3D Quantum Gravity from Loops :

- pure gravity
- particles (massive)
- cosmological constant

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<u>References</u> :

KN and A Perez : CQG 22 (2005) 1739
KN and A Perez : CQG 23 (2005)
KN : in preparation

Presented at the workshop :
3D classical and quantum gravity
Pisa , 2005 .

Introduction

Different quantization schemes (S. carlip)

(i) Spin-Foam Models : Mnf invariants

(ii) Quantum Moduli Spaces : q-groups

(iii) Chern-Simons : knots invariants

etc ...

But 3D techniques and useless in 4D

Loop Quantum Gravity

(i) Formulation close to gauge theory :

theory of G-connections on Mnf

(ii) Canonical and non-perturbative

(iii) Background independent :

makes the approach very attractive

Many problems remain and tests in 3D

I. Pure gravity

Riemannian gravity with $\Lambda = 0$ and no matter
Let $M = \Sigma_g \times I$ and $G = SU(2)$

1. Classical theory (hamiltonian)

a/ Phase space for gravity

(i) $\mathcal{E} = \left\{ (A, E) \;\middle|\; \begin{array}{l} A \text{ is a } G\text{-connection} \\ E \text{ is electric field} \end{array} \right\}$

(ii) Symmetries $\mathcal{G} = \mathcal{C}^\infty(\Sigma_g; S)$

$$S = G \ltimes D$$

where $\begin{cases} G : \text{usual gauge group } SU(2) \\ D \cong \mathbb{R}^3 : \text{generates "diffeomorphisms"} \end{cases}$

$$\Rightarrow \mathcal{P} = \{ X \in \mathcal{E} \mid \mathcal{G}\text{- invariant} \} / \mathcal{G}$$

(iii) Observables $\mathcal{O} \in \mathrm{Fun}(\mathcal{P})$

b/ Relation to Moduli Space

(i) Chern-Simons connection $\Omega = A^a J_a + E^a P_a$
Ω is a S- connection

(ii) Symmetries generated by $F(\Omega) = 0$

2. Quantum theory (canonical)

a/ Different strategies (inequivalent!)

(i) Implementing constraints before

(ii) Implementing constraints after

- Combinatorial quantization
- Loop quantization

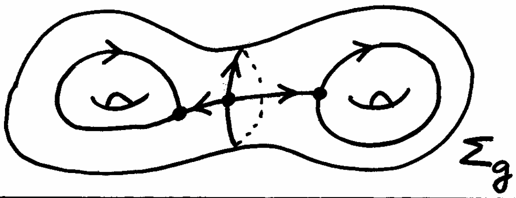
b/ Loop quantization $[\hat{A} ; \hat{E}] = i\hbar$

(i) Choice of polarization:

$$\mathcal{C} = \{ SU(2)\text{-connections on } \Sigma_g \}$$

$\hat{A}$ acts by multiplication;
$\hat{E}$ acts by derivation

(ii) Space of connections on a graph γ



γ : oriented graph
$\begin{cases} V : \# \text{ vertices} \\ E : \# \text{ edges} \end{cases}$

$\mathcal{A}_\gamma$ is a subspace of $\mathrm{Fun}(\mathcal{C})$:

- $\psi \in \mathcal{A}_\gamma$ iff $\exists\ f \in \mathrm{Fun}(G^{\times E})$ s.t. $\forall\ A \in \mathcal{C}$, $\psi(A) = f(\bigotimes_e U_e)$ where $U_e = \mathrm{Hol}_e(A)$.
- Action of $G^{\times V}$ on $\mathcal{A}_\gamma$

Non-physical quantum states are elements of

$$\mathcal{H}_\gamma = L^2(\mathcal{A}_\gamma ; d\mu_\gamma)$$

where $d\mu_\gamma = \bigotimes_{i=1}^{E} d\mu_i$ with $d\mu_i$ a SU(2) Haar

We also denote $\mathcal{A}_\gamma = \mathrm{Cyl}_\gamma$ (cylindrical functions)

(iii) Cylindrical functions on Σ_g

$$\boxed{\mathrm{Cyl}_{\Sigma_g} = \bigcup_\gamma \mathrm{Cyl}_\gamma}$$

Cyl_{Σ_g} admits a natural measure :

$\forall\, \psi_1, \psi_2 \in \mathrm{Cyl}_{\Sigma_g}$; $\exists\, \gamma$ s.t. $\psi_1, \psi_2 \in \mathrm{Cyl}_\gamma$

and $\langle \psi_1, \psi_2 \rangle = \int d\mu_\gamma\, \overline{\psi_1}\, \psi_2$

the graph γ is not unique but $\langle \psi_1, \psi_2 \rangle$ does not depend on the choice of γ.

(iv) Non-physical quantum states

$$\mathcal{H}_{\Sigma_g} = L^2(\mathrm{Cyl}_{\Sigma_g} ; d\mu)$$

$d\mu$ is called Ashtekar-Lewandowski measure.

$d\mu$ satisfies uniqueness property.

c/ Implementing the constraints G

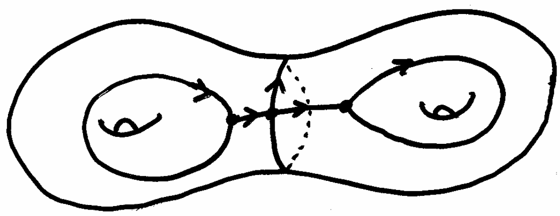
(i) Gauss constraint : invariance under G

$$\mathcal{H}_{kin} = \mathbb{C}\left(Cyl^{inv}_{\Sigma_g} ; d\mu \right)$$

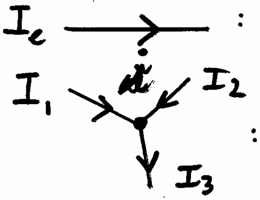
thanks to invariance of $d\mu$.

$$\begin{cases} \forall\, \psi \in Cyl^{inv}_{\Sigma_g} \subset Cyl_{\Sigma_g} ; \\ \exists\ E \in \mathbb{N} \text{ and } f \in \mathcal{F}un\,(G^{\times E}) \\ \text{and } f \circ Adj_{G^{\times V}} = f . \end{cases}$$

(ii) Spin-network basis



$I_e \longrightarrow$: representation of $SU(2)$



: intertwiner $i : I_1 \otimes I_2 \longrightarrow I_3$

A spin-network state is labelled by:
- A graph (oriented) γ
- A coloring $(I_e ; i_v)$ of γ

$\{ |\gamma ; (I_e ; i_v)\rangle \}$ forms an orthogonal basis of $\mathcal{H}_{kin}$.

d/ Implementing the constraints D

(i) General status of solutions

There is no physical states in Cyl_{Σ_g}

Physical states are "distributional":

$$Cyl_{\Sigma_g} \subset \mathcal{H}_{kin} \subset Cyl^*_{\Sigma_g} \ni \Psi_{phys}$$

(ii) The Physical extractor

$$P : Cyl_{\Sigma_g} \longrightarrow Cyl^*_{\Sigma_g}$$

s.t. $\quad P(\Psi_1)(\Psi_2) = P(\Psi_1)(\varphi * \Psi_2)$

$\forall\ \varphi$ a diffeomorphism.

By definition, the physical scalar product is $\boxed{\langle \Psi_1 ; \Psi_2 \rangle_{Phys} = P(\Psi_1)(\Psi_2)}$

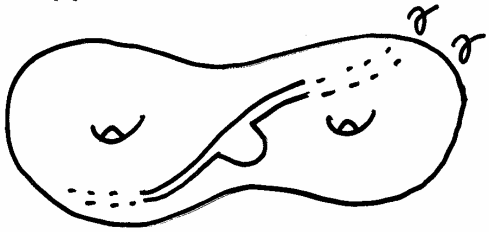
(iii) Explicit expression for P :

Given 2 graphs γ and γ' on Σ_g, ones defines $T_\varepsilon^{\gamma\gamma'}$ (triangulation) and :

$$\langle \Psi_\gamma ; \tilde{\Psi}_{\gamma'} \rangle_{Phys} = \lim_{\varepsilon \to 0} \sum_{j_p} (2j_p+1) \langle \prod_p \chi_{j_p}(U_p) \Psi, \tilde{\Psi}$$

- It imposes $F(A) = 0$
- $|\langle \Psi, \tilde{\Psi} \rangle| \leqslant C \sum_j (2j+1)^{2-2g}$
- Hermiticity and positivity

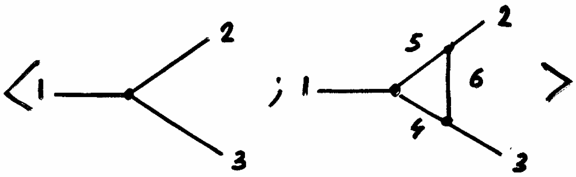
(iv) Diffeomorphism invariance



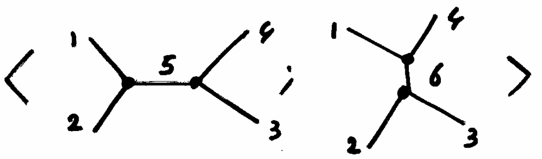
$(\Psi_\gamma - \Psi_\gamma)$ is a null-vector, ie:

$$\forall\ \phi \in Cyl_{\Sigma_g}\ ;\ \langle \Psi_\gamma - \Psi_\gamma\ ;\ \phi \rangle_{phys} = 0 .$$

(v) Relation to Ponzano-Regge model



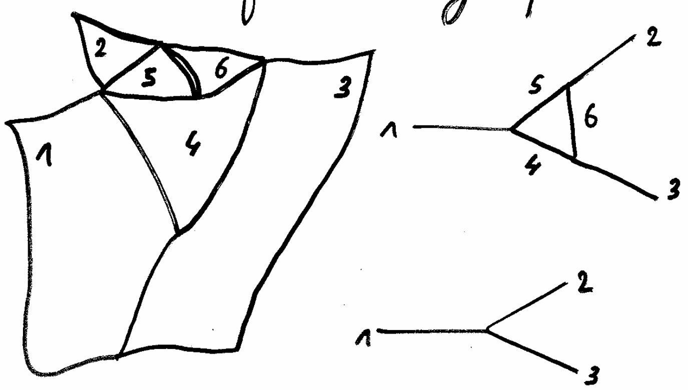
$$\propto \sqrt{\Delta_4 \Delta_5 \Delta_6} \begin{Bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{Bmatrix}$$



$$\propto \sqrt{\Delta_5 \Delta_6} \begin{Bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{Bmatrix}$$

where $\begin{Bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{Bmatrix}$ are $SU(2)$ $(6j)$-symbols

We have the following picture:



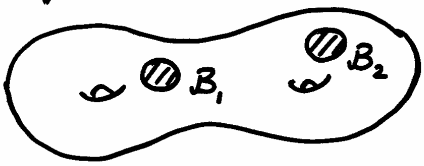
II. Coupling to particles

Purely massive particles (for clarity)
Let $M = \Sigma_{g,n} \times I$ with $2g-2+n>0$.

1. Classical coupling to particles

a/ Kinematical description

(i) Dynamical boundaries



$\forall\, B_i \in \partial \Sigma_{g,n}$, we have $X_i = (\Lambda_i ; q_i)$
$X_i \in S = ISU(2)$ and a mass m_i.

(ii) X_i contains momentum and position of particle.

b/ Dynamics

(i) Action $S[A,E;X] = S_G[A,E] + S_{PP}[X] + S_{coup}[A,E,X]$

(ii) Classical phase space
Dirac bracket between observables defines a sol. of CDYBE.
Relation to Fock-Rosly structure.

2. Quantization à la L.Q.G.

a/ Kinematical description for coupling

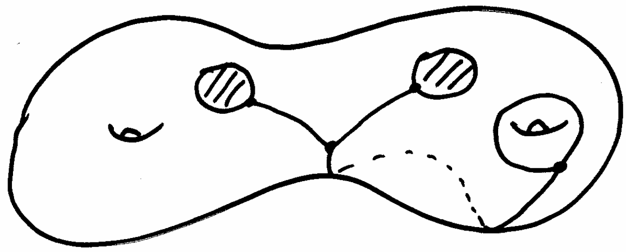
(i) Free euclidean particle

The particle described via $\vec{p} = m \wedge \vec{n}$

$$\mathcal{H}_p = L^2\left(SU(2)/U(1) \, ; \, d\mu \right)$$

(ii) Cylindrical functions on $\Sigma_{g,n}$

$$Cyl_{\Sigma_{g,n}} = Cyl_{\Sigma_g} \otimes \left(\bigotimes_{i=1}^{n} \mathcal{H}_{p_i} \right)$$



(iii) Action of G on vertices

$$\mathcal{H}_{kin} = C\left(Cyl^{inv}_{\Sigma_{g,n}} \, ; \, d\mu \right)$$

$\Longrightarrow$ We have open spin-network with free vertices at location of particles. There is a residual symmetry related to freedom to choose internal frames for particles.

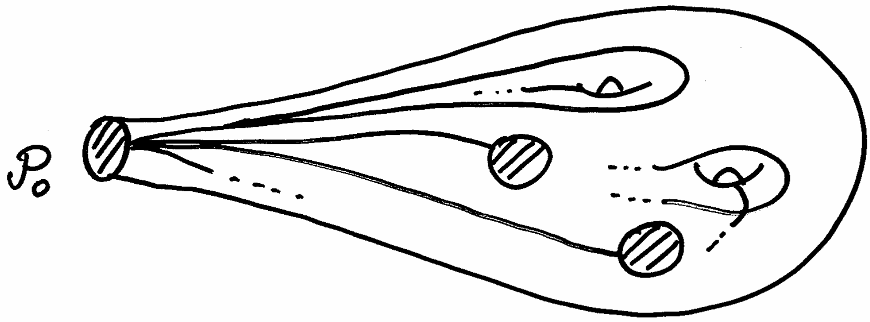
b/ Physical Hilbert space

(i) Construction of "extractor" P

$$P : \mathrm{Cyl}_{\Sigma_{g,n}} \longrightarrow \mathrm{Cyl}^{*}_{\Sigma_{g,n}}$$

$\left\{ \begin{array}{l} \text{- Flatness everywhere but around particles;} \\ \text{- Fixes the masses at boundaries.} \end{array} \right.$

(ii) First step: to the minimal graph



Let Γ_0 be minimal graph

$$\mathcal{H}_{phys} \simeq \mathcal{H}_{phys}(\Gamma_0)$$

Where $\mathcal{H}_{phys}(\Gamma_0) = C\left(P\left(\mathrm{Cyl}_{\Gamma_0}\right)\right)$

$\mathcal{P}_0$ is an observer that captures all physical information.

We break "reparametrization" invariance at the location of $\mathcal{P}_0$.

(iii) Physical scalar product

$$P : Cyl_{\Gamma_0} \longrightarrow Cyl^*_{\Gamma_0}$$

and $Cyl_{\Gamma_0} \simeq F\left((S^2)^{\times n-1} \times SU(2)^{2g} \right)$

then $Cyl^*_{\Gamma_0} \simeq (S^2)^{n-1} \times SU(2)^{2g}$

And, $\forall\ \psi_1, \psi_2 \in Cyl_{\Gamma_0}$,

$$\boxed{\langle \psi_1, \psi_2 \rangle_{phys} = h^{n-1}_{S^2} \otimes h^{2g}_{SU(2)} \left(\overline{P(\psi_1)}\, K\, P(\psi_2) \right)}$$

Where $K \in Cyl^*_{\Gamma_0}$ imposes the constraints.

h_H is Haar measure $: H \to \mathbb{C}$

- It is definite positive
- It is well-defined:

$$\langle \psi_1, \psi_2 \rangle_{phys} \leq \| \psi_1 \psi_2 \| \sum_{k=0}^{\infty} \frac{1}{d_k^{2g+n-1}} \chi_k (h_{m_1} \cdots h_{m_n})$$

(iv) Relation to state-sum models:

The partition function:

$$Z(\Sigma_g ; G) = \sum_{j_e} \prod_e \dim j_e \prod_{e_p} \chi_{j_{e_p}}(m_p) \prod_k G_j(t)$$

Generalization of P-R- amplitude.

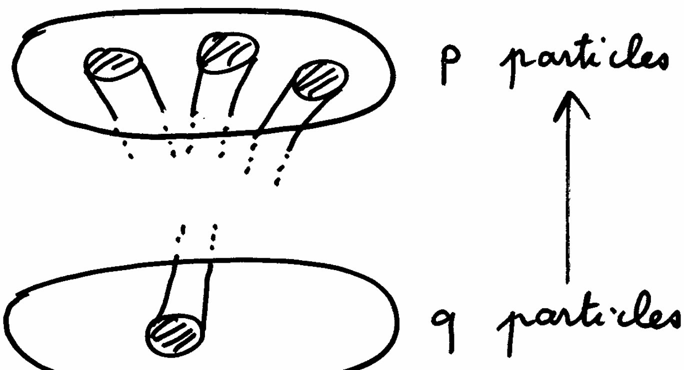
c/ Transition amplitudes

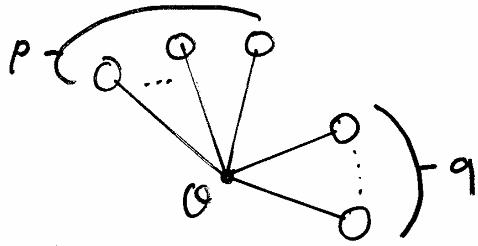
Let's start with the sphere ($g = 0$)

We consider "factorized" states:

$$\psi = \bigotimes_i \psi_i$$

(i) Creation / Anhiliation of particles





$$\langle \psi_1 ; \psi_2 \rangle_{Phys} = h_{S^2}^{n+p} (\overline{\psi}_1 \, K \psi_2)$$

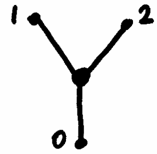
The result does not depend on the choice of the graph. This is a consequence of properties of h.

(ii) Intertwiner and emergence of DSU(2)

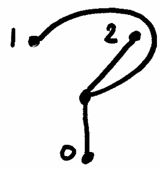
$$\langle \phi ; \bigotimes_{i=1}^{n} \psi_i \rangle = N(m_i) \, i_{BC} \Big(\bigotimes_{i=1}^{n} \psi_i \Big)$$

$$\begin{cases} N(m_i) = \dfrac{\pi}{4} \Big(\prod_i \sin m_i \Big)^{-1/2} \\ i_{BC} : \bigotimes V_{m_i} \longrightarrow \mathbb{C} \end{cases}$$

(iii) Properties of amplitudes

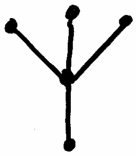


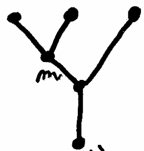
: $3j(m_1, m_2, m_0)$



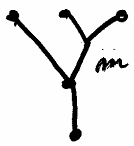
$=$

$3j(m_1, m_2, m_0)\ R(m_1, m_2)$


$= \int d\mu(m)$

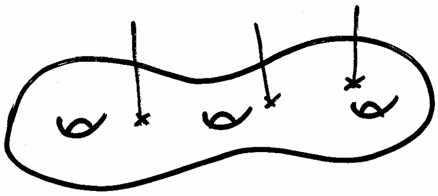


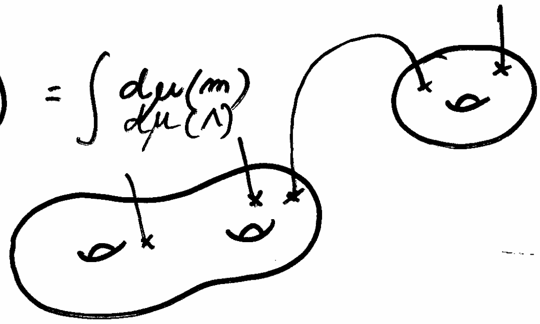
$= \int d\mu(m)$



and m is a "real" mass.

(iv) Factorization properties


$= \int \begin{matrix} d\mu(m) \\ d\mu(\Lambda) \end{matrix}$



This is an immediate consequence of the fact that $\langle \phi ; \bigotimes_i \psi_i \rangle$ is intertwiner.

Many other calculations ...

d/ Relation to QFT

Let $\varphi \in DSU(2)$ and $\varphi \in \mathbb{C}[SU(2)] \subset D$

Let $K \in DSU(2)^{\times 2}$ and $K \in \mathbb{C}[SU(2)]^{\times 2}$

We put $K = 1_{DSU(2)}$

And we define $\underline{S[\varphi] = h^{\times 2}(\varphi_1\, K_{12}\, \varphi_2)}$

$$\langle \varphi(\psi_1) \cdots \varphi(\psi_n) \rangle = \langle \Phi ; \psi_1 \otimes \cdots \otimes \psi_n \rangle_{phys}$$

In the position space

$$\tilde{\tilde{\varphi}}(A) = \int d^3\vec{x}\; e^{i m \Lambda \vec{n}, \vec{x}}\; \tilde{\varphi}(\vec{x})$$

$$S[\tilde{\varphi}] = \int d^3\vec{x}\, d^3\vec{y}\;\; \tilde{\varphi}(\vec{x})\; K_m(\vec{x},\vec{y})\; \tilde{\varphi}(\vec{y})$$

where $K_m(\vec{x},\vec{y}) = \dfrac{\sin(m\,\|\vec{x}-\vec{y}\|)}{m\,\|\vec{x}-\vec{y}\|}$

$\Longrightarrow$ It is a non-local QFT!

Conclusion

Emergence of quantum groups as the result of dynamics.

In the case $\Lambda \neq 0$:

$$\mathcal{H}_{kin}$$

$\downarrow$ imposes $\Lambda \neq 0$

$$\mathcal{H}_{\Gamma_0} \qquad U_q(su(2))$$

$\downarrow$ imposes presence particles

$$\mathcal{H}_{phys} \qquad D(U_q(su(2)))$$