The variational method of solution to the Boltzmann equation: Theory and applications

Silvia Lorenzani

Dipartimento di Matematica, Politecnico di Milano, Italy

In 1969, Carlo Cercignani published, in the Journal of Statistical Physics, a paper entitled: 'A Variational Principle for Boundary Value Problems in Kinetic Theory'. Before this date, variational methods had been used by different authors dealing with kinetic theory. All these authors considered the integral form of the linearized Boltzmann equation. The main idea of this method is the following. Suppose that one has to solve the equation

$$\mathcal{L}h = S \tag{1}$$

where h is the unknown, $\mathcal{L}$ is a linear operator and S a source term. Let us assume that $\mathcal{L}$ is self-adjoint with respect to a certain scalar product $((\cdot,\cdot))$. Then, the functional

$$J(\tilde{h}) = ((\tilde{h}, \mathcal{L}\tilde{h})) - 2((S, \tilde{h})) \tag{2}$$

has the property that, if we set

$$\tilde{h} = h + \eta, \tag{3}$$

the terms of first degree in η disappear and $J(\tilde{h})$ reduces to

$$J(h) + ((\eta, \mathcal{L}\eta)) \tag{4}$$

if and only if h is a solution of Eq. (1).

Thus, a way to look for solutions of Eq. (1) is to look for solutions that make the functional in Eq. (2) stationary. Furthermore, the variational formulation implies that if the functional J is related to some physical quantity, one can compute this quantity with high accuracy, even if we have a poor approximation to h. A disadvantage of this method is that, since the integral form of the linearized Boltzmann equation is available explicitly only for simplified kinetic models, in the more general case of the true linearized collision operator, one has to split such operator into two parts. This fact has the consequence that it is not possible to obtain a closed form expression for the collision operator, which must consequently be approximated by series expansions, and that the basic functional defined in the variational formulation changes from problem to problem.

The novelty introduced by Carlo Cercignani in his paper was the formulation of a new variational principle applied directly to the integrodifferential form of the linearized Boltzmann equation. Within this framework, extremely general boundary conditions and collision terms are allowed.

In this talk, we will discuss about the recent theoretical developments of this variational principle, originally introduced by Carlo Cercignani, and we will present some applications of the method to the computation of parameters employed in classical hydrodynamical equations, when low working pressures impose corrections due to gas rarefaction effects.

References

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