A GEOMETRIC APPROACH FOR CONSTRUCTING SRB AND EQUILIBRIUM MEASURES IN HYPERBOLIC DYNAMICS

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1. Lecture I: Definition and ergodic properties of SRB measures

1.1. **The setting.**

(1) $f\colon M \to M$ a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M;
(2) $U \subset M$ an open subset with the property that $\overline{f(U)} \subset U$; such a set U is called a *trapping region*;
(3) $\Lambda = \bigcap_{n\geq 0} f^n(U)$ a *topological attractor* for f; we allow the case $\Lambda = M$.

Λ is compact, f-invariant, and maximal (i.e., if $\Lambda' \subset U$ is invariant, then $\Lambda' \subset \Lambda$).

1.2. **Physical measures.**

μ an arbitrary probability measure on Λ. The *basin of attraction* of μ is

$$B_\mu = \left\{ x \in U : \frac{1}{n}\sum_{k=0}^{n-1} h(f^k(x)) \to \int_\Lambda h\, d\mu \text{ for any } h \in C^1(M) \right\}$$

μ is a *physical measure* if $m(B_\mu) > 0$. An attractor with a physical measure is often referred to as a *Milnor attractor*.

1.3. **Hyperbolic measures.**

Given $x \in \Lambda$, $v \in T_xM$, the *Lyapunov exponent* of v at x is

$$\chi(x, v) = \limsup_{n\to\infty} \frac{1}{n} \log \|df^n v\|.$$

The function $\chi(x, \cdot)$ takes on finitely many values, $\chi_1(x) \leq \cdots \leq \chi_p(x)$, where $p = \dim M$ which are invariant functions, i.e., $\chi_i(f(x)) = \chi_i(x)$ for every i.

A Borel invariant measure μ on Λ is *hyperbolic* if $\chi_i(x) \neq 0$ and $\chi_1(x) < 0 < \chi_p(x)$; that is

$$\chi_1(x) \leq \dots \chi_k(x) < 0 < \chi_{k+1}(x) \leq \cdots \leq \chi_p(x)$$

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for some $k(x) \geq 1$. If μ is ergodic, then $\chi_i(x) = \chi_i(\mu)$ for a.e. x. The non-uniform hyperbolicity theory ensures that for a hyperbolic measure μ and a.e. $x \in \Lambda$, the following are true.

(1) There is a splitting $T_xM = E^s(x) \oplus E^u(x)$ where

$$E^s(x) = E^s_f(x) = \{v \in T_xM : \chi(x, v) < 0\}, \quad E^u(x) = E^u_f(x) = E^s_{f^{-1}}(x)$$

are *stable* and *unstable subspaces* at x; they satisfy
  (a) $dfE^s(x) = E^s(f(x))$ and $dfE^u(x) = E^u(f(x))$;
  (b) $\angle(E^s(x), E^u(x)) \geq K(x)$ for some Borel function $K(x) > 0$ on Λ that satisfies condition (5) below.

(2) There are *local stable* $V^s(x)$ and *local unstable* $V^u(x)$ manifolds at x; these are $C^{1+\alpha}$ submanifolds given as graphs

$$V^{s,u}(x) = \exp\Big\{(v, \psi^{s,u}(v)) : v \in B^{s,u}(0, r(x)) \subset E^{s,u}(x)\Big\}$$

of $C^{1+\alpha}$ functions $\psi^{s,u}$ defined in the ball $B^{s,u}(0, r(x))$ centered at zero of radius $r(x) > 0$ and mapping it into $E^{u,s}(x)$; we have that $\psi^{s,u}(0) = 0$ and $D\psi^{s,u}(0) = 0$; these functions are constructed via the Stable Manifold Theorem.

(3) The local stable and unstable manifolds at x satisfy

$$d(f^n(x), f^n(y)) \leq C(x)\lambda^n(x)d(x, y), \quad y \in V^s(x),\ n \geq 0,$$
$$d(f^{-n}(x), f^{-n}(y)) \leq C(x)\lambda^n(x)d(x, y), \quad y \in V^u(x),\ n \geq 0$$

for some Borel function $C(x) > 0$ on Λ that satisfies Condition (5) below, and some Borel f-invariant function $0 < \lambda(x) < 1$.

(4) There are the *global stable* $W^s(x)$ and *global unstable* $W^u(x)$ manifolds at x (tangent to $E^s(x)$ and $E^u(x)$, respectively) so that

$$W^s(x) = \bigcup_{n\geq 0} f^{-n}(V^s(f^n x)), \quad W^u(x) = \bigcup_{n\geq 0} f^n(V^u(f^{-n}x));$$

these manifolds are invariant under f, i.e., $f(W^s(x)) = W^s(f(x))$ and $f(W^u(x)) = W^u(f(x))$.

(5) The functions $C(x)$ and $K(x)$ can be chosen to satisfy

$$C(f^{\pm 1}(x)) \leq C(x)e^{\varepsilon(x)}, \quad K(f^{\pm 1}(x)) \geq K(x)e^{-\varepsilon(x)},$$

where $\varepsilon(x) > 0$ is an f-invariant Borel function.

(6) The *size* $r(x)$ of local manifolds satisfies $r(f^{\pm 1}(x)) \geq r(x)e^{-\varepsilon(x)}$.

One can show that $W^u(x) \subset \Lambda$ for every $x \in \Lambda$ (for which the global unstable manifold is defined).

Since $\lambda(x)$ is invariant, it is constant μ-a.e. when μ is ergodic, so from now on we assume that $\lambda(x) = \lambda$ is constant on Λ.

Given $\ell > 1$, define *regular set* of level ℓ by

$$\Lambda_\ell = \Big\{x \in \Lambda : C(x) \leq \ell,\ K(x) \geq \frac{1}{\ell}\Big\}.$$

These sets satisfy:

- $\Lambda_\ell \subset \Lambda_{\ell+1}$, $\bigcup_{\ell \geq 1} \Lambda_\ell = \Lambda$;
- the subspaces $E^{s,u}(x)$ depend continuously on $x \in \Lambda_\ell$; in fact, the dependence is Hölder continuous:
$$d_G(E^{s,u}(x), E^{s,u}(y)) \leq M_\ell d(x,y)^\alpha,$$
where d_G is the Grasmannian distance in TM;
- the local manifolds $V^{s,u}(x)$ depend continuously on $x \in \Lambda_\ell$; in fact, the dependence is Hölder continuous:
$$d_{C^1}(V^{s,u}(x), V^{s,u}(y)) \leq L_\ell d(x,y)^\alpha;$$
- $r(x) \geq r_\ell > 0$ for all $x \in \Lambda_\ell$.

1.4. **SRB measures.** We can choose ℓ such that $\mu(\Lambda_\ell) > 0$. For $x \in \Lambda_\ell$ and a small $\delta_\ell > 0$ set
$$Q_\ell(x) = \bigcup_{y \in B(x,\delta_\ell) \cap \Lambda_\ell} V^u(y).$$
Let ξ_ℓ be the partition of $Q_\ell(x)$ by $V^u(y)$, and let $V^s(x)$ be a local stable manifold that contains exactly one point from each $V^u(y)$ in $Q_\ell(x)$. Then there are *conditional measures* $\mu^u(y)$ on each $V^u(y)$, and a *transverse measure* $\mu^s(x)$ on $V^s(x)$, such that for any $h \in L^1(\mu)$ supported on $Q_\ell(x)$, we have
$$\int h\, d\mu = \int_{V^s(x)} \int_{V^u(y)} h\, d\mu^u(y)\, d\mu^s(x). \tag{1.1}$$
Let $m_{V^u(y)}$ denote the leaf volume on $V^u(y)$.

Definition 1.1. A measure μ on Λ is called an *SRB measure* if μ is hyperbolic and for every ℓ with $\mu(\Lambda_\ell) > 0$, almost every $x \in \Lambda_\ell$ and almost every $y \in B(x,\delta_\ell) \cap \Lambda_\ell$, we have the measure $\mu^u(y)$ is absolutely continuous with respect to the measure $m_{V^u(y)}$.

For $y \in \Lambda_\ell$, $z \in V^u(y)$ and $n > 0$ set
$$\rho_n^u(y,z) = \prod_{k=0}^{n-1} \frac{\mathrm{Jac}(df|E^u(f^{-k}(z)))}{\mathrm{Jac}(df|E^u(f^{-k}(y)))}.$$
One can show that for every $y \in \Lambda_\ell$ and $z \in V^u(y)$ the following limit exists
$$\rho^u(y,z) = \lim_{n\to\infty} \rho_n^u(y,z) = \prod_{k=0}^{\infty} \frac{\mathrm{Jac}(df|E^u(f^{-k}(z)))}{\mathrm{Jac}(df|E^u(f^{-k}(y)))} \tag{1.2}$$
and that $\rho^u(y,z)$ depends continuously on $y \in \Lambda_\ell$ and $z \in V^u(y)$.

Theorem 1.2. *If μ is an SRB measure on Λ, then the density $d^u(x,\cdot)$ of the conditional measure $\mu^u(x)$ with respect to the leaf-volume $m_{V^u(x)}$ on $V^u(x)$ is given by $d^u(x,y) = \rho^u(x)^{-1}\rho^u(x,y)$ where*
$$\rho^u(x) = \int_{V^u(x)} \rho^u(x,y)\, dm^u(x)(y)$$

is the normalizing factor.

In particular, we conclude that the measures $\mu^u(x)$ and $m_{V^u(y)}$ must be equivalent.

The idea of describing an invariant measure by its conditional probabilities on the elements of a continuous partition goes back to the classical work of Kolmogorov and especially later work of Dobrushin on random fields and relation (1.2) can be viewed as an analog of the famous Dobrushin-Lanford-Ruelle equation in statistical physics.

1.5. **Ergodic properties of SRB measures.** Using results of nonuniform hyperbolicity theory one can obtain a sufficiently complete description of ergodic properties of SRB measures.

Theorem 1.3. *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth manifold M with an attractor Λ and let μ be an SRB measure on Λ. Then there is a finite or countable collection of subsets $\Lambda_0, \Lambda_1, \Lambda_2, \dots \subset \Lambda$ such that*

(1) $\Lambda = \bigcup_{i\geq 0} \Lambda_i, \quad \Lambda_i \cap \Lambda_j = \emptyset$*;*
(2) $\mu(\Lambda_0) = 0$ *and* $\mu(\Lambda_i) > 0$ *for* $i > 0$*;*
(3) $f|\Lambda_i$ *is ergodic for* $i > 0$*;*
(4) for each $i > 0$ there is $n_i > 0$ such that $\Lambda_i = \bigcup_{j=1}^{n_i} \Lambda_{i,j}$ where the union is disjoint (modulo μ-null sets), $f(\Lambda_{i,j}) = \Lambda_{i,j+1}$, $f(\Lambda_{n_i,1}) = \Lambda_{i,1}$ and $f^{n_i}|\Lambda_{i,1}$ is Bernoulli;
(5) if μ is ergodic, then the basin of attraction B_μ has positive Lebesgue measure in U.

For smooth measures this theorem was proved by Pesin and an extension to the general case was given by Ledrappier.

Statement 5 of Theorem 1.3 can be paraphrased as follows: *ergodic SRB measures are physical.* The converse is not true; physical measures need not be SRB. Easy examples are given by (1) the Dirac measure on an attracting fixed point, which has all exponents negative and hence is not hyperbolic, and by (2) Lebesgue measure for an irrational circle rotation, which has zero exponent but is still physical.

A more subtle example is given by the time-1 map of the flow illustrated in Figure 1, where the Dirac measure at the hyperbolic fixed point p is a hyperbolic physical measure whose basin of attraction includes all points except q_1 and q_2. (In fact, by slowing down the flow near p, one can adapt this example so that p is an indifferent fixed point and hence, the physical measure is not even hyperbolic.)

1.6. **Characterization of SRB measures.** The following result provides a complete characterization of SRB measures via their entropies.

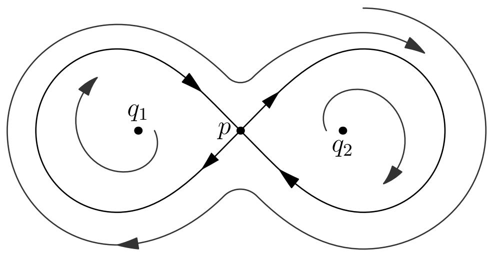

FIGURE 1. A physical measure that is not SRB.

Theorem 1.4. *A measure μ is an SRB measure if and only if the entropy $h_\mu(f)$ of μ is given by the* entropy formula*:*

$$h_\mu(f) = \int_\Lambda \sum_{\chi_i(x)>0} \chi_i(x)\, d\mu(x).$$

For smooth measures (which are a particular case of SRB measures) the entropy formula was proved by Pesin and its extension to SRB measures was given by Ledrappier and Strelcyn. The fact that a hyperbolic measure satisfying the entropy formula is an SRB measure was shown by Ledrappier.

Remark 1.5. Our definition of SRB measure includes the requirement that the measure is hyperbolic. In fact, one can extend the notion of SRB measures to those that have some or even all Lyapunov exponents zero (in the latter case we take $W^u(x) = \{x\}$). It was proved by Ledrappier and Young that a measure satisfies the entropy formula if and only if it is an SRB measure in this more general sense, but we stress that such an SRB measure may not be physical, i.e., its basin may be of zero Lebesgue measure.

It follows from Theorem 1.3 that f admits at most countably many ergodic SRB measures. One can show that a topologically transitive $C^{1+\alpha}$ surface diffeomorphism can have at most one SRB measure but the result is not true in dimension higher than two.

1.7. **An overview of the geometric approach.** The idea of the geometric approach is to follow the classical Bogolyubov-Krylov procedure for constructing invariant measures by pushing forward a given reference measure. In our case the natural choice of a reference measure is the Riemannian volume m restricted to the neighborhood U, which we denote by m_U. We then consider the sequence of probability measures

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} f_*^k m_U. \tag{1.3}$$

Any weak* limit point of this sequence of measures is called a *natural measure* and while in general, it may be a trivial measure, under some additional hyperbolicity requirements on the attractor one obtains an SRB measure.

For attractors with some hyperbolicity one can use a somewhat different approach which exploits the fact that SRB measures are absolutely continuous along unstable manifolds. To this end consider a point $x \in \Lambda$, its local unstable manifold $V^u(x)$, and take the leaf-volume $m_{V^u(x)}$ as the reference measure for the above construction. Thus one studies the sequence of probability measures given by

$$\nu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k m_{V^u(x)}. \tag{1.4}$$

The measures ν_n are spread out over increasingly long pieces of the global unstable manifold of the point $f^n(x)$ and, in some situations, control over the geometry of the unstable manifold makes it possible to draw conclusions about the measures ν_n by keeping track of their densities along unstable manifolds, and ultimately to demonstrate that any weak-star limit of ν_n is in fact an SRB measure.

Every invariant measure μ can be obtained as the limit of the sequence of measures (1.4) where the leaf-volume $m_{V^u(x)}$ is replaced with an appropriately chosen *reference measure*. This follows from a general result that we now present.

Let X be a compact topological space, $f\colon X \to X$ a continuous map, and μ a finite f-invariant ergodic Borel probability measure on X. Let also $Y \subset X$ be a Borel subset of positive measure and ξ a measurable partition of Y. Consider the factor-measure $\tilde{\mu}$ in the factor space $\tilde{Y} := Y/\xi$ and the system of conditional measures $\{\mu_y\}$. Here y runs over a subset $Z \subset \tilde{Y}$ of full $\tilde{\mu}$-measure and for each $y \in Z$ the measure μ_y is supported on the element $C(y)$ of the partition ξ. We have that for any Borel subset $E \subset Y$

$$\mu(E) = \int_{\tilde{Y}} \int_{C(y)} \chi_E(y, z)\, d\mu_y(z)\, d\tilde{\mu}(y). \tag{1.5}$$

Without loss of generality we may assume that μ_y is normalized that is $\mu_y(C(y)) = 1$.

Set $A = \bigcup_{y\in\tilde{Y}\setminus Z} C(y)$. By (1.5), $\mu(A) = 0$. Consider the set B of all *Birkhoff generic points* $x \in Y$, for which

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} h(f^k(x)) = \int_X h\, d\mu \tag{1.6}$$

for every continuous function h on X. Since μ is ergodic we have that B has full measure in Y. By (1.5), there is a set $D \subset B$ of full measure in Y such that for every $x \in D$ we have that $\mu_x(C(x) \cap (A \cup (Y \setminus B))) = 0$.

Fix $x \in D$ and let ν be a probability measure on $C(x)$, which is absolutely continuous with respect to μ_x, i.e., $d\nu(y) = \rho(y)\, d\mu_x(y)$ for some $\rho \in L^1(C(x), \mu_x)$. The measure ν can be extended to the measure $\tilde{\nu}$ on X by setting $\tilde{\nu}(E) := \nu(E \cap C(x))$ for any Borel set $E \subset X$.

Consider a sequence $\{\nu_n\}$ of measures on X defined by

$$\nu_n := \frac{1}{n}\sum_{k=0}^{n-1} f_*^k\tilde{\nu}. \tag{1.7}$$

Theorem 1.6. *The sequence* (1.7) *converges in the weak* topology to the measure* μ.

Proof. Let κ be a weak* limit of (1.7), i.e. there exists a subsequence $\{n_\ell\}_{\ell\in\mathbb{N}}$ such that for every continuous function h on X

$$\int_X h\,d\kappa = \lim_{\ell\to\infty}\int_X h\,d\nu_{n_\ell} = \lim_{\ell\to\infty}\int_{C(x)}\frac{1}{n_\ell}\sum_{k=0}^{n_\ell-1} h\circ f^k d\tilde{\nu}. \tag{1.8}$$

Clearly κ is f-invariant. Since μ is ergodic, to show that $\kappa=\mu$ it suffices to prove that κ is absolutely continuous with respect to μ. To this end consider a nonnegative continuous function h with $\int_X h\,d\mu=\varepsilon$ for some small $\varepsilon>0$. We will show that there exists a constant $K>0$ that depends on $\|h\|$ (the C^0-norm of h) only such that $\int_X h\,d\kappa<K\varepsilon$.

Note that by (1.8), we have for all sufficiently large ℓ that

$$\left|\int_X h\,d\kappa - \int_{C(x)}\frac{1}{n_\ell}\sum_{k=0}^{n_\ell-1} h(f^k(y))\,d\nu\right| < \varepsilon. \tag{1.9}$$

It follows from (1.6) that there is a subset $X_\varepsilon\subset X$ with $\mu(X\setminus X_\varepsilon)\le\varepsilon$ and a number $N=N(\varepsilon)>0$ such that for every $y\in X_\varepsilon$ and $n_\ell>N$,

$$\left|\frac{1}{n_\ell}\sum_{k=0}^{n_\ell-1} h(f^k(y)) - \int_X h\,d\mu\right| < \varepsilon. \tag{1.10}$$

Since the measure ν is absolutely continuous with respect to the measure μ_x, we have that $\nu(C(x)\cap(A\cup(Y\setminus B)))=0$. Therefore, setting $Q:=A\cup(Y\setminus B)$

by (1.9) and (1.10), we obtain for $n_\ell > N$ that

$$\begin{aligned}
\int_X h\,d\kappa &\le \int_{C(x)} \frac{1}{n_\ell}\sum_{k=0}^{n_\ell-1} h(f^k(y))\,d\nu + \varepsilon \\
&= \int_{C(x)\setminus Q} \frac{1}{n_\ell}\sum_{k=0}^{n_\ell-1} h(f^k(y))\,d\nu + \varepsilon \\
&= \int_{(C(x)\setminus Q)\cap X_\varepsilon} \frac{1}{n_\ell}\sum_{k=0}^{n_\ell-1} h(f^k(y))\,d\nu \\
&+ \int_{(C(x)\setminus Q)\setminus X_\varepsilon} \frac{1}{n_\ell}\sum_{k=0}^{n_\ell-1} h(f^k(y))\,d\nu + \varepsilon \\
&\le \int_{C(x)\setminus Q} \left(\int_X h\,d\mu + \varepsilon\right) d\nu + \|h\| \int_{X\setminus X_\varepsilon} \rho\,d\mu_x + \varepsilon \\
&\le \left(\int_X h\,d\mu + \varepsilon\right) + T\|h\|\varepsilon + \varepsilon = (T\|h\| + 3)\varepsilon,
\end{aligned}$$

where $T > 0$ is a constant. The desired result follows. □

2. Lecture II: The geometric approach for constructing SRB measures for uniformly hyperbolic attractors

2.1. **Hyperbolic attractors.** A *(uniformly) hyperbolic attractor* is a topological attractor Λ which is a is *(uniformly) hyperbolic set*. The latter means that for each $x \in \Lambda$ there is a decomposition $T_xM = E^s(x) \oplus E^u(x)$ and constants $c > 0$, $\lambda \in (0,1)$ such that for each $x \in \Lambda$:

(1) $\|d_x f^n v\| \le c\lambda^n \|v\|$ for $v \in E^s(x)$ and $n \ge 0$;
(2) $\|d_x f^{-n} v\| \le c\lambda^n \|v\|$ for $v \in E^u(x)$ and $n \ge 0$.

$E^s(x)$ and $E^u(x)$ are *stable* and *unstable subspaces* at x. They depend continuously on x and in fact, Hölder continuously on x. In particular, $\angle(E^s(x), E^u(x))$ is uniformly away from zero.

For each $x \in \Lambda$ there are $V^s(x)$ and $V^u(x)$ *stable* and *unstable local manifolds* at x. They have uniform size r, depend continuously on x in the C^1 topology.

Since Λ is an attractor and hence, is a locally maximal set within its trapping region U, one can show that $V^u(x) \subset \Lambda$ for any $x \in \Lambda$. This means that the hyperbolic attractor is a union of global unstable manifolds of its points. On the other hand, the intersection of Λ with local unstable manifolds of its points is a Cantor-like set unles $\Lambda = M$.

This is illustrated by the following example of a hyperbolic attractor. Consider the solid torus $P = D^2 \times S^1$ and use coordinates (x, y, θ) on P such that x and y give the coordinates on the disc, and θ is the angular coordinate on the circle.

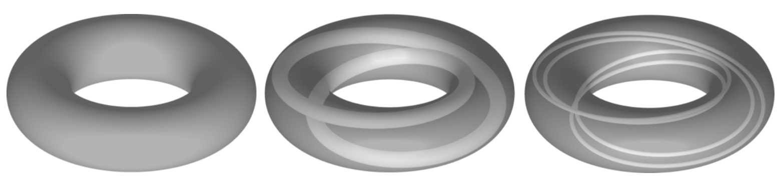

Figure 2. The Smale-Williams solenoid.

Fixing $a \in (0,1)$ and $\alpha, \beta \in (0, \min\{a, 1-a\})$, $\alpha \le \beta$, define a map $f : P \to P$ by

$$f(x, y, \theta) = (\alpha x + a\cos\theta, \beta y + a \sin\theta, 2\theta).$$

P is a trapping region and $\Lambda = \bigcap_{n\ge0} f^n(P)$ is the attractor for f, known as the Smale-Williams solenoid, see Figure 2. Observe that the intersection of the attractor with the transversal disk $\theta = 0$ is a Cantor-like set. One can show that the Hausdorff dimension of this set is $\frac{\log 2}{\log \beta}$.

2.2. **Existence of SRB measures for hyperbolic attractors.**

Theorem 2.1. *Assume that f is $C^{1+\alpha}$ and that Λ is a uniformly hyperbolic attractor. The following statements hold:*

(1) *Every limit measure of either the sequence of measures μ_n (given by* (1.3)*) or the sequence of measures ν_n (given by* (1.4)*) is an SRB measure on Λ.*
(2) *There are at most finitely many ergodic SRB measures on Λ.*
(3) *If $f|\Lambda$ is topologically transitive, then there is a unique SRB-measure μ on Λ, which is the limit of both sequences μ_n and ν_n; moreover, B_μ has full measure in U.*

This theorem was proved by Sinai for the case of Anosov diffeomorphisms, Bowen and Ruelle extended this result to hyperbolic attractors, and Bowen and Ruelle constructed SRB measures for Anosov flows.

We outline a proof of this theorem to demonstrate how the geometric approach works. Note that the geometric proof we give here is not the original proof given by Sinai, Bowen, and Ruelle, who used the symbolic approach. Given $x \in M$, a subspace $E(x) \subset T_xM$, and $a(x) > 0$, the *cone* at x around $E(x)$ with angle $a(x)$ is

$$K(x, E(x), a(x)) = \{v \in T_xM \colon \measuredangle(v, E(x)) < a(x)\}.$$

There exists a neighborhood $\tilde{U} \subset U$ of the attractor Λ and two continuous cone families $K^s(x) = K^s(x, E^s(x), a)$ and $K^u(x) = K^u(x, E^u(x), a)$ such that

$$\overline{df(K^u(x))} \subset K^u(f(x)) \text{ for all } x \in \tilde{U},$$
$$\overline{df^{-1}(K^s(f(x)))} \subset K^s(x) \text{ for all } x \in f(\tilde{U}).$$

Note that the subspaces $E^s(x)$ and $E^u(x)$ for $x \in \tilde{U}$ need not be invariant under df.

Let $W \subset U$ be an *admissible manifold*; that is, a submanifold that is tangent to an unstable cone $K^u(x)$ at some point $x \in U$ and has a fixed size and uniformly bounded curvature. More precisely, fix constants $\gamma, \kappa, r > 0$, and define a (γ, κ)-*admissible manifold of size* r to be $V(x) = \exp_x \operatorname{graph} \psi$, where $\psi : B_{E^u(x)}(0, r) = B(0, r) \cap E^u(x) \to E^s(x)$ is $C^{1+\alpha}$ and satisfies

$$\begin{aligned} &\psi(0) = 0 \text{ and } d\psi(0) = 0, \\ &\|d\psi\| := \sup_{\|v\|<r} \|d\psi(v)\| \le \gamma, \\ &|d\psi|_\alpha := \sup_{\|v_1\|,\|v_2\|<r} \frac{\|d\psi(v_1) - d\psi(v_2)\|}{\|v_1 - v_2\|^\alpha} \le \kappa. \end{aligned} \tag{2.1}$$

Write $\mathbf{I} = (\gamma, \kappa, r)$ for convenience and consider the space of admissible manifolds

$$\mathcal{R}_{\mathbf{I}} = \{\exp_x(\operatorname{graph} \psi) \colon x \in U, \psi \in C^1(B^u(0, r), E^s(x)) \text{ satisfies (2.1)}\}.$$

Given an admissible manifold W, we consider a *standard pair* (W, ρ) where ρ is a continuous "density" function on W. The idea of working with pairs of admissible manifolds and densities was introduced by Chernov and Dolgopyat and is an important recent development in the study of SRB measures via geometric techniques.

Now we fix $L > 0$, write $\mathbf{K} = (\mathbf{I}, L)$, and consider the space of standard pairs

$$\mathcal{R}'_{\mathbf{K}} = \{(W, \rho)\colon W \in \mathcal{R}_{\mathbf{I}}, \rho \in C^{\alpha}(W, [\tfrac{1}{L}, L]), |\rho|_{\alpha} \leq L\}.$$

These spaces are compact in the natural product topology: the coordinates in $\mathcal{R}_{\mathbf{I}}$ are

$$\{x \in M, \psi \in C^1(B^u(0,r), E^s(x)) \text{ with } \|D\psi\| \leq \gamma,\ |D\psi|_{\alpha} \leq \kappa\}$$

and the coordinates in $\mathcal{R}'_{\mathbf{K}}$ are

$$\{x, \psi, \rho \in C^{\alpha}(W) \text{ with } \|\rho\|_{\alpha} \leq L\}.$$

A standard pair determines a measure $\Psi(W, \rho)$ on $\overline{U}$ in the obvious way:

$$\Psi(W, \rho)(E) := \int_{E \cap W} \rho \, dm_W.$$

Moreover, each measure η on $\mathcal{R}'_{\mathbf{K}}$ determines a measure $\Phi(\eta)$ on $\overline{U}$ by

$$\begin{aligned} \Phi(\eta)(E) &= \int_{\mathcal{R}'_{\mathbf{K}}} \Psi(W, \rho)(E) \, d\eta(W, \rho) \\ &= \int_{\mathcal{R}'_{\mathbf{K}}} \int_{E \cap W} \rho(x) \, dm_W(x) \, d\eta(W, \rho). \end{aligned} \tag{2.2}$$

(Compare this to (1.1) in the definition of conditional measures.) Write $\mathcal{M}(\overline{U})$ and $\mathcal{M}(\mathcal{R}'_{\mathbf{K}})$ for the spaces of finite Borel measures on $\overline{U}$ and $\mathcal{R}'_{\mathbf{K}}$, respectively. It is not hard to show that $\Phi\colon \mathcal{M}(\mathcal{R}'_{\mathbf{K}}) \to \mathcal{M}(\overline{U})$ is continuous; in particular, $\mathcal{M}_{\mathbf{K}} = \Phi(\mathcal{M}_{\leq 1}(\mathcal{R}'_{\mathbf{K}}))$ is compact, where we write $\mathcal{M}_{\leq 1}$ for the space of measures with total weight at most 1.

On a uniformly hyperbolic attractor, an invariant probability measure is an SRB measure if and only if it is in $\mathcal{M}_{\mathbf{K}}$ for some $\mathbf{K}$.

Consider now the leaf volume m_W on W that we view as a measure on $\bar{U}$. Its evolution is the sequence of measures

$$\kappa_n = \frac{1}{n} \sum_{k=0}^{n-1} f_*^k m_W. \tag{2.3}$$

By weak* compactness there is a subsequence κ_{n_j} that converges to an invariant measure μ on Λ which is an SRB measure.

Consider the images $f^n(W)$ and observe that for each n, the measure $f_*^n m_W$ is absolutely continuous with respect to leaf volume on $f^n(W)$. For every n, the image $f^n(W)$ can be covered with uniformly bounded multiplicity by a finite number of admissible manifolds W_i, so that

$$f_*^n m_W \text{ is a convex combination of measures } \rho_i \, dm_{W_i}, \tag{2.4}$$

where ρ_i are Hölder continuous positive densities on W_i. . This requires a version of the Besicovitch covering lemma, which is usually formulated for geometrical balls, so one must choose the W_i in such a way that each $f^n(W_i)$ is 'sufficiently close' to being a ball in $f^n(W)$.

We see from (2.4) that $\mathcal{M}_{\mathbf{K}}$ is invariant under the action of f_*, and thus $\kappa_n \in \mathcal{M}_{\mathbf{K}}$ for every n. By compactness of $\mathcal{M}_{\mathbf{K}}$, one can pass to a subsequence κ_{n_k} which converges to a measure $\mu \in \mathcal{M}_{\mathbf{K}}$, and this is the desired SRB measure.

Choosing $W = V^u(x)$, $x \in \Lambda$ we obtain that any limit measure of the sequence ν_n (see (1.4)) is an SRB measure. It is then not difficult to derive from here that any limit measure of the sequence μ_n (see (1.3)) is an SRB measure.

In the particular case when $\Lambda = M$ (that is, f is a $C^{1+\alpha}$ Anosov diffeomorphism), the above theorem guarantees existence of an SRB measure μ for f. In fact, if f is topologically transitive one can show that μ is a unique SRB measure. Reversing the time we obtain the unique SRB measure ν for f^{-1}. One can show that $\mu = \nu$ if and only if μ is a smooth measure.

2.3. **Non-uniformly hyperbolic attractors.** If μ is an SRB measure, then every point in the positive Lebesgue measure set B_μ has non-zero Lyapunov exponents. A natural and interesting question is whether the converse holds true, in essence formulated in the following conjecture by Viana:

Conjecture 2.2. *If a smooth map has only non-zero Lyapunov exponents at Lebesgue almost every point, then it admits an SRB measure.*

The first significant results on SRB measures for non-uniformly hyperbolic systems were those for the attractors for certain special parameters of the Hénon family of maps obtained Benedics and Carleson. These attractors have a fully nonuniformly hyperbolic structure which can be described relatively explicitly and, taking advantage of several specific characteristics of this structure, an SRB measure for the attractors was constructed first by Benedics and Young.

2.4. **Effective hyperbolicity.** We make the following standing assumption.

(H) There exists a forward-invariant set $A \subset U$ of positive volume with two measurable cone families $K^s(x), K^u(x) \subset T_xM$ such that
  (a) $\overline{Df(K^u(x))} \subset K^u(f(x))$ for all $x \in A$;
  (b) $\overline{Df^{-1}(K^s(f(x)))} \subset K^s(x)$ for all $x \in f(A)$.
  (c) $K^s(x) = K(x, E^s(x), a_s(x))$ and $K^u(x) = K(x, E^u(x), a_u(x))$ are such that $T_xM = E^s(x) \oplus E^u(x)$; moreover $d_s = \dim E^s(x)$ and $d_u = \dim E^u(x)$ do not depend on x.

Such cone families automatically exist if f is uniformly hyperbolic on Λ. We emphasize, however, that in our setting $K^{s,u}$ are not assumed to be continuous, but only measurable and the families of subspaces $E^{u,s}(x)$ are not assumed to be invariant.

Let $A \subset U$ be a forward-invariant set satisfying **(H)**. Define

$$\lambda^u(x) = \inf\{\log \|Df(v)\| \mid v \in K^u(x), \|v\| = 1\},$$
$$\lambda^s(x) = \sup\{\log \|Df(v)\| \mid v \in K^s(x), \|v\| = 1\}.$$

Note that if the splitting $E^s \oplus E^u$ is dominated, then we have $\lambda^s(x) < \lambda^u(x)$ for every x. Thus we define the *defect from domination* at x to be

$$\Delta(x) = \tfrac{1}{\alpha} \max(0,\ \lambda^s(x) - \lambda^u(x)),$$

where $\alpha \in (0,1]$ is the Hölder exponent of Df. Roughly speaking, $\Delta(x)$ controls how much the curvature of unstable manifolds can grow as we go from x to $f(x)$. Indeed, if $\lambda^s(x) > \lambda^u(x)$ then the action of Df can push tangent vectors away from E^u and towards E^s, so that the image of an unstable (or admissible) manifold can 'curl up' under the action of f, and $\Delta(x)$ quantifies how much this can happen.

The following quantity is positive whenever f expands vectors in $K^u(x)$ and contracts vectors in $K^s(x)$:

$$\lambda(x) = \min(\lambda^u(x) - \Delta(x), -\lambda^s(x)).$$

The *upper asymptotic density* of $\Gamma \subset \mathbb{N}$ is

$$\overline{\delta}(\Gamma) = \limsup_{N\to\infty} \frac{1}{N} \#\big(\Gamma \cap [0,N)\big).$$

An analogous definition gives the lower asymptotic density $\underline{\delta}(\Gamma)$.

Denote the angle between the boundaries of $K^s(x)$ and $K^u(x)$ by

$$\theta(x) = \inf\{\measuredangle(v,w) \colon v \in K^u(x), w \in K^s(x)\}.$$

We say that a point $x \in A$ is *effectively hyperbolic* if

$$\text{(EH1)} \qquad \liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda(f^k(x)) > 0,$$

$$\text{(EH2)} \qquad \lim_{\bar\theta \to 0} \overline{\delta}\{n \mid \theta(f^n(x)) < \bar\theta\} = 0.$$

Condition (**EH1**) says that not only are the Lyapunov exponents of x positive for vectors in K^u and negative for vectors in K^s, but λ^u gives enough expansion to overcome the 'defect from domination' given by Δ.

Condition (**EH2**) requires that the frequency with which the angle between the stable and unstable cones drops below a specified threshold $\bar\theta$ can be made arbitrarily small by taking the threshold to be small.

If Λ is a hyperbolic attractor for f, then **every** point $x \in U$ is effectively hyperbolic, since there are $\bar\lambda, \bar\theta > 0$ such that $\lambda^s(x) \leq -\bar\lambda$, $\lambda^u(x) \geq \bar\lambda$, and $\theta(x) \geq \bar\theta$ for every $x \in U$, so that $\Delta(x) = 0$ and $\lambda(x) \geq \bar\lambda$.

Let A satisfy **(H)**, and let $S \subset A$ be the set of effectively hyperbolic points. Observe that effective hyperbolicity is determined in terms of a forward asymptotic property of the orbit of x, and hence S is forward-invariant under f.

Theorem 2.3. *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact manifold M, and Λ a topological attractor for f. Assume that*

(1) f admits measurable invariant cone families as in **(H)***;*
(2) the set S of effectively hyperbolic points satisfies $m(S) > 0$.

Then f has an SRB measure supported on Λ.

A similar result can be formulated given information about the set of effectively hyperbolic points on a single 'approximately unstable' submanifold usually called *admissible*. The set of admissible manifolds that we will work with is related to $\mathcal{R}_{\mathbf{I}}$ from the previous lecture, but the precise definition is not needed for the statement of the theorem; all we need here is to have $T_xW \subset K^u(x)$ for 'enough' points x. $W \subset U$. Let d_u, d_s, and A be as in **(H)**, (**EH1**) and (**EH2**), and let $W \subset U$ be an embedded submanifold of dimension d_u.

Theorem 2.4. *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact manifold M, and Λ a topological attractor for f. Assume that*

(1) f admits measurable invariant cone families as in **(H)***;*
(2) there is a d_u-dimensional embedded submanifold $W \subset U$ such that $m_W(\{x \in S \cap W \mid T_xW \subset K^u(x)\}) > 0$.

Then f has an SRB measure supported on Λ.

We outline the proof of this statement to illustrate the geometric approach in the settings of non-uniformly hyperbolic attractors and we follow the same ideas as in the case of uniformly hyperbolic attractors, but there are two major obstacles to overcome.

(1) The action of f along admissible manifolds is not necessarily uniformly expanding.
(2) Given $n \in \mathbb{N}$ it is no longer necessarily the case that $f^n(W)$ contains any admissible manifolds in $\mathcal{R}_{\mathbf{I}}$, let alone that it can be covered by them. When $f^n(W)$ contains some admissible manifolds, we will need to control how much of it can be covered.

To address the first of these obstacles, we need to consider admissible manifolds for which we control not only the geometry but also the dynamics; thus we will replace the collection $\mathcal{R}'_{\mathbf{K}}$ from before with a more carefully defined set (in particular, $\mathbf{K}$ will include more parameters). Since we do not have uniformly transverse invariant subspaces $E^{u,s}$, our definition of an admissible manifold also needs to specify which subspaces are used, and the geometric control requires an assumption about the angle between them.

Given $\theta, \gamma, \kappa, r > 0$, write $\mathbf{I} = (\theta, \gamma, \kappa, r)$ and consider the following set of "(γ, κ)-admissible manifolds of size r with transversals controlled by θ":

$$\mathcal{P}_{\mathbf{I}} = \{\exp_x(\operatorname{graph}\psi) \mid x \in \overline{f(U)},\ T_xM = G \oplus F,\ G \subset \overline{K^u(x)},\\ \angle(G, F) \geq \theta,\ \psi \in C^{1+\alpha}(B_G(r), F) \text{ satisfies (2.1)}\}. \tag{2.5}$$

Elements of $\mathcal{P}_{\mathbf{I}}$ are admissible manifolds with controlled geometry. We also impose a condition on the dynamics of these manifolds. Fixing $C, \overline{\lambda} > 0$, write $\mathbf{J} = (C, \overline{\lambda})$ and consider for each $N \in \mathbb{N}$ the collection of sets

$$\text{(2.6)} \quad \mathcal{Q}_{\mathbf{J},N} = \{f^N(V_0) \mid V_0 \subset U, \text{ and for every } y, z \in V_0, \text{ we have } \\ d(f^j(y), f^j(z)) \leq Ce^{-\overline{\lambda}(N-j)} d(f^N(y), f^N(z)) \text{ for all } 0 \leq j \leq N\}.$$

Elements of $\mathcal{P}_{\mathbf{I}} \cap \mathcal{Q}_{\mathbf{J},N}$ are admissible manifolds with controlled geometry and dynamics in the unstable direction. We also need a parameter $\beta > 0$ that controls the dynamics in the stable direction, and another parameter $L > 0$ that controls densities in standard pairs. Then writing $\mathbf{K} = \mathbf{I} \cup \mathbf{J} \cup \{\beta, L\}$, we obtain a set $\mathcal{R}_{\mathbf{K},N} \subset \mathcal{P}_{\mathbf{I}} \cap \mathcal{Q}_{\mathbf{J},N}$ for which we have the added restriction that we control the dynamics in the stable direction; the corresponding set of standard pairs is written $\mathcal{R}'_{\mathbf{K},N}$.

The set $\mathcal{R}'_{\mathbf{K},N}$ carries a natural product topology in which $\mathcal{R}'_{\mathbf{K},N}$ is compact and the map Φ defined in (2.2) is continuous.

As before, let $\mathcal{M}_{\leq 1}(\mathcal{R}'_{\mathbf{K},N})$ denote the space of measures on $\mathcal{R}'_{\mathbf{K},N}$ with total weight at most 1. The resulting space of measures on U plays a central role:

$$\text{(2.7)} \qquad \mathcal{M}_{\mathbf{K},N} = \Phi(\mathcal{M}_{\leq 1}(\mathcal{R}'_{\mathbf{K},N})).$$

Measures in $\mathcal{M}_{\mathbf{K},N}$ have uniformly controlled geometry, dynamics, and densities via the parameters in $\mathbf{K}$, and $\mathcal{M}_{\mathbf{K},N}$ is compact. However, at this point we encounter the second obstacle mentioned above: because $f(W)$ may not be covered by admissible manifolds in $\mathcal{R}_{\mathbf{K},N}$, the set $\mathcal{M}_{\mathbf{K},N}$ is not f_*-invariant.

To address this, one must establish good recurrence properties to $\mathcal{M}_{\mathbf{K},N}$ under the action of f_* on $\mathcal{M}(\overline{U})$; this can be done via effective hyperbolicity.

Consider for $x \in A$ and $\overline{\lambda} > 0$ the set of *effective hyperbolic times*

$$\text{(2.8)} \quad \Gamma^e_{\overline{\lambda}}(x) = \left\{ n \mid \sum_{j=k}^{n-1} (\lambda^u - \Delta)(f^j(x)) \geq \overline{\lambda}(n-k) \text{ for all } 0 \leq k < n \right\}.$$

Then for every x and almost every effective hyperbolic time $n \in \Gamma^e_{\overline{\lambda}}(x)$, there is a neighborhood $W^x_n \subset W$ containing x such that $f^n(W^x_n) \in \mathcal{P}_{\mathbf{I}} \cap \mathcal{Q}_{\mathbf{J},N}$.

With a little more work, one can produce a "uniformly large" set of points x and times n such that $f^n(W^x_n) \in \mathcal{R}_{\mathbf{K},N}$, and in fact $f^n_* m_{W^x_n} \in \mathcal{M}_{\mathbf{K},N}$. Then this can be used to obtain measures $\nu_n \in \mathcal{M}_{\mathbf{K},N}$ such that

$$\text{(2.9)} \qquad \nu_n \leq \mu_n = \tfrac{1}{n} \textstyle\sum_{k=0}^{n-1} f^k_* m_W \qquad \text{and} \qquad \overline{\lim}_{n\to\infty} \|\nu_n\| > 0.$$

Once this is achieved, compactness of $\mathcal{M}_{\mathbf{K},N}$ guarantees existence of a nontrivial $\nu \in \bigcap_N \mathcal{M}_{\mathbf{K},N}$ such that $\nu \leq \mu = \lim_k \mu_{n_k}$. In order to apply the absolute continuity properties of ν to the measure μ, one must define a collection $\mathcal{M}^{\text{ac}}$ of measures with good absolute continuity properties along admissible manifolds, for which there is a version of the Lebesgue decomposition theorem that gives $\mu = \mu^{(1)} + \mu^{(2)}$, where $\mu^{(1)} \in \mathcal{M}^{\text{ac}}$ is invariant. This measure is non-trivial since $0 \neq \nu \leq \mu^{(1)}$, and the definition of $\mathcal{R}'_{\mathbf{K},N}$ guarantees that the set of points with non-zero Lyapunov exponents has positive measure with respect to ν, and hence also with respect to $\mu^{(1)}$. Thus some ergodic component of $\mu^{(1)}$ is hyperbolic, and hence is an SRB measure.

2.5. **Maps on the boundary of Axiom A: neutral fixed points.** We give a specific example of a map for which the conditions of Theorem 2.4 can be verified. Let $f\colon U \to M$ be a $C^{1+\alpha}$ Axiom A diffeomorphism onto its image with $\overline{f(U)} \subset U$, where $\alpha \in (0,1)$. Suppose that f has one-dimensional unstable bundle.

Let p be a fixed point for f. We perturb f to obtain a new map g that has an indifferent fixed point at p. The case when M is two-dimensional and f is volume-preserving was studied by Katok. We allow manifolds of arbitrary dimensions and (potentially) dissipative maps. For example, one can choose f to be the Smale–Williams solenoid or its sufficiently small perturbation.

We describe a specific perturbation of f for which the conditions of the main theorem can be verified. We suppose that there exists a neighborhood $Z \ni p$ with local coordinates in which f is the time-1 map of the flow generated by

$$\dot{x} = Ax$$

for some $A \in GL(d,\mathbb{R})$. Assume that the local coordinates identify the splitting $E^u \oplus E^s$ with $\mathbb{R} \oplus \mathbb{R}^{d-1}$, so that $A = A_u \oplus A_s$, where $A_u = \gamma \mathrm{Id}_u$ and $A_s = -\beta \mathrm{Id}_s$ for some $\gamma, \beta > 0$. In the Katok example we have $d = 2$ and $\gamma = \beta$ since the map is area-preserving.

Now we use local coordinates on Z and identify p with 0. Fix $0 < r_0 < r_1$ such that $B(0, r_1) \subset Z$, and let $\psi\colon Z \to [0,1]$ be a $C^{1+\alpha}$ function such that (see Figure 3)

(1) $\psi(x) = \|x\|^\alpha$ for $\|x\| \leq r_0$;
(2) $\psi(x) = 1$ for $\|x\| \geq r_1$;
(3) $\psi(x) > 0$ for $x \neq 0$ and $\psi'(x) > 0$.

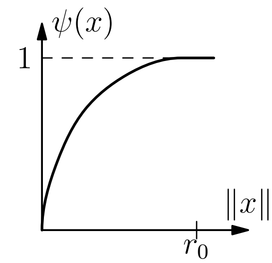

FIGURE 3. The function ψ.

Let $\mathcal{X}\colon Z \to \mathbb{R}^d$ be the vector field given by $\mathcal{X}(x) = \psi(x)Ax$. Let $g\colon U \to M$ be given by the time-1 map of this vector field on Z and by f on $U \setminus Z$. Note that g is $C^{1+\alpha}$ because $\mathcal{X}$ is $C^{1+\alpha}$. One can show that g satisfies the conditions of Theorem 2.4, which proves the following.

Theorem 2.5. *The map g has an SRB measure.*

We also observe that if ψ is taken to be C^∞ away from 0, then g is also C^∞ away from the point p.

3. Lecture III: The geometric approach for constructing equilibrium measures

3.1. **Equilibrium measures.**

(1) X a compact topological space, $f\colon X \to X$ a continuous map, φ a continuous function, the *potential of the system*.

(2) *Bowen's ball* at a point $x \in X$ of radius $r > 0$ and length n:

$$B_n(x,r) = \{y \in X\colon d(f^k(x), f^k(y)) < r \text{ for all } 0 \le k \le n\}.$$

(3) *Topological pressure* $P(\varphi)$:

$$P(\varphi) := \lim_{r\to 0} \liminf_{n\to\infty} \frac{1}{n} \log S(\varphi, n, r),$$

where

$$S(\varphi, n, r) = \inf_{\mathcal{U}} \Big\{ \sum_{B_n(x_i,r)\in\mathcal{U}} \Big(\sup_{y\in B_n(x_i,r)} \sum_{k=0}^{n-1} \varphi(f^k(y)) \Big) \Big\}$$

and the infimum is taken over all finite collections $\mathcal{U}_n = \{B_n(x_i, r)\}$ of Bowen's balls of length n that cover X.

(4) *Free energy*: $E(\mu) = -\big(h_\mu(f) + \int_X \varphi\, d\mu\big)$.

(5) *Variational principle*:

$$P(\varphi) = -\inf E(\mu) = \sup\big\{h_\mu(f) + \int_X \varphi\, d\mu\big\},$$

where $h_\mu(f)$ is the Kolmogorov-Sinai entropy of μ and the infimum and supremum are taken over the set $\mathcal{M}(f, X)$ of all f-invariant Borel probability measures on X.

(6) *Equilibrium measure* μ_φ: an extreme of the variational principle that is

$$P(\varphi) = h_{\mu_\varphi}(f) + \int_X \varphi\, d\mu_\varphi. \tag{3.1}$$

Uniqueness of equilibrium measures is interpreted as "absence of phase transitions".

3.2. **Equilibrium measures on locally maximal hyperbolic sets.**

Let f be a diffeomorphism of compact smooth Riemannian manifold M and Λ a (uniformly) hyperbolic set. We say that Λ is *locally maximal* if there is a neighborhood U of Λ such that if $A \subset U$ is an invariant set, then $A \subset \Lambda$.One can show that Λ is locally maximal if and only if for any $x, y \in \Lambda$ for which the intersection $[x, y] := V^s(x) \cap V^u(y) \neq \emptyset$, then $[x, y] \in \Lambda$.

Throughout this lecture we assume that Λ is a locally maximal(uniformly) hyperbolic set. We say that the map $f|\Lambda$ is *topologically transitive* if there is a point $x \in \Lambda$ whose trajectory $\{f^n(x)\}$ is dense in Λ. One can show that topological transitivity is equivalent to the fact that given two open sets $U, V \subset \Lambda$ there is $n \in \mathbb{Z}$ such that $f^n(U) \cap V \neq \emptyset$.

Consider a Hölder continuous potential φ that is

$$|\varphi(x) - \varphi(y)| \le C d(x, y)^\alpha$$

for some $C > 0$ and $0 < \alpha < 1$.

Theorem 3.1. *Assume that $f|\Lambda$ is topologically transitive. Then there is a unique equilibrium measure μ_φ for φ.*

Uniqueness implies that the measure μ_φ is ergodic. In fact, it has many strong ergodic properties such as the Bernoulli property (up to a rotation), exponential decay of correlations and the Central Limit Theorem.

3.3. **Carathéorody dimension structure.** Let X be a set and $\mathcal{F}$ a collection of subsets of X which we call *admissible* sets. Assume that there exist two set functions $\eta, \psi : \mathcal{F} \to [0, \infty)$ satisfying the following conditions:

(A1) $\emptyset \in \mathcal{F}$; $\eta(\emptyset) = \psi(\emptyset) = 0$ and $\eta(U), \psi(U) > 0$ for any $U \in \mathcal{F}, U \neq \emptyset$;
(A2) for any $\delta > 0$ one can find $\varepsilon > 0$ such that $\eta(U) \le \delta$ for any $U \in \mathcal{F}$ with $\psi(U) \le \varepsilon$;
(A3) there exists $\varepsilon > 0$ such that for any $\varepsilon \ge \varepsilon > 0$, one can find a finite subcollection $\mathcal{G} \subset \mathcal{F}$ covering X such that $\psi(U) = \varepsilon$ for any $U \in \mathcal{G}$.

Let $\xi : \mathcal{F} \to [0, \infty)$ be a set function. We say that the collection of subsets $\mathcal{F}$ and the functions ξ, η, ψ, satisfying Conditions (A1), (A2) and (A3) introduce a *Carathéodory dimension structure* or *C-structure* τ on X and we write $\tau = (\mathcal{F}, \xi, \eta, \psi)$.

For any subcollection $\mathcal{G} \subset \mathcal{F}$ denote by $\psi(\mathcal{G}) := \sup\{\psi(U) | U \in \mathcal{G}\}$. Given a set $Z \subset X$ and numbers $\alpha \in \mathbb{R}$ and $\varepsilon > 0$, define

$$M_C(Z, \alpha, \varepsilon) := \inf_{\mathcal{G}, \psi(\mathcal{G}) \le \varepsilon} \left\{ \sum_{U \in \mathcal{G}} \xi(U) \eta(U)^\alpha \right\},$$

$$R_C(Z, \alpha, \varepsilon) := \inf_{\mathcal{G}, \psi(\mathcal{G}) = \varepsilon} \left\{ \sum_{U \in \mathcal{G}} \xi(U) \eta(U)^\alpha \right\},$$

where the infimum is taken over all finite or countable subcollections $\mathcal{G} \subset \mathcal{F}$ covering Z. Set

$$m_C(Z, \alpha) := \lim_{\varepsilon \to 0} M_C(Z, \alpha, \varepsilon),$$
$$\underline{r}_C(Z, \alpha) := \underline{\lim}_{\varepsilon \to 0} R_C(Z, \alpha, \varepsilon),$$
$$\overline{r}_C(Z, \alpha) := \overline{\lim}_{\varepsilon \to 0} R_C(Z, \alpha, \varepsilon).$$

If $m_C(\emptyset, \alpha) = 0$, then the set function $m_C(\cdot, \alpha)$ becomes an outer measure on X, which induces a measure on the σ-algebra of measurable sets. We call this measure the *α-Carathéodory measure.* In general, this measure may not be σ-finite or it may be a zero measure.

Proposition 3.2. *For any set $Z \subset X$ there exist critical values $\alpha_C \le \underline{\alpha}_C \le \overline{\alpha}_C \in \mathbb{R}$ such that:*

(1) $m_C(Z, \alpha) = \infty$ for $\alpha < \alpha_C$ and $m_C(Z, \alpha) = 0$ for $\alpha > \alpha_C$ (while $m_C(Z, \alpha_C)$ may be 0, ∞, or a finite positive number);

(2) $\underline{r}_C(Z,\alpha) = \infty$ *for* $\alpha < \underline{\alpha}_C$ *and* $\underline{r}_C(Z,\alpha) = 0$ *for* $\alpha > \underline{\alpha}_C$ *(while* $\underline{r}_C(Z,\underline{\alpha}_C)$ *may be* 0*,* ∞*, or a finite positive number);*

(3) $\overline{r}_C(Z,\alpha) = \infty$ *for* $\alpha < \overline{\alpha}_C$ *and* $\overline{r}_C(Z,\alpha) = 0$ *for* $\alpha > \overline{\alpha}_C$ *(while* $\overline{r}_C(Z,\overline{\alpha}_C)$ *may be* 0*,* ∞*, or a finite positive number).*

We call the quantities

$$\dim_C Z = \alpha_C, \ \underline{\mathrm{Cap}}_C Z = \underline{\alpha}_C, \ \overline{\mathrm{Cap}}_C Z = \overline{\alpha}_C.$$

the *Carathéodory dimension* and *lower* and *upper Carathéodory capacities* of the set Z respectively.

A C-structure can be generated by other structures on the set X. For example if X is a metric space, then consider the C-structure given by

$$\mathcal{F} := \{\text{open sets}\}, \quad \xi(U) = 1, \quad \eta(U) = \psi(U) = \operatorname{diam} U.$$

For any $Z \subset X$ we have that

$$\dim_C Z = \dim_H Z, \quad \underline{\mathrm{Cap}}_C Z = \underline{\dim}_B Z, \quad \overline{\mathrm{Cap}}_C Z = \overline{\dim}_B Z,$$

where $\dim_H Z$ is the Hausdorff dimension and $\underline{\dim}_B Z$, $\overline{\dim}_B Z$ are respectively, the lower and upper box dimensions of Z.

3.4. **Carathéodory measures on local unstable manifolds.** We consider again a diffeomorphism f with a locally maximal hyperbolic set Λ and let φ be a Hölder continuous function on Λ. We introduce a particular C-structure on local unstable manifolds associated with φ.

Fix $x_0 \in \Lambda$ and consider the local unstable manifold $V^u := V^u(x_0)$. For $x \in \Lambda$, $n \in \mathbb{N}$, and $\varepsilon > 0$ define

$$B_n^u(x,\varepsilon) := \{y \in V^u(x) \cap \Lambda : d(f^k(y), f^k(x)) < \varepsilon \text{ for } k = 0, \dots, n\},$$

$$B^u(x,\varepsilon) := B_0^u(x,\varepsilon) = \{y \in V^u(x) \cap \Lambda : d(y,x) < \varepsilon\}.$$

The set $B_n^u(x,\varepsilon)$ is Bowen's ball in the local unstable manifold which we call *u-Bowen's ball*.

Recall that a continuous map of a topological space X is *expansive* is there is number $\delta_0 > 0$, called the *expansivity constant*, such that no two trajectories can stay within the distance δ_0 from each other, i.e., if for some $x, y \in \Lambda$ we have that $d(f^n(x), f^n(y) \le \delta_0$ for all $n \in \mathbb{Z}$, then $x = y$. One can show that the map $f|\Lambda$ is expansive.

Set $X := V^u \cap \Lambda$. Let δ_0 be the expansivity constant for f. We fix a small number $0 < r < \delta_0$ and define the collection $\mathcal{F}$ of admissible sets by

$$\mathcal{F} := \{\emptyset\} \cup \{B_n^u(x,r) : x \in V^u \cap \Lambda, n \in \mathbb{N}\}.$$

Given $x \in V^u \cap \Lambda$ and $n \in \mathbb{N}$, we define

$$\xi(B_n^u(x,r)) := \exp\left(\sum_{k=0}^{n-1} \varphi(f^k(x))\right),$$

$$\eta(B_n^u(x,r)) := e^{-n}, \quad \psi(B_n^u(x,r)) := \frac{1}{n},$$

and also set $\eta(\emptyset)=\psi(\emptyset)=\xi(\emptyset)=0$. It is easy to see that the collection of subsets $\mathcal{F}$ and the functions ξ,η,ψ satisfy Conditions (A1), (A2) and (A3), and hence, introduce the Carathéodory dimension structure in X.

Let $P=P(\varphi)$ be the topological pressure of φ and let $m^u:=m_C(\cdot,P)$ denote the P-Carathéodory measure on X. For every set $Z\subset X$ we have that

$$m^u(Z)=\lim_{N\to\infty}\inf_{\{B^u_{n_i}(x_i,r)\}}\left\{\sum_i\exp\left(-Pn_i+\sum_{k=0}^{n_i-1}\varphi(f^k(x_i))\right)\right\}, \tag{3.2}$$

where the infimum is taken over all collections $\{B^u_{n_i}(x_i,r)\}$ of Bowen's u-balls with $x_i\in X$, $n_i\ge N$, which cover Z that is $Z\subset\bigcup_i B^u_{n_i}(x_i,r)$. We have the following.

Theorem 3.3. *The P-Carathéodory measure m^u given by* (3.2) *is finite and non-trivial independently of the choice of the number r provided it is sufficiently small.*

As an immediate corollary of this statement we obtain that for any set $Z\subset X$ of positive m^u-measure we have that $\dim_C Z=P$. One can indeed, show that

$$\dim_C Z=\underline{\mathrm{Cap}}_C Z=\overline{\mathrm{Cap}}_C Z=P.$$

Proof of Theorem 3.3. We need the following three facts about locally maximal hyperbolic sets.

1. A finite set $E\subset\Lambda$ is called (n,δ)-*separated* if for every $x\in\Lambda$ there is a point $y\in E$ such that for all $0\le k\le n$ we have that $d(f^k(x),f^k(y))\le\delta$.

For any $\delta>0$ and a subset $Y\subset\Lambda$ let

$$N(Y,\delta,n):=\sup\left\{\sum_{x\in E}e^{S_n\varphi(x)}\right\}, \tag{3.3}$$

where the supremum is taken over all (δ,n)-separated subsets $E\subset Y$. One can show that

(1) The topological pressure of φ on Λ by

$$P(\varphi):=\lim_{\delta\to0}\lim_{n\to\infty}\frac{1}{n}\log N(\Lambda,\delta,n).$$

(2) For any $0<\delta<\delta_0/3$ there exists $C=C(\delta)>0$ such that

$$C^{-1}e^{nP}\le N(\Lambda,n,\delta)\le Ce^{nP}. \tag{3.4}$$

2. A diffeomorphism f on Λ has the *specification property* that is for every $\delta\ge0$ there is an integer $T(\delta)$ for which the following is true: if $I_1,\dots,I_n$ are intervals of integers contained in $[a,b]$ with $d(I_i,I_j)\ge T(\delta)$ for $i\ne j$ and $x_1,\dots,x_n\in\Lambda$, then there is a point $x\in\Lambda$ with $f^{b-a+T(\delta)}(x)=x$ and $d(f^k(x),f^k(x_i))<\delta$ for $k\in I_i$.

3. A Hölder continuous function φ on Λ has *Bowen's property* that is there exist $K > 0$ and $\varepsilon > 0$ such that for every $x \in \Lambda$ and $n > 0$ if $y \in B_n(x, \varepsilon)$, then $|S_n\varphi(x) - S_n\varphi(y)| \le K$.

We proceed with the proof of the theorem. Fix $x_0 \in \Lambda$ and set $V := V^u(x_0)$. Let E_N be a maximal $(N, \frac{r}{4})$-separated set in Λ. Then $\{B_N(x, \frac{r}{2}) : x \in E_N\}$ is a cover of Λ. In particular, $V \cap \Lambda \subset \bigcup_{i=1}^{\ell} B_N(x_i, \frac{r}{2})$ for some $x_1, \dots, x_\ell \in E_N$. For $i = 1, \dots, \ell$ pick $z_i \in V \cap \Lambda$ such that $z_i \in B_N(x_i, \frac{r}{2})$. Then $V \cap \Lambda \subset \bigcup_{i=1}^{\ell} B^u_N(z_i, r)$. Using Bowen's property, (3.3), and (3.4), we obtain that

$$\begin{aligned} &\inf\left\{\sum_i \exp\left(-Pn_i + S_N\varphi(y_i)\right) : y_i \in V \cap \Lambda, E \subset \bigcup_i B^u_N(y_i, r)\right\} \\ &\le \sum_{i=1}^{\ell} \exp\left(-PN + S_N\varphi(z_i)\right) \le e^{K-PN} \sum_{i=1}^{\ell} e^{S_N\varphi(x_i)} \\ &\le e^{K-PN} N(\Lambda, \frac{r}{4}, N) \le e^{K-PN} C(\frac{r}{4}) e^{NP} \le e^K C(\frac{r}{4}). \end{aligned}$$

We conclude that $m_C(V, P) < \infty$.

We shall prove that the measure of Λ is not zero. To this end consider $y \in \Lambda$, $n \in \mathbb{N}$ and $M > n$. By Lemma 3.4, we have

$$\begin{aligned} e^{S_n\varphi(y) - nP} &\ge e^{S_n\varphi(y)} e^{-MP} C^{-1}(r) N(\Lambda, r, M - n) \\ &\ge e^{S_n\varphi(y)} \frac{C^{-2}(r)}{N(\Lambda, r, M)} N(f^n(B^u_n(y, r)), r, M - n) \\ &\ge e^{-K} \frac{C^{-2}(r)}{N(\Lambda, r, M)} N(B^u_n(y, r), r, M). \end{aligned}$$

Consequently, for any finite cover $\{B^u_{n_i}(y_i, r)\}_i$ of $V \cap \Lambda$ and $M > \max_i\{n_i\}$ we obtain that

$$\begin{aligned} \sum_i \exp(-Pn_i + S_{n_i}\varphi(y_i)) &\ge C \sum_i \frac{N(B^u_{n_i}(y_i, r), r, M)}{N(\Lambda, r, M)} \\ &\ge C \frac{N(\mathcal{O}_r, r, M)}{N(\Lambda, r, M)}, \end{aligned}$$

where

$$\mathcal{O}_r(V) := \left\{ x \in \bigcup_{y \in V} V^s(y) \cap \Lambda : \operatorname{dist}(x, V) < r \right\}.$$

Let $T = T(r)$ be a constant from the specification property and let $M > T$ and $m = M - T$. Consider a maximal $(m, 3r)-$ separated set E_m. For every $x \in E_m$ there exists $p(x) \in \mathcal{O}_r(V)$ such that $f^M(p) = p$ and $d_m(f^T(p(x)), x) \le r$. Note that the set $\{p(x) : x \in E_m\}$ is (M, r)-separated.

We then obtain that

$$\begin{aligned}\frac{N(\mathcal{O}_r, r, M)}{N(\Lambda, r, M)} &\geq \frac{1}{N(\Lambda, r, M)} \sum_{x\in E_m} e^{S_M\varphi(p(x))} \\ &\geq \frac{e^{-K-T\|\varphi\|_{C^0}}}{N(\Lambda, r, M)} \sum_{x\in E_m} e^{S_m\varphi(x)} \\ &= \frac{e^{-K-T\|\varphi\|_{C^0}}}{N(\Lambda, r, M)} N(\Lambda, 3r, m) \\ &\geq e^{-K-T\|\varphi\|_{C^0}} C^{-1}(3r) e^{mP} C^{-1}(r) e^{-MP} \geq \tilde{C}(r).\end{aligned}$$

We conclude that $m_C(V, P) > 0$. To prove the first statement note that both estimates hold if we replace n_i by n and that the outer measure is metric, hence Borel. □

3.5. **The evolution of the measure m^u under the dynamics.** The measure m^u on V^u can be extended to a measure $\tilde{m}^u$ on Λ by setting $\tilde{m}^u(A) := m^u(A \cap V^u)$ for any measurable set $A \subset \Lambda$. Consider the sequence $\{m^u_n\}$ of measures on Λ defined by

$$m^u_n := \frac{1}{n} \sum_{k=0}^{n-1} f^k_* \tilde{m}^u. \tag{3.5}$$

Theorem 3.4. *Let Λ be a locally maximal hyperbolic set for a diffeomorphism f. Assume that $f|\Lambda$ is topologically transitive. Then the sequence* (3.5) *converges in the weak* topology to a measure μ_φ which is the unique equilibrium measure for the potential φ.*

Proof. Fix a number $n > 0$, a point $x_0 \in \Lambda$ and let $V = V^u(x_0)$. Recall that V has a fixed size $\tau > 0$. Denote by $V_n := f^n(V)$ and let m_{V_n} be the P-Carathéodory measure on V_n that is for $Z \subset V_n$,

$$m_{V_n}(Z) = \lim_{N\to\infty} \inf \left\{ \sum_i \exp\left(-Pn_i + S_{n_i}\varphi(x_i)\right) \right\}, \tag{3.6}$$

where the infimum is taken over all $x_i \in V_n \cap \Lambda$ with $n_i \geq N$ such that $Z \subset \bigcup_i B^u_{n_i}(x_i, r)$.

We also consider the measure $f^n_* m_V$ on V_n, where for $E \subset V_n$ we set $f^n_* m_V(E) = m_V(f^{-n}(E))$. For $x \in V_n$ define

$$g_n(x) := \exp(-Pn + \sum_{k=-n}^{-1} \varphi(f^k(x))).$$

Lemma 3.5. *For every Borel set $E \subset V_n$ we have that*

$$\int_E g_n(x) dm_{V_n} = f^n_* m_V(E).$$

We need the following version of the Besicovich Covering Lemma.

Lemma 3.6. *There are points $y_n^{(i)} \in V_n$, $i = 1, \dots, s$ such that*

(1) the sets $V_n^{(i)} := V(y_n^{(i)})$ form an open cover of V_n of finite multiplicity, which depends only on the dimension of V_n;
(2) the balls $B^u(y_n^{(i)}, \frac{1}{4}\tau) \subset V_n^i$ are disjoint.

This result allows us to partition V_n by the sets $\tilde{V}_n^{(1)}, \dots, \tilde{V}_n^{(s)}$, where

$$\tilde{V}_n^{(i)} := V_n^{(i)} \setminus \bigcup_{j=1}^{i-1} V_n^{(j)}. \tag{3.7}$$

Consequently, Lemma 3.5 gives that for every Borel set $E \subset V_n$,

$$\begin{aligned} f_*^n m_V(E) = \int_E g_n(x) dm_{V_n} &= \sum_{i=1}^{s} \int_E g_n(x)\, dm_{\tilde{V}_n^{(i)}} \\ &= \sum_{i=1}^{s} g_n(y_n^{(i)}) \int_E \frac{g_n(x)}{g_n(y_n^{(i)})}\, dm_{\tilde{V}_n^{(i)}}. \end{aligned} \tag{3.8}$$

We have the following statement.

Lemma 3.7. *There exists $L = L(\tau, \dim(V)) > 0$ such that for every $n \in \mathbb{N}$ and every cover as in Lemma 3.6 we have that:*

(1) $\frac{g_n(x)}{g_n(y_n^{(i)})} \in C^\alpha(V_n^{(i)}, [\frac{1}{L}, L])$;
(2) $\left|\frac{g_n(x)}{g_n(y_n^{(i)})}\right|_\alpha \le L$;
(3) $\sum_{i=1}^{s} g_n(y_n^{(i)}) < L$.

For the rest of the proof fix $L > 0$ and let

$$\mathcal{R} := \{W \setminus \bigcup_{i=1}^{p'} V(y_i) : 0 \le p' \le p,\ y_1, y_2, \dots, y_{p'} \in \Lambda\},$$

where p is the bound on multiplicity in Lemma 3.6.

We now define the space of *standard pairs*:

$$\mathcal{R}' := \left\{(\tilde{V}, \rho) : \tilde{V} \in \mathcal{R}^\sqcap, \rho \in C^\alpha(\tilde{V}, [1/L, L]), |\rho|_\alpha \le L\right\}.$$

We endow $\mathcal{R}'$ with the natural product topology of the space of local unstable manifolds $\tilde{V}$ and the space $C^\alpha(\tilde{V}, [1/L, L])$. It is easy to see that $\mathcal{R}'$ is compact in this topology.

A standard pair determines a measure $\Psi(\tilde{V}, \rho)$ on Λ:

$$\Psi(\tilde{V}, \rho)(E) := \int_{E \cap \tilde{V}} \rho\, dm_{\tilde{V}}. \tag{3.9}$$

Moreover, each finite measure η on $\mathcal{R}'$ determines a measure $\Phi(\eta)$ on Λ by

$$\Phi(\eta)(E) := \int_{\mathcal{R}'} \Psi(\tilde{V}, \rho)(E)\, d\eta(\tilde{V}, \rho) = \int_{\mathcal{R}'} \int_{E \cap \tilde{V}} \rho\, dm_{\tilde{V}}\, d\eta(\tilde{V}, \rho). \tag{3.10}$$

Write $\mathcal{M}(\Lambda)$ and $\mathcal{M}(\mathcal{R}')$ for the spaces of finite Borel measures on Λ and $\mathcal{R}'$ respectively. One can see that $\Phi : \mathcal{M}(\mathcal{R}') \to \mathcal{M}(\Lambda)$ is continuous. In particular, $\mathcal{M} := \Phi(\mathcal{M}_{\leq L}(\mathcal{R}'))$ is compact, where we write $\mathcal{M}_{\leq L}$ for the space of measures of total weight at most L.

Consider the sequence of measures (3.5). We see from (3.8) that for every $k \in \mathbb{N}$ and $E \in V_k$,

$$f_*^k m_V(E) = \sum_{i=1}^{s} g_k(y_k^{(i)}) \int_E \frac{g_k(x)}{g_k(y_k^{(i)})}\, dm_{\tilde{V}_k^{(i)}}.$$

Taking $\rho_k^i := \frac{g_n(x)}{g_n(y_n^{(i)})}$, Lemma 3.7 implies that each pair $(\tilde{V}_k^{(i)}, \rho_k^i)$ is in $\mathcal{R}'$. In addition, we can define a measure η_k on $\mathcal{R}'$ by setting

$$\eta_k((\tilde{V}_k^{(i)}, \rho_k^i)) = g_k(y_k^{(i)}).$$

The last statement of Lemma 3.7 implies that $\eta_k(\mathcal{R}') \leq L$. Consequently, $f_*^k m_V(E) \in \mathcal{M}$. The same is true for the average, i.e., $m_n \in \mathcal{M}$ for every n.

By compactness of $\mathcal{M}$ one can pass to a convergent subsequence $m_{n_k} \to \mu \in \mathcal{M}$. We have the following statement.

Lemma 3.8. *The measure μ is an equilibrium measure for φ. Moreover, the conditional measures generated by μ on the local unstable leaves $V^u(x)$ are equivalent to the P-Carathéodory measure m^u on $V^u(x)$.*

This completes the proof of the theorem subject to the proofs of Lemmas 3.5, 3.7, and 3.8.

Proof of Lemma 3.8. The claim that the conditional measures generated by μ on the local unstable leaves $V^u(x)$ are equivalent to the P-Carathéodory measure m^u on $V^u(x)$ immediately follows from the fact that by construction $\mu \in \mathcal{M}$.

Let x be a generic point for μ and let $\delta > 0$. We need the following result.

Lemma 3.9. *There exist $C > 0$ such that for every $n \in \mathbb{N}$,*

$$C^{-1} \exp(-Pn + \sum_{k=0}^{n-1} \varphi(f^k(x))) < \mu(B_n(x, \delta)) < C \exp(-Pn + \sum_{k=0}^{n-1} \varphi(f^k(x))).$$

Proof. Consider the partition ξ of $B(x, \delta)$ by local unstable leaves. There exists a factor-measure $\tilde{\mu}$ in the factor space $B(x, \delta)/\xi$ and the system of conditional measures $\mu_{V(y)}$, $y \in B(x, \delta)/\xi$ such that,

$$\mu(B_n(x, \delta)) = \int_{B(x,\delta)/\xi} \int_{V(y)} \chi_{B_n(x,\delta)}(y, z)\, d\mu_{V(y)}(z)\, d\tilde{\mu}(y).$$

Since $\mu \in \mathcal{M}$, there exists $\rho \in C^{\alpha}(V, [1/L, L])$ such that we can write,

$$\begin{aligned}\mu(B_n(x,\delta)) &= \int_{B(x,\delta)/\xi} \int_{V(y)} \rho \chi_{B_n(x,\delta)}(y,z)\, d\mu_{V(y)}(z)\, d\tilde{\mu}(y) \\ &\le L \int_{B(x,\delta)/\xi} \int_{V(y)} \chi_{B_n(x,\delta)}(y,z)\, dm_{V(y)}(z)\, d\tilde{\mu}(y) \\ &= L \int_{B(x,\delta)/\xi} m_{V(y)} \left(f^{-n}(V(f^n(y)) \cap B(f^n(x),\delta))\right) \, d\tilde{\mu}(y) \\ &= L \int_{B(x,\delta)/\xi} f^n_* m_{V(y)}(B(f^n(x),\delta))\, d\tilde{\mu}(y).\end{aligned}$$

By Lemma 3.5, we may continue

$$= L \int_{B(x,\delta)/\xi} \int_{B(f^n(x),\delta)} g_n(z)\, dm_{V(f^n(y))}(z)\, d\tilde{\mu}(y)$$

(3.11)

$$\le L \int_{B(x,\delta)/\xi} e^K \exp(-Pn + \sum_{k=0}^{n-1} \varphi(f^k(x))) m_{V(f^n(y))}(B(f^n(x),\delta)) \, d\tilde{m}(y).$$

By Theorem 3.3, we have that

$$m_{V(f^n(y))}(B(f^n(x),\delta)) \le m_{V(f^n(y))}(B(f^n(y),2\delta)) \le C(2\delta).$$

Therefore, (3.11) can be estimated by $C \exp(-Pn + S_n\varphi(x))$ with some $C = C(x,\delta) > 0$.

From the above argument it is clear how to obtain the reverse inequality in (3.11). Namely, we have that

$$\begin{aligned}&\mu(B_n(x,\delta)) \\ &\quad\ge 1/L \int_{B(x,\delta)/\xi} e^{-K} \exp(-Pn + S_n\varphi(x)) m_{V(f^n(y))}(B(f^n(x),\delta))\, d\tilde{\mu}(y) \\ &\quad\ge 1/L \int_{B(x,\frac{1}{2}\delta)/\xi} e^{-K} \exp(-Pn + S_n\varphi(x)) m_{V(f^n(y))}(B(f^n(x),\delta))\, d\tilde{\mu}(y).\end{aligned}$$

Observe that if $f^n(y) \in B(f^n(x), \frac{1}{2}\delta)$, then

$$V(f^n(y)) \cap B(f^n(y), \frac{1}{2}\delta) \subset V(f^n(y)) \cap B(f^n(x),\delta).$$

Consequently, Theorem 3.3 implies that

$$m_{V(f^n(y))}(B(f^n(x),\delta)) \ge m_{V(f^n(y))}(B(f^n(y), \frac{1}{2}\delta)) \ge C^{-1}(\frac{1}{2}\delta).$$

We then conclude that

$$\mu(B_n(x,\delta)) \ge C^{-1} \exp(-Pn + S_n\varphi(x))$$

and the desired result follows. □

To finish the proof of Lemma 3.8 observe that by Lemma 3.9,

$$\begin{aligned}\lim_{n\to\infty}\frac{-\log\mu(B_n(x,\delta))}{n} &= \lim_{n\to\infty}\frac{-\log\exp(-Pn+S_n\varphi(x))}{n}\\ &= \lim_{n\to\infty}\left(\frac{Pn}{n}-\frac{1}{n}\sum_{k=0}^{n-1}\varphi(f^k(x))\right) = P-\int_\Lambda\varphi\,d\mu.\end{aligned}$$

By the Brin-Katok formula for the metric entropy, this implies that

$$P = h_\mu(f)+\int_\Lambda\varphi\,d\mu$$

and hence, μ is an equilibrium measure. □

□

4. Lecture IV: The geometric approach for constructing SRB measures for uniformly hyperbolic attractors with singularities

4.1. **Maps with singularities.** Let M be a smooth compact manifold, $U \subset M$ an open bounded connected subset, *the trapping region*, $N \subset U$ a closed subset and $f\colon U \setminus N \to U$ a C^2 diffeomorphism such that

$$\begin{aligned} \|d^2 f_x\| &\le C_1 d(x, \mathcal{S}^+)^{-\alpha_1} \text{ for any } x \in U \setminus N, \\ \|d^2 f_x^{-1}\| &\le C_2 d(x, \mathcal{S}^-)^{-\alpha_2} \text{ for any } x \in f(U \setminus N), \end{aligned} \tag{4.1}$$

where $\mathcal{S}^+ = N \cup \partial U$ is the *singularity set for* f and $\mathcal{S}^- = f(\mathcal{S}^+)$ that is

$$\mathcal{S}^- = \{y \in U : \text{there is } z \in \mathcal{S}^+, z_n \in U \backslash \mathcal{S}^+ \text{ such that } z_n \to z, f(z_n) \to f(z)\}$$

is the *singularity set for* f^{-1}. We will assume that $m(\mathcal{S}^+) = m(\mathcal{S}^-) = 0$.

Define

$$U^+ = \{x \in U : f^n(x) \notin \mathcal{S}^+, n = 1, 2, \dots\}$$

and the *topological attractor with singularities*

$$D = \bigcap_{n \ge 0} f^n(U^+), \quad \Lambda = \bar{D}.$$

Given $\varepsilon > 0$ and $\ell > 1$, set

$$\begin{aligned} D^+_{\varepsilon,\ell} &= \{z \in \Lambda : d(f^n(z), \mathcal{S}^+) \ge \ell^{-1} e^{-\varepsilon n}, n = 0, 1, 2, \dots\}, \\ D^-_{\varepsilon,\ell} &= \{z \in \Lambda : d(f^n(z), N^-) \ge \ell^{-1} e^{-\varepsilon n}, n = 0, 1, 2, \dots\}, \\ D^0_{\varepsilon,\ell} &= D^+_{\varepsilon,\ell} \bigcap D^-_{\varepsilon,\ell}, \\ D^0_{\varepsilon} &= \bigcup_{\ell \ge 1} D^0_{\varepsilon,\ell}. \end{aligned}$$

The set D^0_ε is the *core* of the attractor and it may be an empty set as it may be the set D.

Theorem 4.1. *Assume that there are $C > 0$ and $q > 0$ such that for any $\varepsilon > 0$ and $n > 0$*

$$m(f^{-n}(\mathcal{U}(\varepsilon, \mathcal{S}^+) \cap f^n(U^+))) \le C\varepsilon^q, \tag{4.2}$$

where $\mathcal{U}(\varepsilon, \mathcal{S}^+)$ is a neighborhood of the (closed) set $\mathcal{S}^+$. Then there is an invariant measure μ on Λ such that, $\mu(D^0_\varepsilon) > 0$, in particular, the core is not empty.

4.2. **Hyperbolic attractors with singularities.** We say that a topological attractor with singularities Λ is *hyperbolic*, if there exist two families of stable and unstable cones

$$K^s(x) = K(x, E_1(x), \theta(x)),\ K^u(x) = K(x, E_2(x), \theta(x)),\ x \in U \setminus \mathcal{S}^+$$

such that

(1) the angle $\angle(E_1(x), E_2(x)) \ge$ const. ;

(2) $df(K^s(x)) \subset K^s(f(x))$ for any $x \in U \setminus \mathcal{S}^+$ and $df^{-1}(K^u(x)) \subset K^u(f(x))$ for any $x \in f(U \setminus \mathcal{S}^+)$;
(3) for some $\lambda > 1$
   (a) $\|df_x v\| \geq \lambda \|v\|$ for $x \in U \setminus \mathcal{S}^+$ and $v \in K^u(x)$;
   (b) $\|df_x^{-1} v\| \geq \lambda \|v\|$ for $x \in f(U \setminus \mathcal{S}^+)$ and $v \in K^s(x)$.

Theorem 4.2. *Let Λ be a hyperbolic attractor with singularities for a $C^{1+\alpha}$ map and assume that Condition* (4.2) *holds. Then f admits an SRB measure on Λ.*

4.3. **Examples of hyperbolic attractors with singularities.** We describe the following three examples of hyperbolic attractors with singularities which satisfy requirements (4.1) and (4.2) and thus possess SRB measures.

The geometric Lorenz attractor. Let $I = (-1, 1)$, $U = I \times I$, $N = I \times 0 \subset U$ and $f : U \setminus N \to U$ is given by

$$f(x, y) = ((-B|y|^{\nu_0} + B\text{sign}(y)|y|^{\nu} + 1)\text{sign}(y),\ ((1 + A)|y|^{\nu_0} - A)\text{sign}(y)),$$

where

$$0 < A < 1, \quad 0 < B < \frac{1}{2}, \quad \nu > 1, \quad \frac{1}{1 + A} < \nu_0 < 1.$$

This attractor models the behavior of the Poicaré map on an appropriately chosen cross-section for the flow generated by the Lorenz system of ODE:

$$\dot{x} = -\sigma x + \sigma y, \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz$$

for the values of the parameters $\sigma = 10$, $b = \frac{8}{3}$ and $r \sim 24.05$. **The Lozi attractor**. Let $I = (-c, c)$ for some $0 < c < 1$ and let $U = I \times I$, $N = 0 \times I \subset U$ and $f \colon U \setminus N \to U$ is given by

$$f(x, y) = (1 + by - a|x|, x),$$

where $0 < a < a_0$ and $0 < b < b_0$ for some small $a_0 > 0$ and $b_0 > 0$.

Up to a change of coordinates this map was introduced by Lozi as a simple version of the famous Hénon map in population dynamics.

The Belykh attractor. Let $I = (-1, 1)$, $U = I \times I$, $N = \{(x, y) : y = kx\} \subset U$ and $f : U \setminus N \to U$ is given by

$$f(x, y) = \begin{cases} (\lambda_1(x - 1) + 1,\ \lambda_2(y - 1) + 1) & \text{for } y > kx, \\ (\mu_1(x + 1) - 1,\ \mu_2(y + 1) - 1) & \text{for } y < kx, \end{cases}$$

where

$$0 < \lambda_1, \mu_1 < \frac{1}{2}, \quad 1 < \lambda_2, \quad \mu_2 < \frac{2}{1 - |k|}, \quad |k| < 1.$$

In the case $\lambda_1 = \mu_1$ and $\lambda_2 = \mu_2$ this map was introduced by Belykh as one of the simplest models in the phase synchronization theory in radiophysics.

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