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Grafting and Poisson structure in (2+1)-gravity with $\Lambda = 0$

Workshop Classical and quantum gravity in 3 dimensions
Pisa, September 5-11 2005

References:

1. C. Meusburger, **Grafting and Poisson structure and symmetry in (2+1)-gravity with vanishing cosmological constant**, gr-qc/0508004
2. C. Meusburger, B. J. Schroers: **Mapping class group actions in Chern-Simons theory with gauge group $G \ltimes \mathfrak{g}^*$**, Nucl. Phys. B 706 (2005) 569-597, hep-th/0312049

Contents:

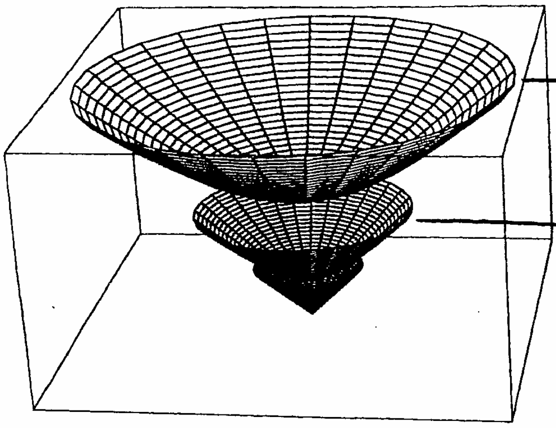
1 Construction of (2+1)-spacetimes via grafting

[Benedetti, Bonsante, Guadagnini]

spacetimes: $M \approx \mathbb{R} \times S_g$, $g \geq 2 \quad \Rightarrow \quad M = U/\pi_1(S_g)$, $U \subset \mathbb{M}^3$

1.1 Static spacetimes

1. Foliate interior of lightcone U by hyperboloids H_T



hyperboloid

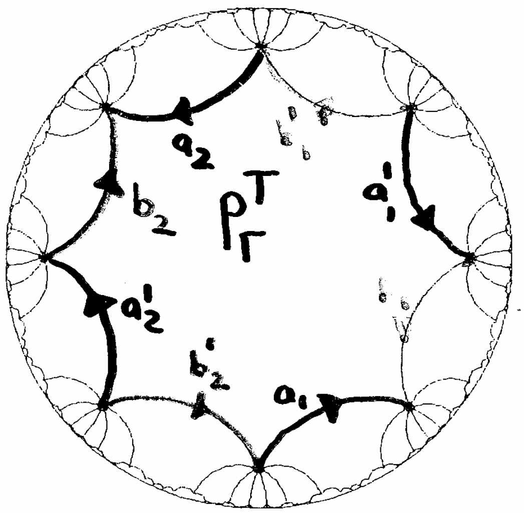
$H_T = \{\boldsymbol{x} \in \mathbb{M}^3 \mid x_0^2 - x_1^2 - x_2^2 = T^2,\ x^0 > 0$

hyperbolic space H_1

2. Cocompact Fuchsian group Γ

$SO(2,1) \supset \Gamma = \langle v_{A_1}, v_{B_1}, \ldots, v_{A_g}, v_{B_g}\,;\, [v_{B_g}, v_{A_g}^{-1}] \cdots [v_{B_1}, v_{A_1}^{-1}] = 1 \rangle \cong \pi_1(S_g)$

$\Rightarrow$ tesselation of H_T by geodesic arc $4g$-gons



fundamental polygon $P_\Gamma^T \subset H_T$

generators of Γ: $v_{A_i} : a_i \mapsto a_i'$, $v_{B_i} : b_i \mapsto b_i'$

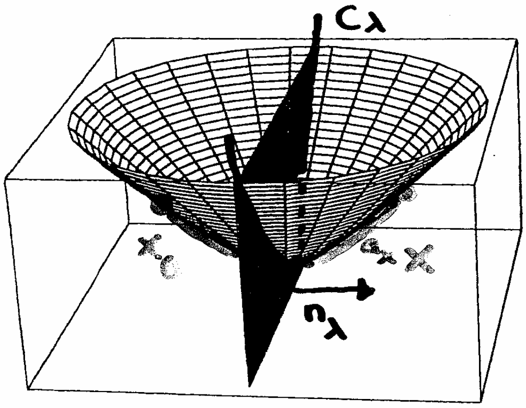
3. spacetime: $M = U/\Gamma$

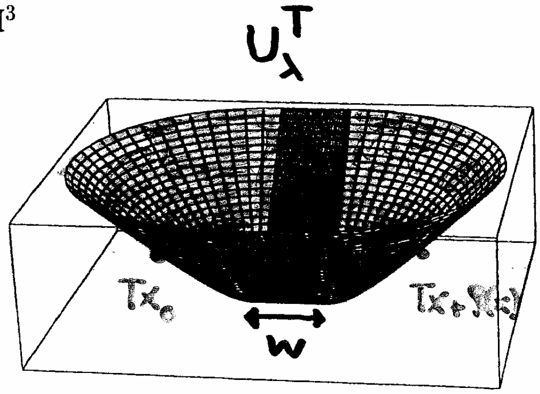
1.2 Grafting

ingredients: Γ, closed simple geodesic λ on $S_\Gamma = H_1/\Gamma$ with weight $w > 0$

1. λ lifts to Γ-invariant multicurve $M_\lambda = \{vc_\lambda | v \in \Gamma\} \subset H_1$

2. Grafting: $M_\lambda \Rightarrow$ regular domain $U_\lambda \subset \mathbb{M}^3$





- choose basepoint $\boldsymbol{x}_0 \in H_1 \setminus M_\lambda$

- translation for $\boldsymbol{x} \in H_1 \setminus M_\lambda$: $\quad \rho(\boldsymbol{x}) = w \sum_{v\in\Gamma:\, a_x \cap vc_\lambda \neq \emptyset} \epsilon_{v,x}\, v\boldsymbol{n}_\lambda$

- domain: $\quad U_\lambda = \bigcup_{T \in \mathbb{R}_0^+} U_\lambda^T$

$$U_\lambda^T = \underbrace{\{T\boldsymbol{x} + \rho(\boldsymbol{x}) | \boldsymbol{x} \in H_1 \setminus M_\lambda\}}_{\text{translated pieces of hyperboloid } H_T} \cup \underbrace{\{T\boldsymbol{x} + t\rho_+(\boldsymbol{x}) + (1-t)\rho_-(\boldsymbol{x}) | \boldsymbol{x} \in M_\lambda, t \in [0,1]\}}_{\text{hyperbolic cylinders}}$$

T=cosmological time

3. Action of $\Gamma \cong \pi_1(S_g)$ on U_λ: $f : \Gamma \to ISO(2,1)$, $f(v) = (v, \rho(vx_0))$
   $\Rightarrow$ leaves U_λ invariant, free, properly discontinuous

4. Grafted spacetime: $\quad M = U_\lambda / \Gamma_f$

2 Phase space and Poisson structure in the Chern-Simons formulation

gauge group: $ISO(2,1) = SO(2,1) \ltimes \mathbb{R}^3$:

- generators $J_a, P_a \in iso(2,1)$: $[J_a, J_b] = \epsilon_{abc}J^c$ $[J_a, P_b] = \epsilon_{abc}P^c$ $[P_a, P_b] = 0$

- parametrisation: $(u, \boldsymbol{a}) = (u, -u\boldsymbol{j})$, $u = e^{-p^a J_a} \in SO(2,1)$, $\boldsymbol{a}, \boldsymbol{j} \in \mathbb{R}^3$
  $(u_1, \boldsymbol{a}_1)(u_2, \boldsymbol{a}_2) = (u_1 u_2, \boldsymbol{a}_1 + u_1 \boldsymbol{a}_2)$

- formal parameter θ, $\theta^2 = 0 \Rightarrow$ representation $(P_a)_{bc} = \theta(J_a)_{bc} = -\theta\epsilon_{abc}$
  $(u, \boldsymbol{a}) \leftrightarrow (1 + \theta a^b J_b)u$

gauge field: $A = e^a P_a + \omega^a J_a = A_0 dx^0 + A_S$

equations of motion:

$$F_S = d_S A_S + A_S \wedge A_S = 0 \quad \partial_0 A_S = d_S A_0 + [A_S, A_0]$$

observables for $\lambda \in \pi_1(S_g)$:

conjugation invariant functions of holonomy $H_\lambda = (e^{-p^a_\lambda J_a}, -e^{-p^a_\lambda J_a}\boldsymbol{j}_\lambda)$

$$m_\lambda^2 = -\boldsymbol{p}_\lambda^2 \quad m_\lambda s_\lambda = \boldsymbol{p}_\lambda \boldsymbol{j}_\lambda$$

phase space:

parametrised by holonomies A_i, B_i of generators a_i, b_i of $\pi_1(S_g)$

$\mathcal{M}_g = \{(A_1, B_1, \ldots, A_g, B_g) \in ISO(2,1)^{2g} \mid [B_g, A_g^{-1}] \cdots [B_1, A_1^{-1}] = 1\}/ISO(2,1)$

Poisson structure: from symplectic potential on $ISO(2,1)^{2g}$

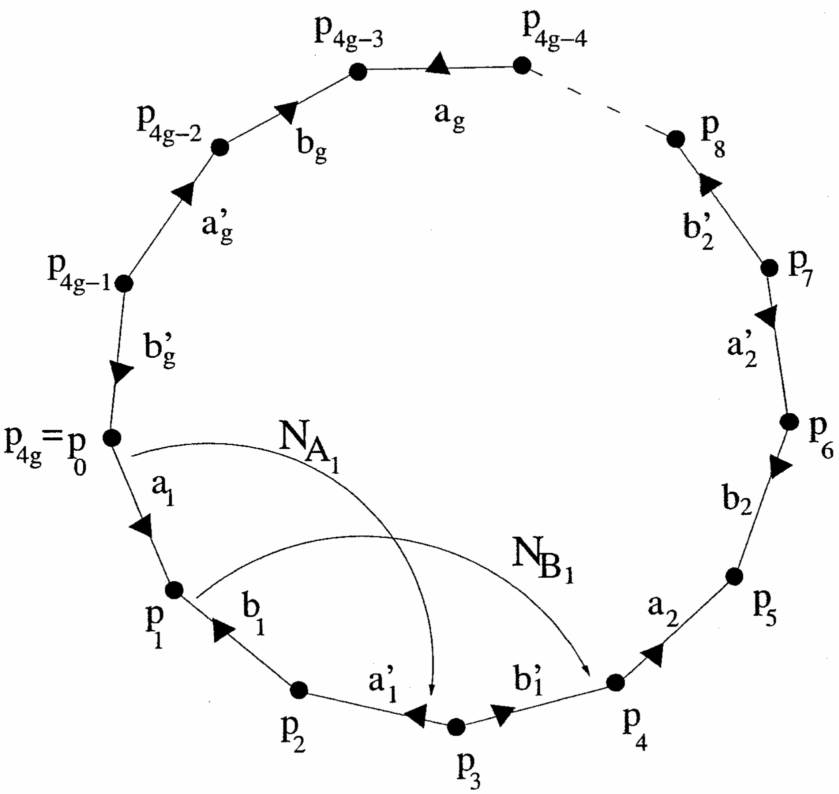
by imposing the constraint $[B_g, A_g^{-1}] \cdots [B_1, A_1^{-1}] = 1$ and dividing by the associated gauge transformations (simultaneous conjugation with $ISO(2,1)$

Trivialisation and embedding

trivialisation: on simply connected region $R \subset \mathbb{R} \times S_g$:

$$A_S = \gamma d_S \gamma^{-1}, \quad \gamma = (v, \boldsymbol{x}) : R \to ISO(2,1)$$
$$\boldsymbol{x} : R \to \mathbb{R}^3 = \text{embedding into } \mathbb{M}^3$$

maximal simply connected region by cutting S_g along the generators $a_i, b_i \in \pi_1(S_g) \Rightarrow 4g$-gon P_g



overlap condition:

$\gamma^{-1}|_{a_i'} = N_{A_i}\gamma^{-1}|_{a_i} \quad \gamma^{-1}|_{b_i'} = N_{B_i}\gamma^{-1}|_{b_i}$ with constants $N_{A_i}, N_{B_i} \in ISO(2,1)$
$\Rightarrow$ determined completely by embedding of sides a_i, a_i', b_i, b_i'

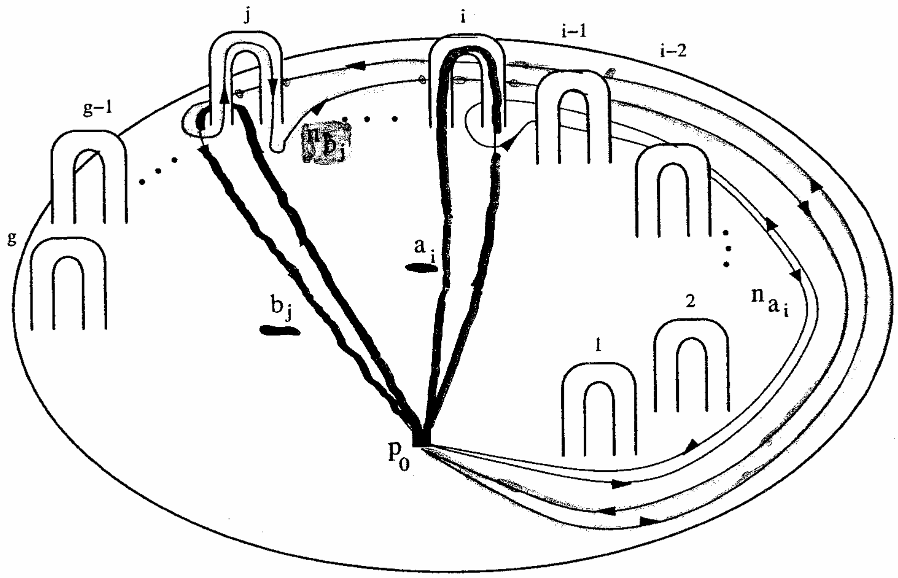
holonomies: $A_i = \gamma(p_{4i-3})\gamma^{-1}(p_{4i-4}) \qquad B_i = \gamma(p_{4i-3})\gamma^{-1}(p_{4i-2})$

Holonomies and dual holonomies

overlap condition

$$A_i = \gamma(p_0)[N_{A_1}^{-1}, N_{B_1}] \cdots [N_{A_i}^{-1}, N_{B_i}] \cdot N_{B_i} \cdot [N_{B_{i-1}}, N_{A_{i-1}}^{-1}] \cdots [N_{B_1}, N_{A_1}^{-1}]\gamma^{-1}(p_0)$$
$$B_i = \gamma(p_0)[N_{A_1}^{-1}, N_{B_1}] \cdots [N_{A_i}^{-1}, N_{B_i}] \cdot N_{A_i} \cdot [N_{B_{i-1}}, N_{A_{i-1}}^{-1}] \cdots [N_{B_1}, N_{A_1}^{-1}]\gamma^{-1}(p_0)$$

$$N_{A_i} = \gamma^{-1}(p_0)[{A_1}^{-1}, B_1] \cdots [{A_i}^{-1}, B_i] \cdot B_i \cdot [B_{i-1}, {A_{i-1}}^{-1}] \cdots [B_1, {A_1}^{-1}]\gamma(p_0)$$
$$N_{B_i} = \gamma^{-1}(p_0)[{A_1}^{-1}, B_1] \cdots [{A_i}^{-1}, B_i] \cdot A_i \cdot [B_{i-1}, {A_{i-1}}^{-1}] \cdots [B_1, {A_1}^{-1}]\gamma(p_0)$$



$\Rightarrow$ up to conjugation:
N_{A_i}, N_{B_i} = holonomies of dual set of generators $n_{a_i}, n_{b_i} \in \pi_1(S_g)$

$$A_i = H[a_i], \quad B_i = H[b_i] \qquad\qquad N_{A_i} = H[n_{a_i}], \quad N_{B_i} = H[n_{b_i}]$$

3 Grafting in the Chern-Simons formalism

idea: $x^0 = T \Rightarrow$ embedding on surfaces of constant cosmological time T
. $\Rightarrow$ determine holonomies from embedding of sides of polygon P_g

static case: polygon P_g embedded onto polygon P_Γ^T in tesselation of H_T

$$\boldsymbol{x}_{st}(T,\cdot): P_g \mapsto P_\Gamma^T \subset H_T \Rightarrow \quad N_{A_i}^{st} = (v_{A_i}, 0), N_{B_i}^{st} = (v_{B_i}, 0)$$

grafted spacetime: $\boldsymbol{x}(T,\cdot): P_g \to U_\lambda^T \quad \Rightarrow \quad N_{A_i}{=}(v_{A_i}, ?),\ N_{B_i}{=}(v_{B_i}, ?)$

$\Rightarrow$ translation of corners: $\quad \boldsymbol{x}(T,p_i) = \boldsymbol{x}_{st}(T,p_i) + \rho(p_i)$

$$\rho(p_i) = \rho(\boldsymbol{x}_{st}(1,p_i)) = w \sum_{v\in\Gamma:\, a_x \cap vc_\lambda \neq \emptyset} \epsilon_{v,x}\, v\boldsymbol{n}_\lambda$$

transformation of holonomies under grafting along λ

$$\begin{aligned} Gr_{w\lambda}: A_i^{st} &\mapsto A_i = \gamma(p_{4i-3})\gamma^{-1}(p_{4i-4}) = A_i^{st}\cdot(1, \rho(p_{4i-4}) - \rho(p_{4i-3})) \\ B_i^{st} &\mapsto B_i = \gamma(p_{4i-3})\gamma^{-1}(p_{4i-2}) = B_i^{st}\cdot(1, \rho(p_{4i-2}) - \rho(p_{4i-3})) \end{aligned}$$

For transformation of holonomies: determine intersections of of sides a_i, b_i with geodesics in M_λ:

1. $\lambda \subset H_1/\Gamma$ closed $\Rightarrow \exists v = e^{-p_\lambda^a J_a} \in \Gamma:\ vc_\lambda = c_\lambda,\ \boldsymbol{n}_\lambda = -\hat{\boldsymbol{p}}_\lambda = -\frac{1}{m_\lambda}\boldsymbol{p}_\lambda$
2. express v in terms of the generators of Γ:

$$v = v_{X_r}^{\alpha_r}\cdots v_{X_1}^{\alpha_1} \text{ with } v_{X_i} \in \{v_{A_1}, \ldots, v_{B_g}\}, \alpha_i \in \{\pm 1\} \tag{1}$$

- geodesics in M_λ intersecting P_Γ^T

$$c_\lambda,\quad v_{X_1}^{\alpha_1}c_\lambda,\quad v_{X_2}^{\alpha_2}v_{X_1}^{\alpha_1}c_\lambda,\quad \ldots,\quad v_{X_{r-1}}^{\alpha_{r-1}}\cdots v_{X_1}^{\alpha_1}c_\lambda$$

- intersections with sides a_i $(b_i) \Leftarrow 1:1 \Rightarrow$ factors $v_{A_i}^{\pm1}$ $(v_{B_i}^{\pm1})$ in (1)

$\Rightarrow$ Explicit formula for grafting map $Gr_{w\lambda}: ISO(2,1)^{2g} \to ISO(2,1)^{2g}$,
$\Rightarrow$ transformation of general holonomies H_η, $\eta \in \pi_1(S_g)$

The grafting transformation

$$Gr_{w\lambda}: \quad u_{A_i} \mapsto u_{A_i} \qquad u_{B_i} \mapsto u_{B_i}$$

$$\begin{aligned} \boldsymbol{j}_{A_i} \mapsto \boldsymbol{j}_{A_i} &+ w\mathrm{Ad}(v_0^{-1} v_{H_1}^{-1} \cdots v_{H_{i-1}}^{-1}) \sum_{k: X_k = A_i, \alpha_k = 1} \mathrm{Ad}(v_{X_{k-1}}^{\alpha_{k-1}} \cdots v_{X_1}^{\alpha_1})\, \boldsymbol{n}_\lambda \\ &- w\mathrm{Ad}(v_0^{-1} v_{H_1}^{-1} \cdots v_{H_{i-1}}^{-1}) \sum_{k: X_k = A_i, \alpha_k = -1} \mathrm{Ad}(v_{X_k}^{\alpha_k} \cdots v_{X_1}^{\alpha_1})\, \boldsymbol{n}_\lambda \end{aligned}$$

$$\begin{aligned} \boldsymbol{j}_{B_i} \mapsto \boldsymbol{j}_{B_i} &- w\mathrm{Ad}(v_0^{-1} v_{H_1}^{-1} \cdots v_{H_{i-1}}^{-1} v_{A_i}^{-1} v_{B_i}) \sum_{k: X_k = B_i, \alpha_k = 1} \mathrm{Ad}(v_{X_{k-1}}^{\alpha_{k-1}} \cdots v_{X_1}^{\alpha_1})\, \boldsymbol{n}_\lambda \\ &+ w\mathrm{Ad}(v_0^{-1} v_{H_1}^{-1} \cdots v_{H_{i-1}}^{-1} v_{A_i}^{-1} v_{B_i}) \sum_{k: X_k = B_i, \alpha_k = -1} \mathrm{Ad}(v_{X_k}^{\alpha_k} \cdots v_{X_1}^{\alpha_1})\, \boldsymbol{n}_\lambda \end{aligned}$$

where $\gamma^{-1}(p_0) = (v_0, \boldsymbol{x}_0)$ and

$$\begin{aligned} v_{A_i} &= v_0 u_{H_1}^{-1} \cdots u_{H_i}^{-1} \cdot u_{B_i} \cdot u_{H_{i-1}} \cdots u_{H_1} v_0^{-1} \\ v_{B_i} &= v_0 u_{H_1}^{-1} \cdots u_{H_i}^{-1} \cdot u_{A_i} \cdot u_{H_{i-1}} \cdots u_{H_1} v_0^{-1} \end{aligned}$$

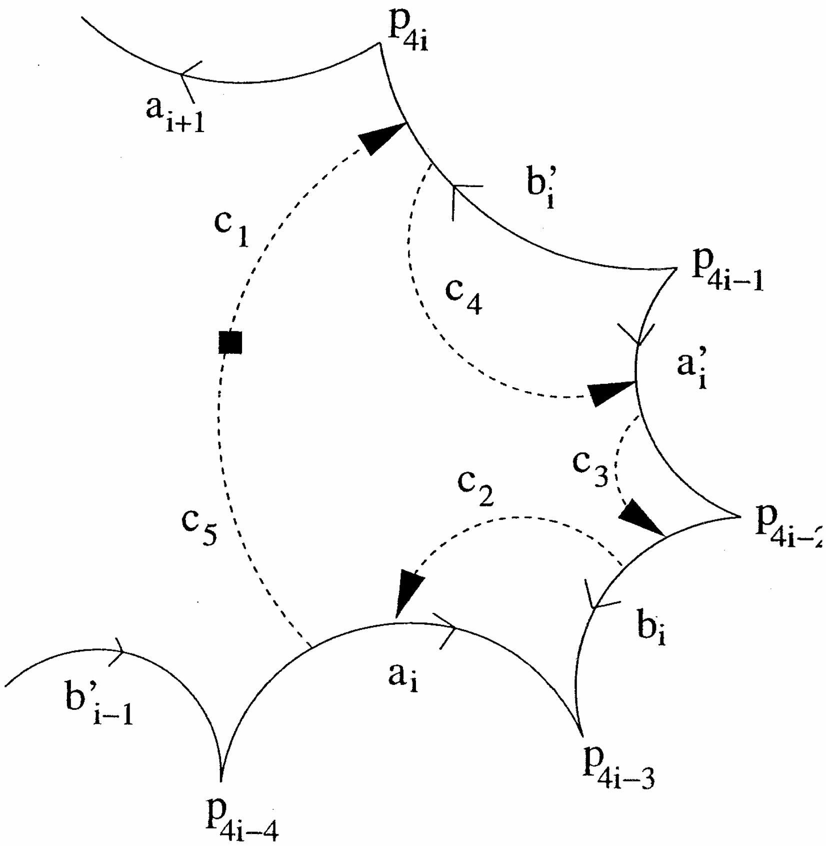
$$\begin{aligned} u_{A_i} &= v_0^{-1} v_{H_1}^{-1} \cdots v_{H_i}^{-1} \cdot v_{B_i} \cdot v_{H_{i-1}} \cdots v_{H_1} v_0 \\ u_{B_i} &= v_0^{-1} v_{H_1}^{-1} \cdots v_{H_i}^{-1} \cdot v_{A_i} \cdot v_{H_{i-1}} \cdots v_{H_1} v_0 \end{aligned}$$

Example:

$\lambda = b_i \circ a_i^{-1} \circ b_i^{-1} \circ a_i$

Geodesics in M_λ intersecting the polygon $P_\Gamma^1 \subset H_1$ (only non-trivial intersection points)

$$c_1 = c_\lambda, \qquad c_2 = v_{B_i}^{-1} c_\lambda, \qquad c_3 = v_{A_i} v_{B_i}^{-1} c_\lambda$$
$$c_4 = v_{B_i} v_{A_i}^{-1} v_{B_i}^{-1} c_\lambda, \qquad c_5 = [v_{A_i}^{-1}, v_{B_i}] c_\lambda = c_\lambda,$$



Transformation of the holonomies

$$\boldsymbol{j}_{A_i} \mapsto \boldsymbol{j}_{A_i} + t(1 - \mathrm{Ad}(u_{A_i}^{-1}))\hat{\boldsymbol{p}}_\lambda \qquad \boldsymbol{j}_{B_i} \mapsto \boldsymbol{j}_{B_i} + t(1 - \mathrm{Ad}(u_{B_i}^{-1}))\hat{\boldsymbol{p}}_\lambda$$

$$Gr_{tm_\lambda \lambda}: \ A_i \mapsto H_\lambda^{-\theta t} A_i H_\lambda^{\theta t} \qquad B_i \mapsto H_\lambda^{-\theta t} B_i H_\lambda^{\theta t}$$
$$D_{t\lambda}: \quad A_i \mapsto H_\lambda^{-t} A_i H_\lambda^{t} \qquad B_i \mapsto H_\lambda^{-t} B_i H_\lambda^{t} \qquad H_\lambda = [B_i, A_i^{-1}] = e^{-(p_\lambda^a + \theta k_\lambda^a) J_a}$$

4 Grafting and Poisson structure

Theorem The grafting transformation $Gr_{w\lambda} : ISO(2,1)^{2g} \to ISO(2,1)^{2g}$ is generated via the Poisson bracket by the mass m_λ

$$F \circ Gr_{w\lambda} = -\{wm_\lambda, F\} \qquad \forall F \in \mathcal{C}^\infty(ISO(2,1)^{2g}).$$

$\Rightarrow$ **Properties of the grafting transformation** $Gr_{w\lambda}$:

1. Defined for general (not necessarily simple) elements $\lambda \in \pi_1(S_g)$

2. Poisson isomorphism

$$\{F \circ Gr_{w\lambda}, G \circ Gr_{w\lambda}\} = \{F, G\} \circ Gr_{w\lambda} \qquad \forall F, G \in \mathcal{C}^\infty(ISO(2,1)^{2g})$$

3. Leaves constraint $[B_g, A_g^{-1}] \cdots [B_1, A_1^{-1}] \approx 1$ invariant and commutes with the associated gauge transformations by simultaneous conjugation

4. The grafting transformations $Gr_{w\lambda}$ for different $\lambda \in \pi_1(S_g)$, $w \in \mathbb{R}^+$ commute and

$$F \circ Gr_{w_r\lambda_r} \circ \ldots \circ Gr_{w_1\lambda_1} = \left\{ \sum_{i=1}^{r} w_i m_{\lambda_i}, F \right\} \qquad \forall \lambda_i \in \pi_1(S_g), w_i \in \mathbb{R}^+$$

5. Implies general relation for the Poisson brackets of masses and spins $\{m_\lambda, s_\eta\} = \{s_\lambda, m_\eta\}$ $\forall \lambda, \eta \in \pi_1(S_g)$.

5 Grafting and Dehn twists

Dehn twists:

(infinitesimal) Dehn twists along simple curves $\lambda \in \pi_1(S_g)$
$\Rightarrow$ transformation $D_{w\lambda} : ISO(2,1)^{2g} \to ISO(2,1)^{2g}$

1. infinitesimally generated via the Poisson bracket by observable $m_\lambda s_\lambda$

$$\frac{d}{dw}|_{w=0} F \circ D_{w\lambda} = \{m_\lambda s_\lambda, F\} \qquad \forall F \in \mathcal{C}^\infty(ISO(2,1)^{2g})$$

2. Poisson isomorphism: $\{F \circ D_{w\lambda}, G \circ D_{w\lambda}\} = \{F, G\} \circ D_{w\lambda}$

3. Leaves constraint invariant and commutes with gauge transformations

4. Explicit formula for action on holonomy H_η, $\eta \in \pi_1(S_g)$:

- write curves as product in generators $a_i, b_i \in \pi_1(S_g)$

$$\lambda = x_r^{\alpha_r} \circ \ldots \circ x_1^{\alpha_1},\ \eta = y_s^{\beta_s} \circ \ldots \circ y_1^{\beta_1} \quad x_i, y_j \in \{a_1, ..., b_g\}, \alpha_i, \beta_j \in \{\pm 1\}$$

- intersection point between factors $x_{k+1}^{\alpha_{k+1}}$ and $x_k^{\alpha_k}$ on λ, $y_{l+1}^{\beta_{l+1}}$ and $y_l^{\alpha_l}$ on η

$$D_{w\lambda} : H_\eta \mapsto Y_s^{\beta_s} \cdots Y_{l+1}^{\beta_{l+1}} \cdot (X_k^{\alpha_k} \cdots X_1^{\alpha_1}) \cdot H_\lambda^{\epsilon w} \cdot (X_1^{-\alpha_1} \cdots X_k^{-\alpha_k}) \cdot Y_l^{\beta_l} \cdots Y_1^{\beta_1}$$
$$H_\lambda^{\epsilon w} = e^{-\epsilon w (p_\lambda^a J_a + k_\lambda^a P_a)},\ H_\lambda = e^{-(p_\lambda^a J_a + k_\lambda^a P_a)} = X_r^{\alpha_r} \cdots X_1^{\alpha_1}$$

Grafting:

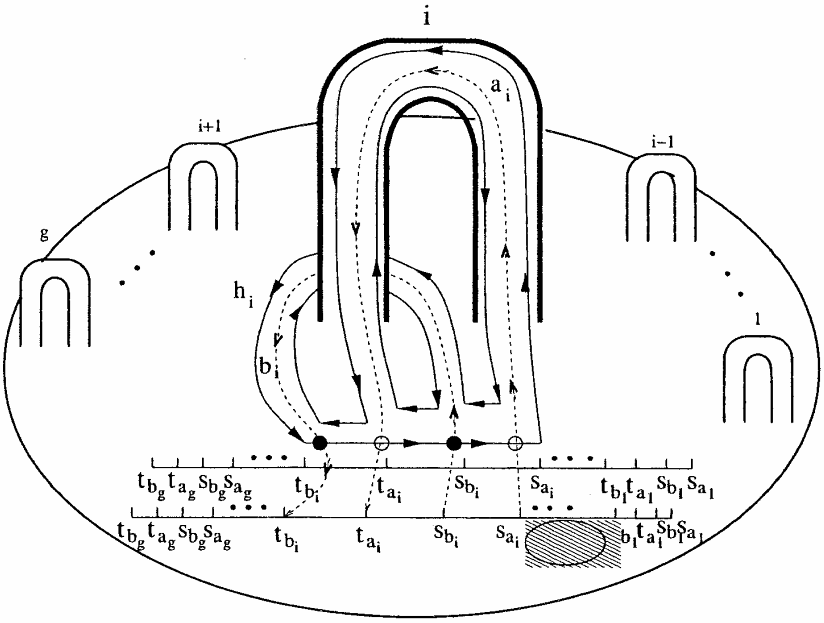
$Gr_{wm_\lambda \lambda}$:

$$H_\eta \mapsto Y_s^{\beta_s} \cdots Y_{l+1}^{\beta_{l+1}} \cdot (X_k^{\alpha_k} \cdots X_1^{\alpha_1}) \cdot (1, -w\epsilon \boldsymbol{p}_\lambda) \cdot (X_1^{-\alpha_1} \cdots X_k^{-\alpha_k}) \cdot Y_l^{\beta_l} \cdots Y_1^{\beta_1}$$
$$= Y_s^{\beta_s} \cdots Y_{l+1}^{\beta_{l+1}} \cdot (X_k^{\alpha_k} \cdots X_1^{\alpha_1}) \cdot H_\lambda^{\epsilon w \theta} \cdot (X_1^{-\alpha_1} \cdots X_k^{-\alpha_k}) \cdot Y_l^{\beta_l} \cdots Y_1^{\beta_1}$$

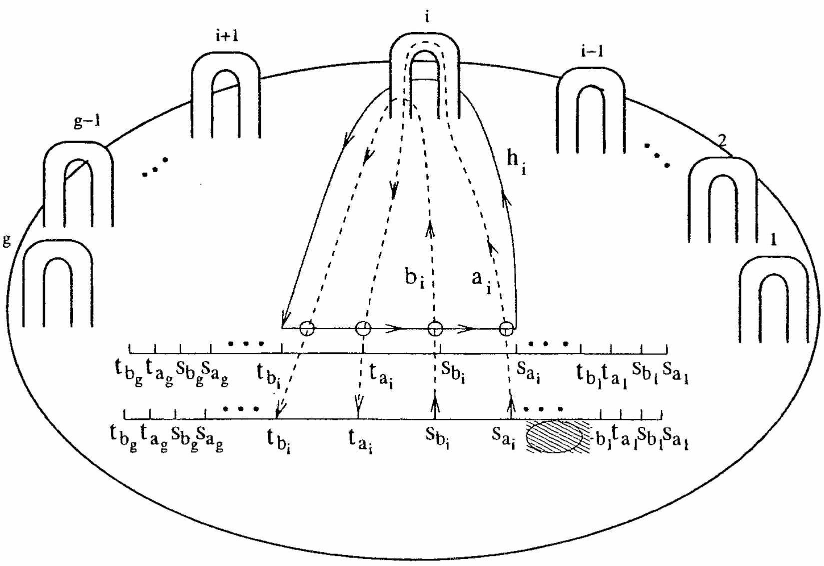
Grafting along λ =inf. Dehn twist along λ with formal parameter θ, $\theta^2 = 0$

$$\boxed{Gr_{wm_\lambda \lambda} = D_{\theta w \lambda}}$$

Graphical procedure for the determination of the intersection points



The decomposition of $[b_i, a_i^{-1}]$ (full line) and its intersection points with a_i, b_i (dashed lines)



The decomposition of $[b_i, a_i^{-1}]$ (full line) and its intersection points with a_i, b_i (dashed lines), simplified representation without horizontal segments that do not contain intersection points

6 Outlook and Conclusions

Relation between geometrical construction of (2+1)-spacetimes via grafting and phase space and Poisson structure in the Chern-Simons formulation of (2+1)-dimensional gravity for $\Lambda = 0$, spacetimes $M \approx \mathbb{R} \times S_g$

- implementation of grafting along closed, simple $\lambda \in \pi_1(S_g)$ in the Chern-Simons formalism $\Rightarrow$ grafting transformation $Gr_{w\lambda}$ on Poisson manifold $(ISO(2,1)^{2g}, \Theta)$
- generated via the Poisson bracket by gauge invariant observable m_λ
- Poisson isomorphism, respects constraint, commutative
- general relation for Poisson brackets of mass and spin $\{m_\lambda, s_\eta\} = \{s_\lambda, m_\eta\}$
- can be viewed as (infinitesimal) Dehn twist along λ with formal parameter θ, $\theta^2 = 0$

$\Rightarrow$ **Physical interpretation of gauge invariant observables:**

m_λ: generates grafting: cuts spatial surface along λ and translates sides of the cut

$m_\lambda s_\lambda$: generates inf. Dehn twist: cuts spatial surface along λ and rotates sides of the cut

Open questions

- Other cases of cosmological constant $\Lambda > 0$, $\Lambda < 0$?
- Manifestation of Wick rotation [Benedetti, Bonsante] on phase space ?

The symplectic potential Θ on the extended phase space $ISO(2,1)^{2g}$

Θ in terms of the holonomies $A_i = (u_{A_i}, -u_{A_i}\boldsymbol{j}_{A_i})$, $B_i = (u_{B_i}, -u_{B_i}\boldsymbol{j}_{B_i})$

$$\begin{aligned}\Theta = &\sum_{i=1}^{g} \langle \boldsymbol{j}_{A_i}, \delta(u_{H_{i-1}} \cdots u_{H_1})(u_{H_{i-1}} \cdots u_{H_1})^{-1}\rangle \\ &\qquad - \langle \boldsymbol{j}_{A_i}, \delta(u_{A_i}^{-1} u_{B_i}^{-1} u_{A_i} u_{H_{i-1}} \cdots u_{H_1})(u_{A_i}^{-1} u_{B_i}^{-1} u_{A_i} u_{H_{i-1}} \cdots u_{H_1})^{-1}\rangle \\ &+ \sum_{i=1}^{g} \langle \boldsymbol{j}_{B_i}, \delta(u_{A_i}^{-1} u_{B_i}^{-1} u_{A_i} u_{H_{i-1}} \cdots u_{H_1})(u_{A_i}^{-1} u_{B_i}^{-1} u_{A_i} u_{H_{i-1}} \cdots u_{H_1})^{-1}\rangle \\ &\qquad - \langle \boldsymbol{j}_{B_i}, \delta(u_{B_i}^{-1} u_{A_i} u_{H_{i-1}} \cdots u_{H_1})(u_{B_i}^{-1} u_{A_i} u_{H_{i-1}} \cdots u_{H_1})^{-1}\rangle\end{aligned}$$

with $u_{H_i} = [u_{B_i}, u_{A_i}^{-1}]$, $\boldsymbol{j}_{A_i} = j^a_{A_i} P_a$, $\boldsymbol{j}_{B_i} = j^a_{B_i} P_a$ and pairing

$$\langle J_a, P^b\rangle = \delta_a^b \qquad \langle J_a, J_b\rangle = \langle P^a, P^b\rangle = 0$$

Θ in terms of Lorentz components v_{A_i}, v_{B_i} of dual holonomies N_{A_i}, N_{B_i}

$$\Theta = \sum_{i=1}^{g} \langle \boldsymbol{l}_{A_i}, v_{A_i}^{-1}\delta v_{A_i}\rangle + \langle \boldsymbol{l}_{B_i}, v_{B_i}^{-1}\delta v_{B_i}\rangle$$

with

$$\begin{aligned}\boldsymbol{l}_{A_i} =& \mathrm{Ad}(v_{H_{i-1}} \cdots v_{H_1} v_0)\boldsymbol{j}_{A_i} = \mathrm{Ad}(v_0 u_{H_1}^{-1} \cdots u_{H_{i-1}}^{-1})\boldsymbol{j}_{A_i} \\ \boldsymbol{l}_{B_i} =& -\mathrm{Ad}(v_{B_i}^{-1} v_{A_i} v_{H_{i-1}} \cdots v_{H_1} v_0)\boldsymbol{j}_{B_i} = -\mathrm{Ad}(v_0 u_{H_1}^{-1} \cdots u_{H_{i-1}}^{-1} u_{A_i}^{-1} u_{B_i})\boldsymbol{j}_{B_i},\end{aligned}$$

Poisson brackets

$$\{l_a^X, l_b^X\} = -\epsilon_{abc} l_X^c \quad \{l_a^X, v_X\} = -v_X J_a \qquad X \in \{A_1, \ldots, B_g\}.$$