Surgery formulae for 3–manifold invariants defined via configuration space

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M. Kontsevich proposed a topological construction for an invariant Z of oriented rational homology 3-spheres using configuration space integrals [Ko]. (Other constructions were proposed by Axelrod, Singer, Bott, Cattaneo... following Witten [AS1, AS2, BC1, BC2, Wi].) G. Kuperberg and D. Thurston proved that Z is a universal real finite type invariant for integral homology spheres in the sense of Ohtsuki, Habiro and Goussarov. They also showed that the degree one part of Z is the invariant defined by Casson in 1984 (as an algebraic number of $SU(2)$-representations of the π_1 [AM, M]) for integral homology 3-spheres. See [KT].

Here, I propose to give an elementary presentation of the Casson invariant following the Kontsevich-Kuperberg-Thurston ideas, and to use it to illustrate some generalisations of the Kuperberg-Thurston results.

1 A topological invariant as a triple intersection in the two-point configuration space of a homology sphere

Let M be an oriented rational homology 3-sphere (a closed 3-manifold such that $H_*(M;\mathbb{Q}) = H_*(S^3;\mathbb{Q})$). Identify an open ball in M to a neighborhood of ∞ in $S^3 = \mathbb{R}^3 \cup \infty$ with the help of an embedding of $S^3 \setminus B^3(1)$ into M, where $B^3(1)$ is the unit ball of $\mathbb{R}^3$. Consider the open configuration space of ordered pairs of distinct points

$$\breve{C}_2(M) = (M \setminus \infty)^2 \setminus \text{diagonal}$$

and its compactification $C_2(M)$
that is obtained from M^2 by blowing-up (∞,∞) and the closures of $\infty \times (M \setminus \infty)$, $(M \setminus \infty) \times \infty$ and $\text{diag}(M \setminus \infty)$, successively. Here to blow-up a submanifold amounts to replace it by its unit normal bundle.
This compactification $C_2(M)$ is a compact 6-dimensional manifold with the same homotopy type as $\breve{C}_2(M)$ and with the same rational homology as the sphere S^2. A *good* trivialisation for M is a trivialisation τ_M of the tangent bundle of $M \setminus \infty$ that coincides with the trivialisation of the tangent bundle of $\mathbb{R}^3$ near ∞. Such a trivialisation gives rise to a projection map

$$p(\tau_M) : \partial C_2(M) \longrightarrow S^2$$

defined as follows. When $M = S^3$, and when $S^3 \setminus \infty = \mathbb{R}^3$ is equipped with its standard trivialisation τ_0, $p(\tau_0)$ is continuously defined over the whole configuration space $C_2(S^3)$, and

$$p(\tau_0)\left((x,y) \in \breve{C}_2(S^3)\right) = \frac{y-x}{||y-x||}.$$

Now, for any closed 3-manifold M that is equipped with a good trivialisation, $p(\tau_M)$ can similarly be defined as the "limit direction of a vector from one point to another one" on $\partial C_2(M)$.
(There are essentially three ways for a pair (x,y) of points in $\breve{C}_2(M)$ to approach $\partial C_2(M)$, either x approaches ∞ and $p(\tau_M)$ approaches the direction of $(-x)$ (if y is not in the fixed neighborhood of ∞), or y approaches ∞, or x and y become close to each other and $p(\tau_M)$ approaches the direction of the vector between them given by the trivialisation.)
For any point a in S^2, there exists a rational 4-chain Σ_a (and even a genuine 4-manifold when M is a homology sphere) such that

$$\partial\Sigma_a = p(\tau_M)^{-1}(a) = \Sigma_a \cap \partial C_2(M).$$

Furthermore, Σ_a is well-determined up to cobordism in the interior of $C_2(M)$. Pick three distinct points a, b, c of S^2, then define $6\lambda(M, \tau_M)$ as the algebraic intersection of the 3 codimension 2 submanifolds Σ_a, Σ_b and Σ_c. It is clear that $\lambda(M, \tau_M)$ is a topological invariant of (M, τ_M).

Now, Pontrjagin classes provide a natural map

$$p_1 : \{ \text{ homotopy classes of good trivialisations of } M \ \} \longrightarrow \mathbb{Z}.$$

This map is defined as the relative first Pontrjagin class of a signature 0 cobordism W with corners between the rational homology ball B_M and the ball B^3 whose boundary $\partial W = B_M \cup (-[0,1] \times S^2) \cup -B^3(1)$ is equipped with a trivialisation naturally induced by τ_M on B_M and by τ_0 on $(-[0,1] \times S^2) \cup -B^3(1)$.

When M is a homology sphere, p_1 is a bijection from the set of good trivialisations of $T(M \setminus \infty)$ (up to homotopy) to $4\mathbb{Z}$, and

$$\lambda(M) = \lambda(M, p_1^{-1}(0))$$

is a topological invariant of M. For rational homology spheres, there is not always such a preferred homotopy class of good trivialisations. Nevertheless, it can be shown that in any case

$$\lambda(M) = \lambda(M, \tau_M) - \frac{p_1(\tau_M)}{24}$$

is a topological invariant of rational homology spheres, and that

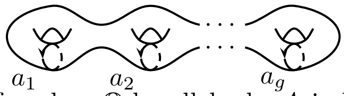
$$\lambda(-M) = -\lambda(M).$$

(So far, I have only given a dual version of the original Kontsevich definition, further explained by D. Thurston and G. Kuperberg in [KT]. See also [L3, 6.5].)

2 A characteristic property of λ

The computation of λ is allowed by the following *finite type* type property.

A *genus g $\mathbb{Q}$-handlebody* is an (oriented, compact) 3-manifold A with the same homology with rational coefficients as the standard (solid) handlebody H_g below.

$H_g =$ 

Note that the boundary of such a $\mathbb{Q}$-handlebody A is homeomorphic to the boundary ($\partial H_g = \Sigma_g$) of H_g. The intersection form on a surface Σ is denoted by $\langle , \rangle_\Sigma$. For a $\mathbb{Q}$-handlebody A, $\mathcal{L}_A$ denotes the kernel of the map induced by the inclusion:

$$H_1(\partial A; \mathbb{Q}) \longrightarrow H_1(A; \mathbb{Q}).$$

It is a Lagrangian of $(H_1(\partial A; \mathbb{Q}), \langle , \rangle_{\partial A})$, we call it the *Lagrangian* of A.

Now, consider some 2–component lagrangian-preserving surgery data that is some

$$(M; (A, A'), (B, B'))$$

where

1. M is a rational homology sphere,
2. for any $C \in \{A, B\}$, C and C' are $\mathbb{Q}$-handlebodies whose boundaries are identified by implicit diffeomorphisms (we shall write $\partial C = \partial C'$) so that $\mathcal{L}_C = \mathcal{L}_{C'}$,
3. the disjoint union $A \sqcup B$ is embedded in M

For a subset I of $\{A, B\}$, let M_I be the manifold obtained from M by replacing C by C', for any C in I. The computation of

$$\lambda(M_{\{AB\}}) - \lambda(M_A) - \lambda(M_B) + \lambda(M)$$

is very intuitive, and can be sketched as follows.
We can compute the invariants using 4-chains Σ^I_a, that coincide wherever it makes sense (for example, they coincide over the set of pairs of elements of $M \setminus (A \sqcup B)$). In this way the only triple intersection points that may contribute to the above alternate sum must involve points in (A or A') and in (B or B'). Therefore, the support of the contributing triple intersections is the union of $A \times B$, $A \times B'$, $A' \times B$, $A' \times B'$, and its symmetric under the exchange of points in pairs. In order to fix the 4-chains Σ there, we shall introduce more notation.
Consider a set of curves $\{a_1, a_2, \ldots, a_{g_A}\}$ like in the picture made of representatives of a basis of $\mathcal{L}_A$. Each a_i bounds a 2-chain $S(a_i)$ in A and a 2-chain $S'(a_i)$ in A'. Let $\{z_1, z_2, \ldots, z_{g_A}\}$ be a set of curves on the boundary of A such that the intersection numbers $\langle a_i, z_j \rangle_{\partial A}$ equal the Kronecker δ_{ij}. Similarly, fix a basis $\{b_1, b_2, \ldots, b_{g_B}\}$ for $\mathcal{L}_B$, 2-chains $S(b_i)$ in B, $S'(b_i)$ in B', with $\partial S(b_i) = \partial S'(b_i) = b_i$, and dual curves y_j. Now, coming back to the definition of the 4-dimensional chains Σ that are Poincaré dual to a canonical generator of $H^2(C_2(M); \mathbb{Q})$, there are several ways to see that for any pair of disjoint

knots (J,K) in $M \setminus \infty$, the algebraic intersection of the corresponding torus $J \times K \subset C_2(M)$ with Σ is the linking number $\ell(J,K)$ of J and K in M. In particular, it is reasonable to expect that Σ intersects $A \times B$ as

$$\sum_{(i,j)\in\{1,2,\dots,g_A\}\times\{1,2,\dots,g_B\}} \ell(z_i,y_j) S(a_i) \times S(b_j).$$

This can indeed be achieved, and this gives rise to the formula

$$\lambda(M_{\{AB\}}) - \lambda(M_A) - \lambda(M_B) + \lambda(M)$$

$$= \frac{2}{6}\left(\sum_{(i,j)\in\{1,2,\dots,g_A\}\times\{1,2,\dots,g_B\}} \ell(z_i,y_j)(S(a_i)\cup -S'(a_i)) \times (S(b_j)\cup -S'(b_j))\right)^3$$

The third power should be understood as the triple intersection of 4-manifolds in the compact 6-dimensional manifold $(A \cup -A') \times (B \cup -B')$, and in this case can be expressed from products of triple intersections in $(A \cup -A')$ and $(B \cup -B')$. Consider the triple intersection of surfaces in $(A\cup -A')$ as the linear form $\mathcal{I}_{AA'}$ in $(\otimes^3 \mathcal{L}_A)^* = \otimes^3 \mathcal{L}_A^*$ that maps (a_i,a_j,a_k) to the intersection of $S(a_i)\cup -S'(a_i)$, $S(a_j)\cup -S'(a_j)$ and $S(a_k)\cup -S'(a_k)$.

Theorem 1 *[L3]*

$$\lambda(M_{\{AB\}}) - \lambda(M_A) - \lambda(M_B) + \lambda(M) =$$

$$-\frac{2}{6}\left(\sum_{\substack{(i_1,i_2,i_3)\in\{1,2,\dots,g_A\}^3\\ (j_1,j_2,j_3)\in\{1,2,\dots g_B\}^3}} \prod_{k=1}^{3} \ell(z_{i_k},y_{j_k}) \mathcal{I}_{AA'}(a_{i_1},a_{i_2},a_{i_3}) \mathcal{I}_{BB'}(b_{j_1},b_{j_2},b_{j_3})\right)$$

In the special case when $(A\cup -A')$ and $(B \cup -B')$ are both genus 3 Heegaard splittings of $(S^1)^3$, such that $\ell(y_i,z_j)$ is the Kronecker δ_{ij}, this number is exactly (-2) and we already see that λ is non trivial.

When B is obtained from A by perturbing the boundary identification by a Torelli homeomorphism (a homeomorphism that induces the identity in homology), the above theorem was observed by Kuperberg and Thurston in [KT] and allowed them to identify λ with the Casson invariant. Indeed, this formula (together with the property $\lambda(S^3)=0$) is sufficient to characterize the Casson invariant. I had proved the formula of the theorem for the Walker generalisation of the Casson invariant in 1994 [L1]. In [L3], I proved that together with the property that $\lambda(-M) = -\lambda(M)$, it determines the Walker invariant. In particular, the constructed invariant λ is the Casson-Walker invariant of rational homology spheres.

A more direct identification with the Casson-Walker invariant can be obtained through the following surgery formula that can be seen as a direct consequence of the above theorem.

If K is a null-homologous knot in M, and if a p/q-surgery is performed on it to transform M into $M_{(K;p/q)}$, the above formula allows for a quick computation of

$$\lambda(M_{(K;p/q)}) - \lambda(M \sharp L(p,-q))$$

where $L(p,-q)$ is the lens space obtained by p/q-surgery on the trivial knot and $\sharp$ stands for the connected sum. Let $(x_i, y_i)_{i=1,\dots,g}$ be a symplectic basis for a Seifert surface F_K of K ($\partial F_K = K$), and let z^+ stand for a parallel of z pushed away from F_K in the direction of the positive normal of F_K.

$$\lambda(M_{(K;p/q)}) - \lambda(M \sharp L(p,q)) = \frac{q}{p} \sum_{(i,j)\in\{1,\dots,g\}^2} \ell(x_i^+, x_j)\ell(y_i^+, y_j^-) - \ell(x_i^+, y_j^-)\ell(y_i^+, x_j^-).$$

The right-hand side is $\frac{q\Delta(K)''(1)}{2p}$ where $\Delta(K)$ is the Alexander polynomial of the knot and this formula is the Casson surgery formula when $p = 1$ and in the case of homology spheres. These results generalise to the higher degree Kontsevich-Kuperberg-Thurston invariants. The generalisation of the theorem is written in [L3]. The Casson surgery formula generalises to a formula for n-component boundary links for degree n configuration space invariants of rational homology spheres as follows.

3 Generalisations to higher degree invariants

All the real-valued finite type invariants of integral homology spheres are in the algebra generated by specific linear combinations of configuration space integrals associated to trivalent graphs. Slightly more precisely, to any trivalent graph Γ with $2n$ oriented vertices (*oriented* means equipped with a cyclic orientation of the three adjacent edges) we may associate a configuration space integral or an algebraic intersection number $I_\Gamma(\omega)$ roughly defined as follows. Consider the space $C_{2n}(M)$ of configurations of $2n$ distinct points (the $2n$ vertices of the graph Γ). Each edge e of Γ defines a projection p_e from $C_{2n}(M)$ onto $C_2(M)$, the preimage of a 4-chain Σ under p_e is then a codimension 2 chain in $C_{2n}(M)$, and the algebraic intersection of all the codimension 2 chains corresponding to edges is a rational number. Dually, (to make the picture more symmetric and to give a specific statement), rather than considering 4-chains, consider a dual closed antisymmetric 2-form ω on $C_2(M)$ that restricts to the boundary as $p(\tau_M)^*(\omega_2)$ where ω_2 is the homogeneous two-form on S^2 with total volume one, and associate with every graph Γ the configuration space integral

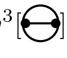
$$I_\Gamma(\omega) = \int_{C_{2n}(M)} \bigwedge_{e \text{ edge of } \Gamma} p_e^*(\omega)$$

that is the integral over $C_{2n}(M)$ of the product over the edges e of the forms $p_e^*(\omega)$. Then the degree n part of the Kontsevich-Kuperberg-Thurston invariant

Z reads

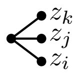
$$Z_n(M,\tau_M) = \sum_{\Gamma} \frac{1}{\sharp \mathrm{Aut}(\Gamma)} I_\Gamma(\omega)[\Gamma]$$

where the sum runs over all trivalent graphs Γ with $2n$ oriented vertices without looped edges, and $\sharp\mathrm{Aut}(\Gamma)$ is the number of automorphisms of such a Γ. $Z_n(M,\tau_M)$ takes values in the space $\mathcal{A}_n(\emptyset)$ of the Jacobi diagrams generated by vertex-oriented trivalent graphs with $2n$ vertices and quotiented by the IHX and AS relations. For example,

$$Z_1(M,\tau_M) = \frac{1}{12}\int_{C_2(M)} \omega^3 [$$  $$].$$

Again $Z_n(M,\tau_M)$ can be corrected with the help of the map p_1 to give rise to an invariant $Z_n(M)$ of rational homology spheres. (See [L2] for more specific statements.)

Then the lagrangian-preserving surgery formula for the Casson-Walker invariant has a natural generalisation for Z_n, and for an alternate sum involving $2n$ lagrangian-preserving surgeries that can be stated as follows. Represent the triple intersection forms $\mathcal{I}_{AA'}$ on $H_2(A \cup -A';\mathbb{Q})$ corresponding to a replacement of a $\mathbb{Q}$-handlebody A by another such A' with identical boundary and lagrangian as the following tripod $G(\mathcal{I}_{AA'})$ whose three univalent vertices form an ordered set:

$$G(\mathcal{I}_{AA'}) = \sum_{\{\{i,j,k\}\subset\{1,2,\dots,g_A\};i<j<k\}} \mathcal{I}_{AA'}(a_i,a_j,a_k)$$ 

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When G is a graph with $2n$ trivalent vertices and with univalent vertices decorated by curves of M, define its contraction as the sum

$$\langle\langle G\rangle\rangle_n = \sum_p \ell(G_p)[G_p]$$

that runs over all the ways p of gluing the univalent vertices two by two in order to produce a vertex-oriented trivalent graph G_p without looped edge, where $\ell(G_p)$ is the product over the pairs of glued univalent vertices in p of the linking numbers of the corresponding curves. The contraction $\langle\langle . \rangle\rangle_n$ is linearly extended to linear combinations of graphs, and the disjoint union of combinations of graphs is bilinear.

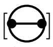
Then the generalisation of Theorem 1 reads

Theorem 2 *For any $2n$–component lagrangian-preserving surgery data*

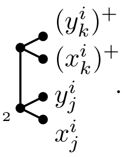
$$(M;(A_i,A'_i)_{i\in\{1,\dots,2n\}})$$

in a rational homology sphere M,

$$\sum_{I\subset\{1,\dots,2n\}} (-1)^{\sharp I} Z_n(M_I) = \langle\langle \bigsqcup_{i\in\{1,\dots,2n\}} G(\mathcal{I}_{A_iA'_i})\rangle\rangle_n.$$

Since λ is obtained from Z_1 by mapping [] to 2, this formula is consistent with Theorem 1. It easily implies the following surgery formula on n-component boundary links.

Theorem 3 *Consider a link $(K_1, K_2, \ldots, K_n)$ where all the K_i bound disjoint oriented surfaces F^i. Let p_i/q_i be a surgery coefficient for K_i, and let $(x^i_j, y^i_j)_{j=1,\ldots,g(F^i)}$ be a symplectic basis for the Seifert surface F^i. Define*

$$G(F^i) = \sum_{(j,k)\in\{1,2,\ldots,g(F^i)\}^2}$$ 

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For $I \subset 1, \ldots, n$, let

$$M_I = M_{(K_i;p_i/q_i)_{i\in I}} \sharp\sharp_{j\notin I} L(p_j, -q_j)$$

denote the connected sum of the manifold obtained from M by surgery on $(K_i; p_i/q_i)_{i\in I}$ and the lens spaces $L(p_j, -q_j)$ for $j \notin I$. Then

$$2^n \sum_{I\subset\{1,\ldots,n\}} (-1)^{n-\sharp I} Z_n(M_I) = \langle\langle \bigsqcup_{i\in\{1,\ldots,n\}} \frac{q_i}{p_i} G(F^i) \ \rangle\rangle_n \ .$$

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