Canonical foliations of quasi-fuchsian manifolds

(mostly a survey of some recent results)
joint with K. Krasnov

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- Is there any canonical foliation of q-fuchsian metrics ???

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AdS analogs: GHMC AdS 3-mflds (G. Mess).

Two descriptions of the Fock metric

Consider the following metric on $S\times\mathbb{R}$ (V. Fock):

$$ds^2 = dr^2+(e^\phi\cosh^2(r)+t\bar{t}e^{-\phi}\sinh^2(r))|dz|^2+(tdz^2+\bar{t}d\bar{z}^2)\cosh(r)\sinh(r)\,,$$

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Thus S_0 is a minimal surface, and the S_r are equidistant surfaces. So it can be written also as:

$$dr^2 + I_0((\cosh(r)E + \sinh(r)B)\cdot, (\cosh(r)E + \sinh(r)B)\cdot)\,,$$

where B is the shape operator of the minimal surface.

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Note: q-fuchsian 3-mflds always contain a min surface, which is area minimizing.

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The shape operator B^* satisfies the Codazzi equation and a modified Gauss equation: $K_\infty = 1 - tr(B^*)$. Then set $b := 2B^* - 1$. There is also an expression in terms of conformal structure and QHD.

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Critical points

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Proof: clear from the Hessian of the area functional.
But no geometric explanation. Moreover (Taubes) the degenerate directions are not the same.

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THM (B. Andrews): any surface with $k < 1$ can be deformed to a minimal surface with $k < 1$.

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Uses the existence of hyperbolic 3-mfld N which fibers over S^1, with no foliation by min surfaces (Hass-Thurston, Rubinstein ?).

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It also follows that for some q-fuchsian mflds the foliation by equidistants from a minimal surface does not even cover the convex core.

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Inequalities in the other directions ??

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which is str. convex because $det_g h \leq 0$.

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So: $T^*\mathcal{T}_g$ = germs of max surfaces in AdS = GHMC AdS mflds.
Provides *limited* Wick rotation: from "good" q-fuchsian mflds to GHMC AdS mflds.

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What about hyperbolic 3-mflds ?

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Also yields a foliation for the dS mflds obtained by duality (quotients of max convex subset of dS by surfaces groups, etc) by constant Gauss curvature surfaces.
However neither the equidistant foliation from a min surface nor the CMC foliation provides (yet ?) a nice canonical foliation of q-fuchsian mflds.