To match or not to match

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Entropy and matching for 3 different families

On this poster 3 families of Continued fraction expansions are shown. We are interested in the entropy as a function of α. In the first two cases (matching) intervals can be found on which the entropy is monotonic. On these intervals a matching condition holds i.e there are N, M such that

$$T_\alpha^N(\alpha) = T_\alpha^M(\alpha - 1)\ .$$

Depending on the sign of $N - M$ on this interval the entropy is increasing, decreasing or, in the case $N = M$, constant. The proofs depend on the fact that rationals have a finite expansion (and therefore match). However, this is not true for the third case. Another key ingredient is the fact that a certain algebraic condition holds. We do not know if this is the true in the last case. For the α-expansions a lot is already done and published. The entropy of the Ito α-continued fractions is work in progress with Carlo Carminati. The N α-expansions is work in progress with Cor Kraaikamp.

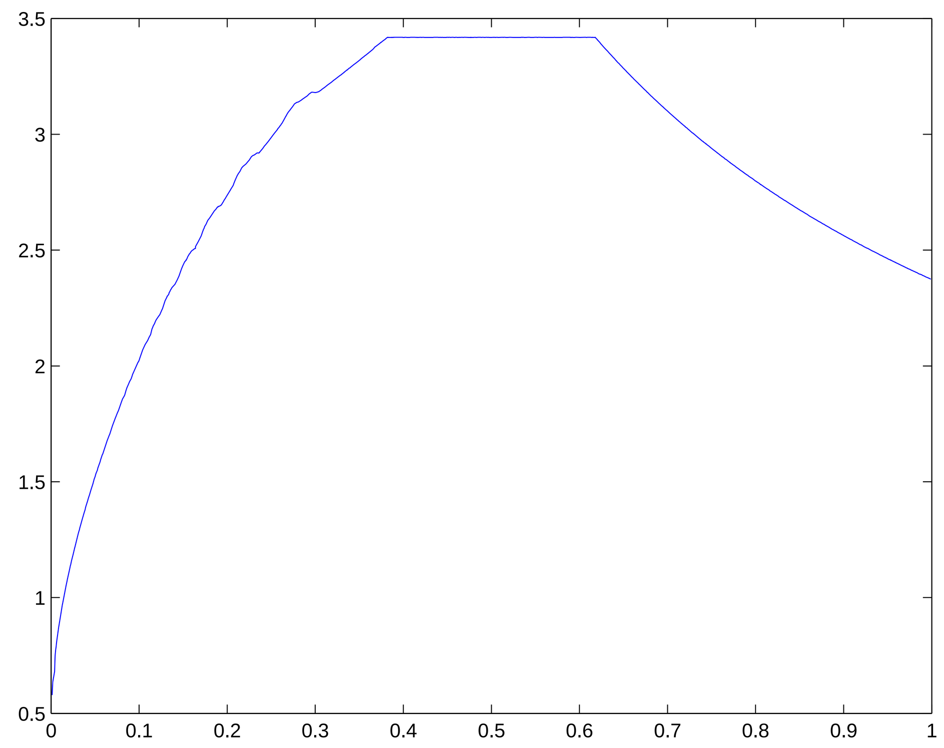
α-expansions

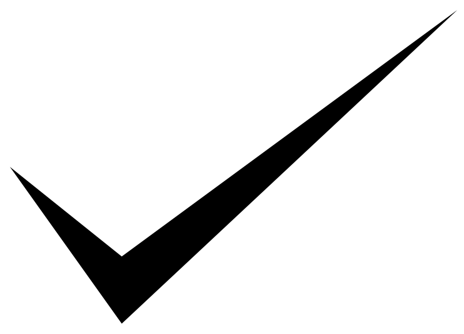
We define $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ with $\alpha \in [0, 1]$ by

$$T_\alpha = \frac{1}{|x|} - \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor \quad for \quad x \neq 0 \quad and \quad T_\alpha(0) = 0 \qquad (1)$$

- All rationals have a finite expansion
- 1 possible algebraic conditions for all rationals
- Matching holds almost everywhere
- The union of all matching intervals have full measure
- For every $\delta > 0$ there is an interval $[a, b] \subset (0, \delta)$ on which the entropy is decreasing, an interval on which the entropy is constant and an interval on which the entropy is increasing.

For references see [1, 2, 3, 4, 6]





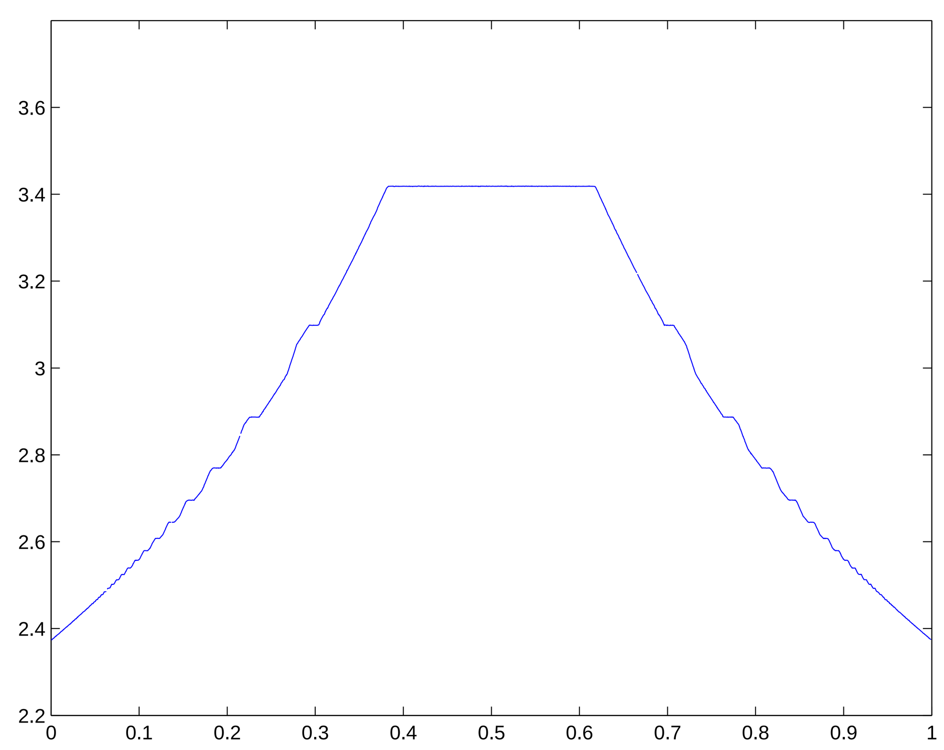
ITO α-expansions

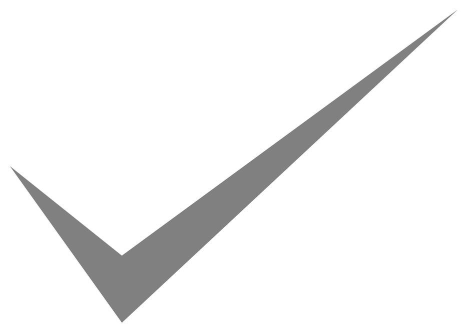
We define $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ with $\alpha \in [0, 1]$ by

$$T_\alpha = \frac{1}{x} - \left\lfloor \frac{1}{x} + 1 - \alpha \right\rfloor \quad for \quad x \neq 0 \quad and \quad T_\alpha(0) = 0 \qquad (2)$$

- All rationals have a finite expansion
- 6 possible algebraic conditions for all rationals
- 3 out of 6 conditions give entropy monotonic in neighbourhood
- We can proof matching holds on a dense open set
- If matching holds almost everywhere is an open question

For reference see [5]





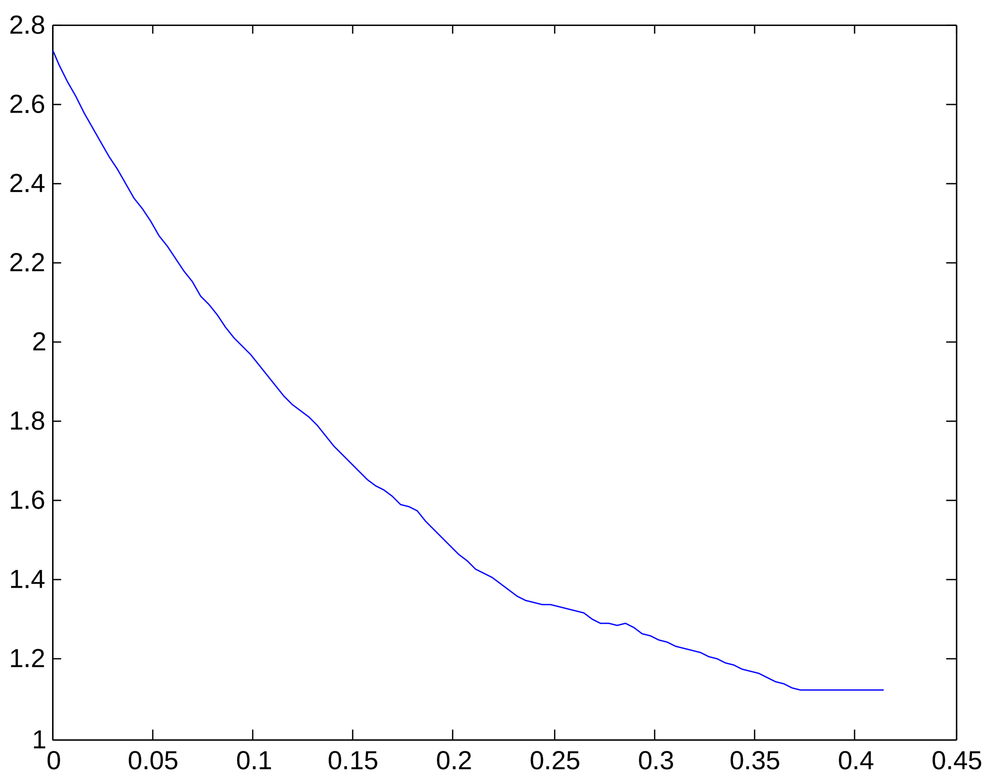
N α-expansions

We define $T_{N,\alpha} : [\alpha, \alpha + 1] \to [\alpha, \alpha + 1]$ with $\alpha \in (0, \sqrt{N} - 1]$ by

$$T_\alpha = \frac{N}{x} - \left\lfloor \frac{N}{x} - \alpha \right\rfloor \qquad (3)$$

- All rationals have a infinite expansion
- We do not know if any algebraic condition holds (neither we know if this will help proving monotonicity on a matching interval)
- We do not know if for every N there is at least 1 matching interval
- We do not know for which α to expect matching
- We do not know ...

$N = 2$



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References

[1] C. Carminati, S. Marmi, A. Profeti and G. Tiozzo. The entropy of α-continued fractions: numerical results.*Nonlinearity* 23 (2010), 2429-2456

[2] C. Carminati and G. Tiozzo. A canonical thickening of and the entropy of α-continued fraction transformations. *Erdogic theory and dynamical systems* 32 (2012), 1249-1269

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