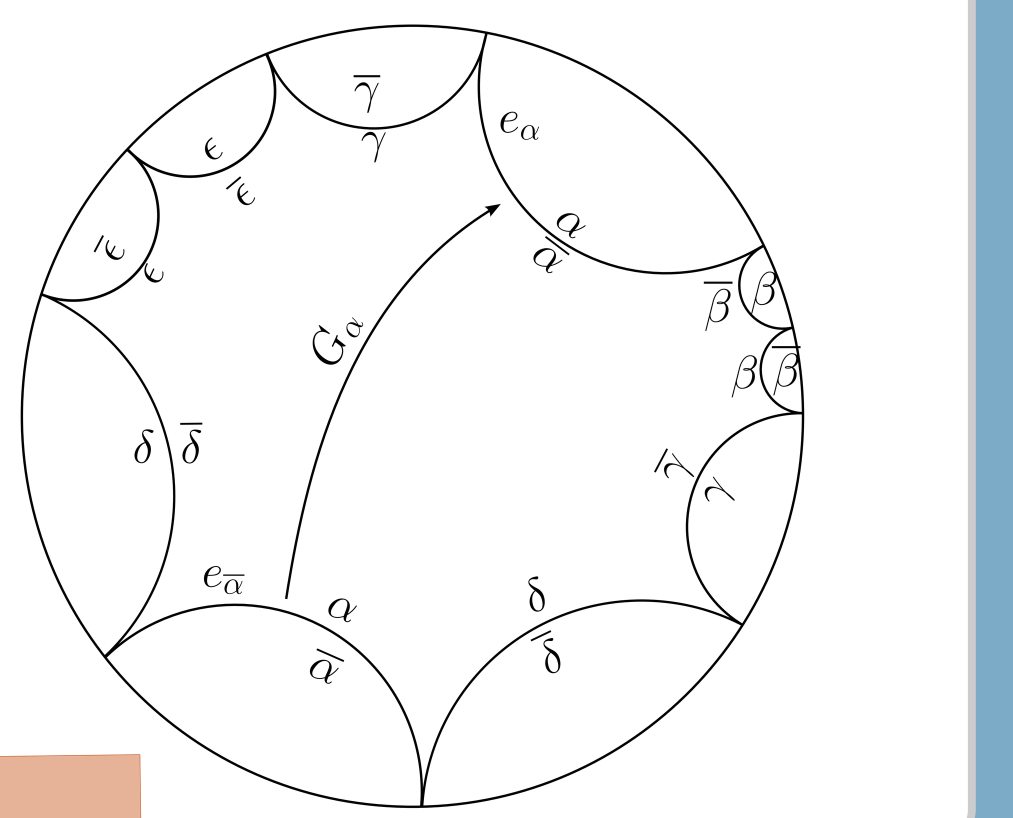
Since S is Veech, the group $V(S)$ formed by derivatives of affine diffeomorphisms is a *lattice* in $\mathrm{SL}(2, \mathbb{R})$. We code the hyperbolic geodesic whose endpoint is $\theta \in \partial\mathbb{D}$ using the coding for $\mathbb{D}/V(S)$ developed by Bowen and Series in [2]. Fixing a direction θ, that is not a vertex of an ideal polygon in the tessellation of $\mathbb{D}$ given by $V(S)$, we construct a series of wedges, formed by saddle connections on S that contain the direction θ and are shrinking on it. We show that wedges can be used to compute the Lagrange values higher than an explicit constant $L_0(S)$ that depends on the geometry of S.

Fixing a letter α in the alphabet we are using to code geodesics, we define two continued fractions-like expansions, and use them to prove a formula that gives the areas of the wedges and that has a similar structure to the classic one:

$$\frac{1}{\mathrm{Area}(v_r^{(n)}, \theta)} \sim c \cdot \left([\alpha_{n-1}, \dots, \alpha_0]^-_\alpha + \mu_\alpha + [\alpha_{n+p+1}, \alpha_{n+p+2}, \dots]^+_\alpha\right),$$

where c is a fixed constant and μ_α is the shearing factor of the parabolic transformation associated to α.

We construct two Cantor sets on $\mathbb{R}$ using the continued fractions-like expressions $[\alpha_0, \alpha_1, \dots]^\pm_\alpha$ and we show that their sum is an interval of length larger than μ_α. We can now repeat the final part of the classic proof.



References

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[2] R. Bowen, and C. Series, *Markov maps associated with Fuchsian groups*, Institut des Hautes Études Scientifiques. Publications Mathématiques, **50** (1979), 153–170.

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[4] M. Hall, *On the sum and products of continued fractions*, Annals of Mathematics (2), **48** (1947), 966–993.

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