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CAUSAL ADS-SPACETIMES

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Classical and quantum gravity in 3 dimensions.

AdS : anti-de Sitter space ($n=3$)
time oriented, oriented $\quad G = SO_0(2,2)$

$M^{(2+1)}$ locally AdS (Einstein in the void, $\Lambda < 0$)

Exemple: $\quad M = {}_{\Gamma}\backslash \Omega \qquad \Omega \subset \widetilde{AdS}$
$\Gamma \subset G$

Our concerns:

① Given $\Gamma \subset G$ discrete, find $\Omega \subset \widetilde{AdS}$ such that ${}_{\Gamma}\backslash \Omega$ manifold

② Description of BTZ (multi-) black-holes
$\Leftrightarrow$ ($\mathbb{S}^2$, $PSL_2(\mathbb{C})$)-structure $\Leftrightarrow$ Ku-Pi measured laminations

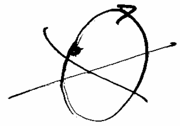
For ①: Comparison with $\mathbb{H}^3$:
If $\Gamma \subset SO_0(1,3) \simeq Isom(\mathbb{H}^3)$:
${}_{\Gamma}\backslash \mathbb{H}^3$ (complete) $\mathbb{H}^3$- manifold

+ limit set $\Lambda_\Gamma \subset \partial \mathbb{H}^3 = \mathbb{S}^2$

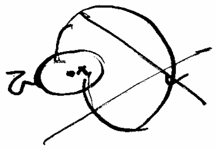
for $\Gamma \subset G = Isom(\widetilde{AdS})$: NO!

CAUSALITY NOTIONS

- CAUSAL SPACETIMES non-spacelike closed curves forbidden:

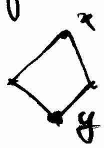


- STRONGLY CAUSAL SPACETIMES
  every point x admits a "causally convex" neighborhood U
  $\Updownarrow$



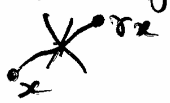
  Alexandrov Topology
  $\cap$
  manifold Topology

- Globally hyperbolic spacetimes: (maximality) $\exists$ Cauchy hypersurface
  $\Updownarrow$
  compactness of

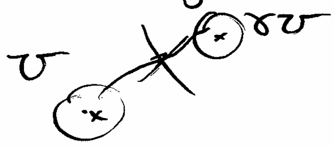


Γ acting on strongly causal spacetime:

- CAUSAL ACTION: for every x, every γ:



- STRONGLY CAUSAL ACTION:
  every point x admits a neighborhood U



MEANINGFUL FOR EVERY PSEUDO-RIEMANNIAN METRIC.

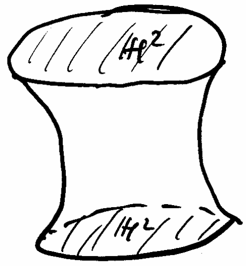
- ALL RIEMANNIAN MANIFOLDS ARE GLOBALLY HYPERBOLIC
- FOR RIEMANNIAN ACTIONS:
  causal action $\Leftrightarrow$ free
  strongly causal action $\Leftrightarrow$ proper

anti-de Sitter AdS

$$Q = -u^2 - v^2 + x_1^2 + x_2^2 \quad \text{on} \quad \mathbb{R}^{2,2}$$

$$AdS = \{Q = -1\}$$

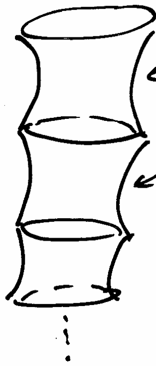
projects on $\mathbb{ADS}$ in $\mathbb{RP}^3$ (Klein model)



hyperboloid

AdS is NOT causal

$\widetilde{AdS}$



"affine domains"

$\widetilde{AdS}$ is STRONGLY CAUSAL

(but not globally hyperbolic)

$$\widetilde{AdS} \hookrightarrow \widetilde{Ein}_3 \approx \underset{ds_0^2 - dt^2}{\mathbb{S}^2 \times \mathbb{R}}$$

Einstein Universe

lightlike geodesics

CONFORMAL EMBEDDING

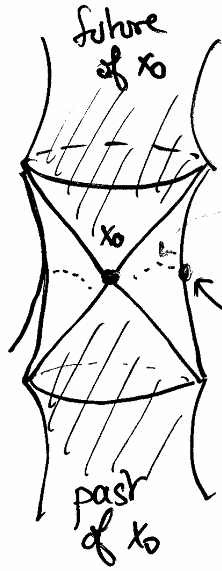
i.e.: respect causality relation.

$$\partial \widetilde{AdS} \approx \widehat{Ein}_2 \subset \widetilde{Ein}_3$$ "CONFORMAL BOUNDARY"

(compare: $\partial \mathbb{H}^3 \approx \mathbb{S}^2 \subset \mathbb{S}^3$)

$\widehat{Ein}_n$ is GLOBALLY HYPERBOLIC.

$$\Gamma \subset \mathrm{Isom}(\widetilde{AdS}) \approx \widetilde{SO_0(2,2)}$$



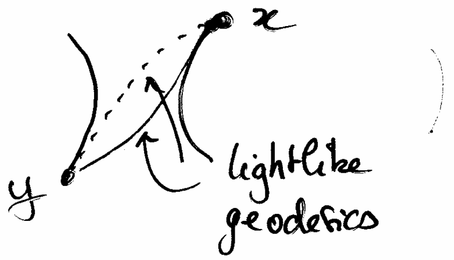
If $\gamma x_0 \not< x_0$ in $\widetilde{AdS}$ $\forall \gamma \in \Gamma$
non causally related

orbit $\Gamma \cdot x_0$

$$\Lambda_\Gamma = \overline{\Gamma \cdot x_0} \cap \partial \widetilde{AdS}$$

DEFINITION: Γ admissible if Γ preserves some generic achronal closed subset Λ of $\widehat{Ein}_2$

(Generic: does not contain



ELEMENTARY CASES:

- $\Lambda =$ 0, 1 or 2 points
- $\Lambda =$  (lightlike segment) extreme
- $\Lambda =$  (2 lightlike segments) conical

$$\mathrm{Isom}(\mathbb{ADS}) \simeq SO_0(2,2) \simeq PSL(2,\mathbb{R}) \times PSL_2(\mathbb{R})$$

$$\begin{tikzcd}
 & PSL_2 \simeq PSL_2 \times \{id\} \arrow[d, hook] \\
\Gamma \arrow[ur, "\rho_L"] \arrow[r, hook] \arrow[dr, "\rho_R"'] & \mathrm{Isom}(\mathbb{ADS}) = PSL_2 \times PSL_2 \\
 & PSL_2 \simeq \{id\} \times PSL_2 \arrow[u, hook]
\end{tikzcd}$$

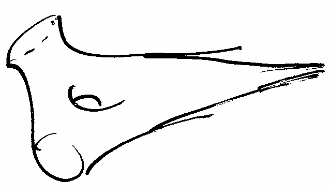
(Γ non abelian) Admissible

$\Updownarrow$

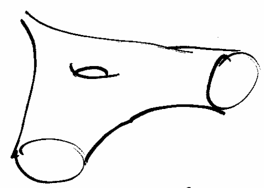
ρ_L, ρ_R faithful, discrete, semi-conjugate

means

$$\Sigma_L = \rho_L(\Gamma) \backslash \mathbb{H}^2 \overset{\text{homeo.}}{\simeq} \Sigma_R = \rho_R(\Gamma) \backslash \mathbb{H}^2$$



(marked surface)

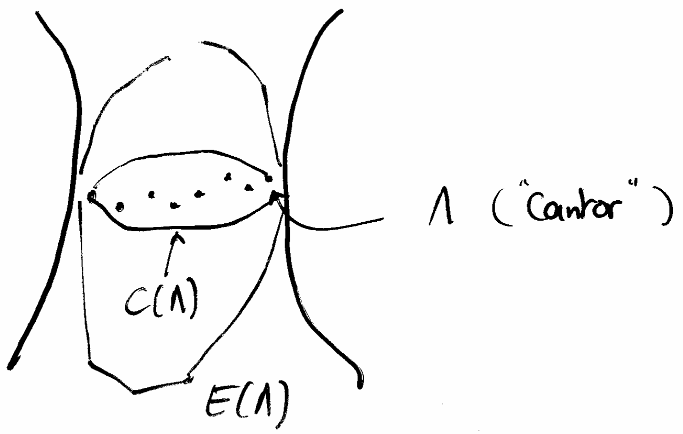


(marked surface)

$(\Sigma_L, \Sigma_R) \in \underline{\text{Teich} \times \text{Teich}}$

$\Lambda \subset \widehat{\mathcal{E}in_2}$ closed achronal generic

$\Rightarrow \Lambda \subset$ affine domain



$C(\Lambda)$: convex hull of Λ

$E(\Lambda)$: Invisible domain from Λ in AdS

$(E(\Lambda) = \{ x \in AdS \,/\, \forall p \in \Lambda ,\ x \not\leq p \})$

$\Omega(\Lambda)$: Invisible domain from Λ in $\mathcal{E}in_2$

$\overline{E}(\Lambda) = E(\Lambda) \cup \Omega(\Lambda)$

<u>THEOREM</u>: Λ generic achronal non-elementary subset preserved by Γ.

THEN, THE ACTION OF Γ ON $\overline{E}(\Lambda)$ IS PROPERLY-DISCONTINUOUS AND STRONGLY CAUSAL

(Γ torsion-free $\Leftrightarrow$ free action)

$$M_\Lambda(\Gamma) = \Gamma \backslash E(\Lambda)$$
$$\cap \qquad\qquad \cap$$
$$\overline{M}_\Lambda(\Gamma) = \Gamma \backslash \overline{E}(\Lambda)$$

Λ ELEMENTARY? CASE-BY-CASE ARGUMENT, NOT ALWAYS TRUE. IMPLY Γ abelian

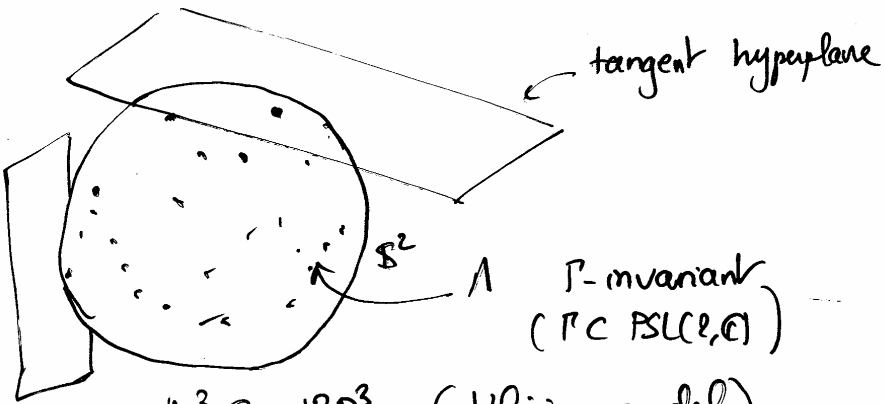
ANYWAY:

IF Γ is NON-abelian, then

$\Lambda_\Gamma = \overline{\{\text{attractive fixed points}\}}$ is

THE MINIMAL Γ-INVARIANT ACHRONAL SUBSET

COMPARISON WITH RIEMANNIAN CONTEXT:



$\mathbb{H}^3 \subset \mathbb{RP}^3$ (Klein model)

$\mathbb{RP}^3 \setminus \overline{\mathbb{H}^3} \approx$ de Sitter space(time)

$\Rightarrow E(\Lambda) = \mathbb{H}^3 \cup \underset{S^2 \setminus \Lambda}{\overset{\shortparallel}{\Omega(\Lambda)}} \cup \underset{E(\Lambda) \subset dS_3}{E(\Lambda)}$

$\Gamma \backslash E(\Lambda)$ Globally hyperbolic dS-spacetime (Scannell)

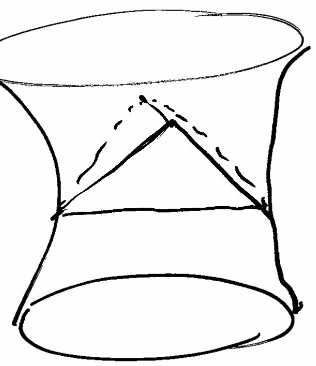
AdS - case :

Λ topological circle $\Leftrightarrow \Omega(\Lambda) = \emptyset$

$\Updownarrow$

$M_\Lambda(\Gamma)$ GLOBALLY HYPERBOLIC

Example Torus Universes:



$C(\Lambda) = E(\Lambda) =$ Tetraedron

Γ abelian $\subset \mathbb{Z}^2$

$\Gamma = \mathbb{Z}^2$: torus universe

THM (Bonsante - Benedetti, Barbot)

$\left. \begin{array}{l} M_\Lambda(\Gamma) \\ \Lambda \text{ topological circle} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{maximal globally hyperbolic} \\ \text{AdS - spacetimes with} \\ \text{Cauchy - complete Cauchy} \\ \text{surface} \end{array} \right.$

① $\leftarrow$: Mess : $\Sigma \hookrightarrow AdS \Rightarrow \Sigma \hookrightarrow \mathbb{H}^2$ distance increasing

② $\rightarrow$: $M_\Lambda(\Gamma)$ contains a Cauchy surface?

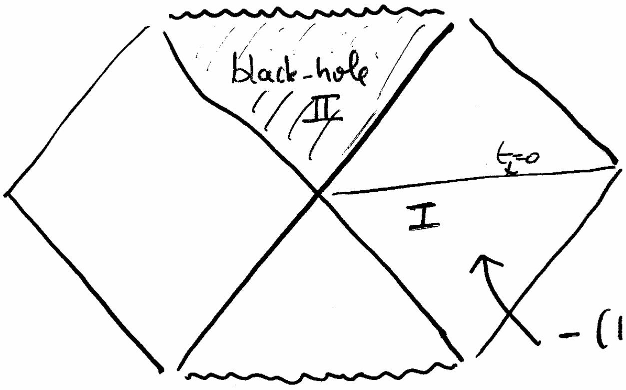
(My proof: $\Sigma \hookrightarrow AdS$ spacelike $\Longrightarrow \Sigma \hookrightarrow \mathbb{H}^2 \times \mathbb{H}^2$)

Σ convex "$\Longrightarrow$" $\Sigma \hookrightarrow \mathbb{H}^2 \times \mathbb{H}^2$ distance decreasing

BLACK HOLES:

A quick insight into Schwarzschild black-hole
Kerr

$S^2 \times$
ds_0

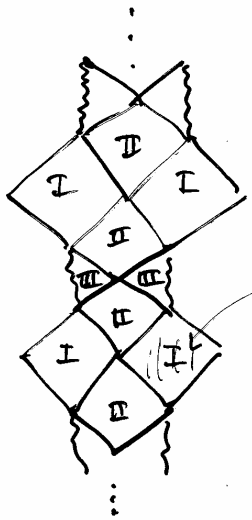


$$-\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 ds_0^2$$

$r \approx \infty$ metric $\approx -dt^2 + dr^2 + r^2 ds_0^2$: flat

OBSERVORS AT CONFORMAL BOUNDARY
VERY FAR

Kerr:



$$-\left(1-\frac{2Mr}{\rho^2}\right)dt^2 + \sin^2\theta\,\frac{4Mra}{\rho^2}d\phi\,dt + \rho^2 d\theta^2$$

$$+\left(r^2+a^2+\sin^2\theta\,\frac{2Mra}{\rho^2}\right)\sin^2\theta\,d\phi^2 + \frac{\rho^2}{\Delta^2}dr^2$$

$\Delta = r^2 - 2Mr + a^2$

$\rho^2 = r^2 + a^2\cos^2\theta$

M: mass

a: angular momentum per unit mass

CONFORMAL BOUNDARIES

M lorentzian $\Rightarrow$ $\overline{M} = M \cup \underbrace{\partial M}_{\text{conformal boundary}}$

"Universal":

$\overline{M}_{univ}$ if for every $\overline{M} = M \cup \partial M$

$M \hookrightarrow \overline{M}$ extends to injection $\overline{M} \hookrightarrow \overline{M}_{univ}$

IN GENERAL ?

FOR AdS-spacetimes:

$E \subset AdS$, define $\partial E = \mathrm{Int}(\overline{E} \cap \mathrm{Ein}_2)$

and $\overline{E} = E \cup \partial E$

$(AdS, \mathrm{Ein}_2, \mathrm{Iso}(AdS))$-structure

THM: Any AdS-spacetime admits a universal conformal boundary

(complete case: Frances)

$\Rightarrow$ can define observers, black-holes

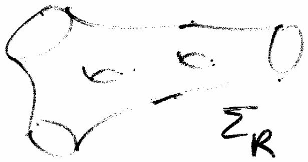
BTZ - multi black-holes

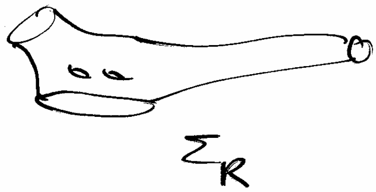
Bañados
Teitelbaum
Zanelli
+ Ammineborg, Brill,
Peldán, etc...

$\Gamma \subset SO_0(2,2)$ admissible

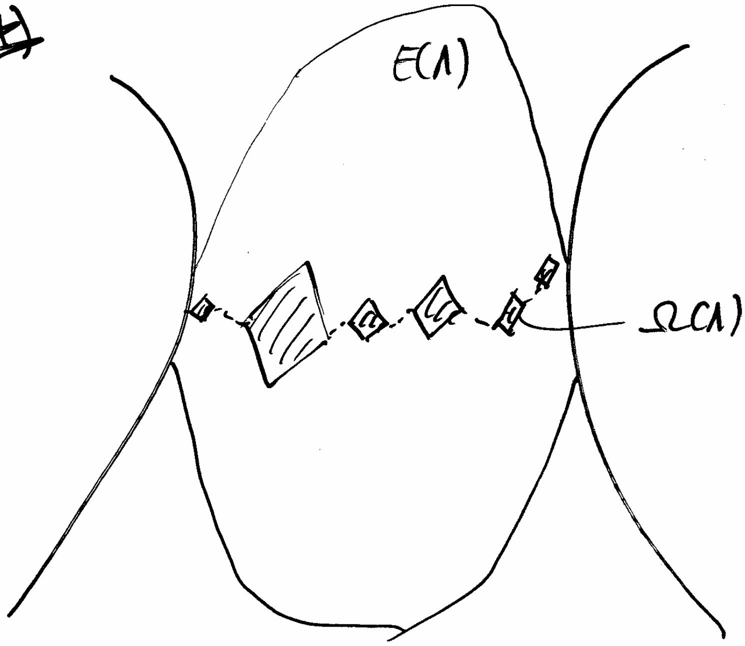
$\Updownarrow$

$\rho_L, \rho_R \in \mathrm{Rep}^{\alpha}(\Gamma, PSL_2)$



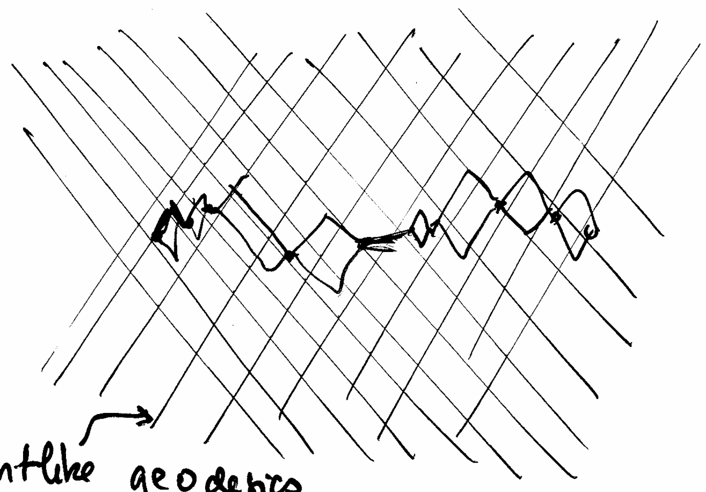


$\Lambda = \Lambda_\Gamma$ limit set



$M_\Lambda(\Gamma) \subset \overline{M}_\Lambda(\Gamma)$

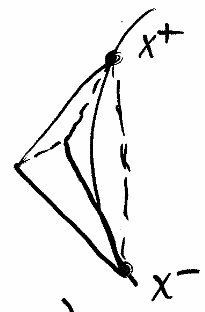
$\partial M_\Lambda(\Gamma) = \Omega(\Lambda)$ = observers



Λ^+
Λ^- $\subset \mathrm{Ein}_2$

$$E(\Lambda) = \underset{\text{non-disjoint}}{E(\Lambda^+) \cup E(\Lambda^-)} \cup \underline{\underline{\text{Ends}}}$$

End: AdS $\cap$ Tetraedron
||
what can see x^-
and can be seen from x^+
(generator of Alexandrov Topology)



$E(\Lambda^+)$: Black-hole (Globally hyperbolic)

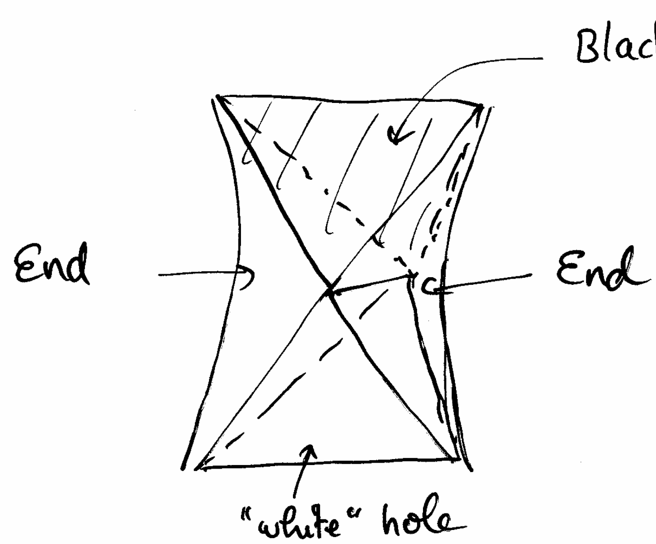
On end:

Kerr-like metric: $-(N^\perp)^2 dt^2 + (N^\perp)^{-2} dr^2 + r^2 \left(d\varphi + \frac{J}{2r^2} dt\right)^2$

with $N^\perp = r^2 - M + \left(\frac{J}{2r}\right)^2$

~~Non~~ STATIC Single BTZ black-hole

($J=0$)



$$\Gamma = \left(e^{\pi r_0 \Delta} , e^{\pi r_0 \Delta} \right)$$

$$\Delta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

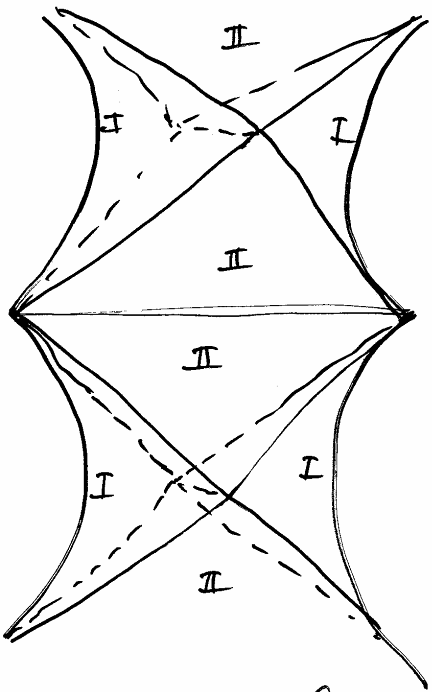
$$M = 2 r_0^2$$

$$J = 0$$

Single BTZ black-hole :

$\Gamma = \langle \gamma_0 \rangle$ where $\gamma_0 = \left(\exp(\pi(r_+ - r_-)\Delta), \exp(\pi(r_+ + r_-)\Delta) \right)$

Non Static



$M = r_+^2 + r_-^2$

$J = 2 r_+ r_-$

+ regions (III) : $D(\Gamma) = \{ x \in AdS \,/\, Z(x) \text{ spacelike} \}$

where $\gamma = \exp(Z)$

Description similar to Kerr maximal spacetime

("extreme" case, ~ isolated)

In general:

Γ non abelian, admissible

$\forall \gamma \in \Gamma, \quad D(\gamma) = \{ x \,/\, Z_\gamma(x) \text{ spacelike} \}$

$$D(\Gamma) = \bigcap_{\gamma \in \Gamma} D(\gamma)$$

<u>THM:</u> $\boxed{D(\Gamma) = E(\Lambda_\Gamma)}$

(Particular cases in Amminebörg, Bengtsson, Holst "A spinning anti-de Sitter wormhole")

Last comments:

- Higher dimensions
- wick rescaling / rotation $\to$ NON-COMPLETE HYPERBOLIC MANIFOLDS (IF "MOMENTA IS TOO BIG")
- Observables: Vol(Conv Λ), etc...