1) Singular Local Objects

⇒ "equational" divergence + resurgence

⇒ first Bridge Equation (BE_1)

‖ L'Equation du Pont et la classification analytique des Objets Locaux (Orsay 1985)

‖ Six lectures on Transseries, Analysable Functions and the Constructive Proof of Dulac's Conjecture (Kluwer, 1993)

2) Singular Parameters

⇒ "coequational" divergence + resurgence

⇒ second and third Bridge Equations (BE_2, BE_3)

‖ Weighted products and parametric resurgence (Travaux en Cours, 47, 1994)

‖ Recent advances in the analysis of divergence and singularities (Kluwer, 2004)

⇓

3) Arithmetical "dimorphy".

⇒ the "flexion structure"

⇒ the quest for canonical irreducibles.

The basic Bridge Equation.

BE ~ BE$_1$ $\boxed{\Delta_\omega\, y(z,u) = \mathbb{A}_\omega\, y(z,u)}$

Ingredient 1: the formal integral $y(z,u)$
with critical variable z and a complete set of parameters $u = (u_1, \dots, u_{\nu-1})$.
For a local differential // difference // functional equation // system, y is simply a parameter-saturated formal solution.
For a local vector field // diffeomorphism, it is the formal change of chart $(y_1, \dots, y_\nu) \to (z, u_1, \dots, u_{\nu-1})$ that takes it to the form $\partial/\partial z$ or $z \to z+1$.

Ingredient 2: the alien derivations Δ_ω.
They are exotic derivation operators which:
* measure the Stokes constants (in the multiplicative plane z)
* evaluate the singularities over ω (in the Borel-conjugate plane ζ)

Ingredient 3: the holomorphic invariants $\mathbb{A}_\omega$.
They are ordinary (first order) differential operators in z and u.
They are:
* analytic invariants, i.e. invariant under analytic changes of chart
* holomorphic invariants, i.e. they depend holomorphically on the local object (within a given formal class)
* non-formal invariants, i.e. they depend effectively on a infinity of Taylor coefficients (no finite jet of the local object suffices).

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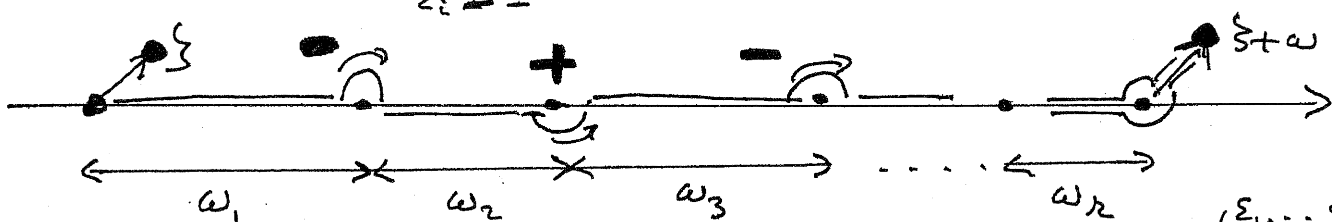
(2) Standard // organic alien derivations. (2)

Borel transform (inverse Laplace tr.): $\begin{cases} \overset{(\sim\infty)}{z} \longrightarrow \overset{(\sim 0)}{\zeta} \\ z^{-n} \longrightarrow \zeta^{n-1}/(n-1)! \end{cases}$

$$\varphi(z) \rightarrow \hat{\varphi}(\zeta) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \varphi(z)\, e^{z\zeta}\, dz$$

General alien derivations. Index $\omega \in \mathbb{C}_\bullet$.

$$\Delta_\omega\, \hat{\varphi}(\zeta) = \frac{1}{2\pi i} \sum_{\varepsilon_i = \pm} d^{\binom{\varepsilon_1 \dots \varepsilon_r}{\omega_1 \dots \omega_r}} . \hat{\varphi}^{\binom{\varepsilon_1 \dots \varepsilon_r}{\omega_1 \dots \omega_r}} (\zeta + \omega)$$



Definite algebraic constraints on the coefficient $d^{\binom{\varepsilon_1 \dots \varepsilon_r}{\omega_1 \dots \omega_r}}$

Standard alien derivations: don't depend on the increments ω_i

$$d^{\binom{\varepsilon_1 \dots \varepsilon_r}{\omega_1 \dots \omega_r}} := \varepsilon_r \frac{p!\, q!}{(p+q+1)!} = \varepsilon_r \frac{p!\, q!}{r!} \qquad \begin{cases} p := \text{nb of } + \text{ in } \varepsilon \\ q := \text{nb of } - \text{ in } \varepsilon \end{cases}$$

Organic alien derivations: preserve exponential growth.

$$d^{\binom{\varepsilon_1 \dots \varepsilon_r}{\omega_1 \dots \omega_r}} := \begin{cases} \frac{1}{2} \varepsilon_r \cdot \dfrac{\omega_{p+1}}{\omega_1 + \dots + \omega_r} & \text{if } \varepsilon = (\overbrace{+ \dots +}^{p} \overbrace{- \dots -}^{q} \varepsilon_r) \\ \frac{1}{2} \varepsilon_r \cdot \dfrac{\omega_{q+1}}{\omega_1 + \dots + \omega_r} & \text{if } \varepsilon = (\overbrace{- \dots -}^{q} \overbrace{+ \dots +}^{p} \varepsilon_r) \\ 0 & \text{otherwise.} \end{cases}$$

P₁. The operators Δ_ω are derivations, i.e.

$$\Delta_\omega (\hat{\varphi}_1 * \hat{\varphi}_2) = (\Delta_\omega \hat{\varphi}_1) * \hat{\varphi}_2 + \hat{\varphi}_1 * (\Delta_\omega \hat{\varphi}_2) \quad (\text{in } \zeta\text{-plane})$$

$$\Delta_\omega (\varphi_1\ \varphi_2) = (\Delta_\omega \varphi_1) . \varphi_2 + \varphi_1 . (\Delta_\omega \varphi_2) \quad (\text{in } z\text{-plane})$$

P₂ Setting $\mathbf{\Delta}_\omega := e^{-\omega z} \Delta_\omega$ we have $[\mathbf{\Delta}_\omega, \partial_z] = 0$ (in z-plane)

P₃ Standard // organic: free + couple systems. Conversion rule.

General statements.

1 – Shape of the index reservoir Ω (Resurgence "lattice").

* Always enumerable. * Spanned by "multipliers". λ_i
* For a vector field : $\omega = \sum n_i \lambda_i$ with $n_i \geq -1$
* For a diffeo : $\omega = n_0 \lambda_0 + \sum n_i \lambda_i$ with $\lambda_0 = 2\pi i$, $n_0 \in \mathbb{Z}$.

2 – Shape of the invariants A_ω. $\boxed{E(y)=0 \Rightarrow E_\omega(y, \Delta_\omega y) = 0}$

Two formal constraints: must preserve the general shape of the formal integral ("entire" dependence in u; exponential factors $e^{-\omega z}$ accompanying the u_i). + Growth constraints in ω.

3 – Complete systems of holomorphic invariants:
The **B.E.** always yields such systems.

4 – Free systems of holomorphic invariants.
The **B.E.** relative to ${}^{org}\Delta_\omega$ ("organic" choice) yields invariants ${}^{org}A_\omega$ subject only to having
(i) the proper shape (ii) exponential growth in ω.

5 – Applications: sectorial normalisation; fractional iteration, etc etc.

6 – "Display" and transcendence. (order reversal!).

$$\{\Delta_\omega y = A_\omega y \text{ and } [\Delta_{\omega_1}, A_{\omega_2}] = 0\} \Rightarrow \{\Delta_{\omega_r} \dots \Delta_{\omega_1} y = A_{\omega_1} \dots A_{\omega_r} y\}$$

$$\Rightarrow \text{display}(y) = y + \sum_{r \geq 1} \sum_{\omega_i} \Delta_{\omega_r} \dots \Delta_{\omega_1} y \; Z^{\omega_1, \dots, \omega_r}$$

$$\Rightarrow \{R(y_1, \dots, y_n) = 0\} \Rightarrow \{R(\text{display}(y_1), \dots, \text{display}(y_n)) = 0\}$$

7 – Canonical Object Synthesis with "twist" c: by "reversing **B.E.**"

8 – Obstruction to "twistless" synthesis ($c = 0$). Given by the "Drawbridge Equation" **B.E*** strikingly similar to **BE**.

Ex1 and 2 : simply resonant vector fields // diffeos.

Singular local Object: vector field (**Ex1**) or diffeo (**Ex2**).

Ex1 : $$Y = \sum_{j=1}^{\nu} \left(\lambda_j y_j + o(y)\right) \partial_{y_j} \quad \text{with} \begin{cases} \lambda_1 \dots \lambda_{\nu-1} \ \mathbb{Q}\text{-indep.} \\ \lambda_\nu = 0 \end{cases}$$

Ex2 : $$f : \begin{cases} y_j \to \ell_j y_j + o(y) & (j = 1 \dots \nu-1) \\ y_\nu \to y_\nu + o(y) & (\ell_j = e^{\lambda_j}) \end{cases}$$

Formal integral: **Ex1** and **Ex2** :

$$y(z,u) = \{ y_j(z, u_1, \dots, u_{\nu-1}) \in \mathbb{C}[[z^{-1}, e^{\lambda_1 z} u_1, \dots, e^{\lambda_{\nu-1} z} u_{\nu-1}]]$$

(or a slight modification if $\rho \neq 0$; ρ = iteration residue).

Index reservoir Ω:

Ex1 : $$\omega = n_1 \lambda_1 + \dots + n_{\nu-1} \lambda_{\nu-1}$$

Ex2 : $$\omega = n_1 \lambda_1 + \dots + n_{\nu-1} \lambda_{\nu-1} + n_0 \lambda_0 \quad (\lambda_0 := 2\pi i)$$

with $n_0 \in \mathbb{Z}$; $n_i \in \mathbb{N} \cup \{-1\}$; and at most one n_i equal to -1

Holomorphic invariants:

Ex1 : $$\mathbb{A}_\omega = u_1^{n_1} \dots u_{\nu-1}^{n_{\nu-1}} \left\{ \sum_{i=1}^{\nu-1} A_\omega^i \, u_i \, \partial_{u_i} \right\}$$

Ex2 : $$\mathbb{A}_\omega = u_1^{n_1} \dots u_{\nu-1}^{n_{\nu-1}} e^{-n_0 \lambda_0 z} \cdot \left\{ A_\omega^0 \, \partial_z + \sum_{i=1}^{\nu-1} A_\omega^i \, u_i \, \partial_{u_i} \right\}$$

with scalar coefficients $A_\omega^i \in \mathbb{C}$.

Applications: Criteria for analytic conjugacy; fractional iteration etc... Sectorial resummation; description of "borderline behaviour" etc...

Ex 3 : multiple resonance. Identity-tangent diffeos.

Local, identity-tangent diffeo on $\mathbb{C}^\nu_{,0}$:

$$f : y_j \longmapsto y_j + Q_j(y) + o(y^2) \qquad (i=1,\dots,\nu)$$

with generic quadratic forms $Q_1,\dots,Q_\nu$.

Eigenradii γ : they are the fixed points $\gamma=(\gamma_1,\dots,\gamma_\nu)$ of the mapping $y_i \mapsto Q_i(y)$; $\mathbb{P}^{\nu-1}(\mathbb{C}) \longrightarrow \mathbb{P}^{\nu-1}(\mathbb{C})$.

Formal integral : to each γ there answers a formal integral $y = {}^\gamma y$

$$y(z,u) = \{ y_i(z,u_1,\dots,u_{\nu-1}) \in \mathbb{C}[[\underset{\bullet}{z}^{-1}, u_1 \underset{\bullet}{z}^{\lambda_1},\dots,u_{\nu-1}\underset{\bullet}{z}^{\lambda_{\nu-1}}]] \}$$

with $z \sim \underset{\bullet}{z}$; $z \equiv \underset{\bullet}{z} + \rho \log \underset{\bullet}{z}$; and well-defined formal invariants $\lambda_j = \lambda_j(\gamma)$

Index reservoir Ω : $\qquad \omega = n_0 \lambda_0$ with $\lambda_0 = 2\pi i$ and $n \in \mathbb{Z}^*$.

Holomorphic invariants :

$$\mathbb{A}_\omega = e^{-\omega z} \left\{ A^0_\omega(u)\, \partial_z + \sum_{i=1}^{\nu-1} A^i_\omega(u)\, \partial_{u_i} \right\}$$

with $A^0_\omega(u),\dots,A^{\nu-1}_\omega(u) \in \mathbb{C}[[u_1,\dots,u_{\nu-1}]]$.

Completeness and conversion rule.

Each system $\{{}^\gamma\mathbb{A}_\omega, \omega\in\Omega\}$ attached to a given eigenradius γ is complete, but the conversion rule:

$$\{{}^\gamma\mathbb{A}_\omega\} \longleftrightarrow \{{}^{\gamma'}\mathbb{A}_\omega\}$$

is non-algebraic (it involves a complex procedure of analytic continuation)

Applications: all geometric aspects of f. For instance : criteria for the existence of analytic trajectories.

Ex 4: polycritical differential system.

Local analytic system:

$$* \quad \frac{1}{p_i}\, t^{1+p_i}\, \partial_t \mathbf{y}_i + \lambda_i \mathbf{y}_i = b_i(t,\mathbf{y}) \qquad \begin{cases} t \sim 0 \\ 1 \le i \le \nu \end{cases}$$

For simplicity, assume formal conjugacy to the normal system:

$$** \quad \frac{1}{p_i}\, t^{1+p_i}\, \partial_t \mathbf{y}_i + \lambda_i \mathbf{y}_i = 0$$

with distinct "levels" $p_1 \le p_2 \le \dots \le p_\nu$ and the corresponding "critical times" $\mathbf{z_i = t^{-p_i}}$ $(t \sim 0;\ z_i \sim \infty)$

The formal integral has the predictable shape:

$$\mathbf{y}(t,u) = \{\mathbf{y}_i(t,u_1,\dots,u_\nu) \in \mathbb{C}[[t, u_1 e^{\lambda_1 t^{-p_1}}, u_2 e^{\lambda_2 t^{-p_2}}, \dots]]$$

but actual resummation is via a step-wise process, known as accelero-summation, which takes us through all successive critical times z_i, from "slowest" to "fastest".

Each critical time z_i gives rise to its own BE:

$$\overset{(z_i)}{\mathbf{\Delta}}_\omega\ \mathbf{y}(z_i,u) = \overset{(z_i)}{\mathbb{A}}_\omega\ \mathbf{y}(z_i,u).$$

with indices ω spanned by the corresponding multipliers λ_j (i.e. $p_j \equiv p_i$) and with invariants $\mathbb{A}_\omega$ involving (inside their coefficients) all the parameters u_j linked to the previous "slower" critical times (i.e. $p_j < p_i$).

Together, all these $\mathbb{A}_\omega$ constitute a complete + free system of holomorphic invariants.

Remark: polarisations.

Gist of proofs.

Proofs are mainly based on explicit expansions of the formal integral and its holomorphic invariants:

$$(*)\quad y(z,u) = \Big\{ id + \sum_{r\geq 1} \sum_{\omega_i \in \Omega} \mathcal{W}^{\omega_1,\dots,\omega_r}(z)\, \mathbb{B}_{\omega_r} \cdots \mathbb{B}_{\omega_1} \Big\} . z$$

$$(**)\quad \mathbb{A}_\omega = \sum_{r\geq 1} \frac{1}{r} \sum_{\omega=\omega_1+\dots+\omega_r} \mathbf{W}^{\omega_1,\dots,\omega_r} [\mathbb{B}_{\omega_r} \dots [\mathbb{B}_{\omega_2}, \mathbb{B}_{\omega_1}]]$$

These expansions involve three ingredients:

(i) the Taylor coefficients of the Local Object under investigation, suitably encoded as ordinary differential operators $\mathbb{B}_\omega$ (of order 1).

(ii) resurgence monomials $\mathcal{W}^{\bullet}(z)$ that behave simply under alien // ordinary derivation.

(iii) resurgence monics $\mathbf{W}^{\bullet}$ which are transcendental constants produced by the resurgence monomials under alien derivation.

Convergence in these expansions is usually straightforward (though it sometimes calls for a suitable re-ordering of terms known as "arborification") because the whole "divergence" is carried by the critical variable z and absorbed into the resurgence monomials.

Remark: the derivation of the Bridge Equation itself is even simpler:

$$\| . R(f) = 0 \Rightarrow \| . R^*_*(f, \Delta_\omega f) = 0 \|$$

Hyperlogarithmic monomials // moulds. "Dimorphy".

Simplest, most basic system. Splits into 2 classes: ∂-friendly // Δ-friendly.

∂-friendly monomials $\mathcal{V}^\bullet(z)$. Characterised by induction:

$$\left(\partial_z + \omega_1 + \dots + \omega_r\right)\mathcal{V}^{\omega_1,\dots,\omega_r}(z) = -\mathcal{V}^{\omega_1,\dots,\omega_{r-1}}(z)\cdot\frac{1}{z}$$

Associated moulds $V^\bullet$. Characterised by:

$$\Delta_{\omega_0}\mathcal{V}^{\omega_1,\dots,\omega_r}(z) = \sum_{\omega_0=\omega_1+\dots+\omega_i} V^{\omega_1,\dots,\omega_i}\,\mathcal{V}^{\omega_{i+1},\dots,\omega_r}(z)$$

Δ-friendly monomials $\mathcal{U}^\bullet(z)$.

Characterised by: $\mathcal{U}^\bullet(z) \circ V^\bullet = \mathcal{V}^\bullet(z)$

Δ-friendliness: $$\Delta_{\omega_0}\mathcal{U}^{\omega_1,\dots,\omega_r}(z) = \begin{cases} 0 & \text{if } \omega_0 \neq \omega_1 \\ \mathcal{U}^{\omega_2,\dots,\omega_r}(z) & \text{if } \omega_0 = \omega_1 \end{cases}$$

Associated moulds $U^\bullet$:

Characterised by : $U^\bullet \circ V^\bullet = I^\bullet$

Symmetries: the monomials $\mathcal{V}^\bullet(z), \mathcal{U}^\bullet(z)$ are symmetral

the moulds $V^\bullet, U^\bullet$ are alternal

This basic system of monomials // moulds is called "hyperlogarithmic" because their actual calculation (in the convolutive plane) involves integrals of that type.

The moulds admit numerous encodings, each one manifesting a specific symmetry. But in the end everything reduces to a fundamental arithmetical dimorphy

Canonical Object Synthesis.

"Analysis": Object $\to$ Invariants. "Synthesis": Invariants $\to$ Object.

Local Object: simply resonant vector field (for ex.)

Bridge equation:
$$\Delta_\omega \; y(z,u) = \mathbb{A}_\omega \; y(z,u) \qquad (*)$$

Variant:
$$[\Delta_\omega, \Theta] = -\Theta \cdot \mathbb{A}_\omega \qquad (**)$$

with Θ denoting the normalising substitution operator $(y_i \to y_i^{nor})$

"Formal" Object Synthesis: For any system of Δ-friendly resurgence monomials $\mathcal{U}e^{\bullet}$, i.e. such that

$$\begin{cases} \Delta_{\omega_0} \mathcal{U}e^{\omega_1,\dots,\omega_r}(z) = \mathcal{U}e^{\omega_2,\dots,\omega_r}(z) \text{ if } \omega_0 = \omega_1 \; (\text{resp. } 0 \text{ if } \omega_0 \neq \omega_1) \\ \mathcal{U}e^{\bullet}(z) \text{ symmetral} \end{cases}$$

the expansion
$$\Theta = 1 + \sum_{1 \leq r} \sum_{\omega_i \in \Omega} (-1)^r \, \mathcal{U}e^{\omega_1,\dots,\omega_r}(z) \, \mathbb{A}_{\omega_r} \cdots \mathbb{A}_{\omega_1}$$

defines a formal substitution operator (ie $\Theta(\varphi.\psi) \equiv (\Theta\varphi).(\Theta\psi)$) which formally verifies $(**)$.

Effective, canonical ("spherical") Object Synthesis.

There is a canonical choice, corresponding to the "spherical monomials":

$$\begin{cases} \mathcal{U}e_c^{\omega_1,\dots,\omega_r}(z) = \underline{SPA}\displaystyle\int_0^\infty \frac{dy_1 \dots dy_r}{(y_r - y_{r-1}) \cdots (y_2 - y_1)(y_1 - z)} \times \dots \\ \times \exp\left\{ -\displaystyle\sum_{i=1}^{r} \omega_i \cdot (y_i - z) - \sum_{i=}^{r} c^2 \, \overline{\omega_i} \cdot (y_i^{-1} - z^{-1}) \right\} \end{cases}$$

for a large enough value of the twist $c > 0$.

Remark 1: Antipodal involution. Explain "spherical".

Remark 2: limit $c \to 0$.

Twistless Object Synthesis. The "Drawbridge Equation".

The twistless resurgence monomials ($c=0$) correspond to the classical, hyperlogarithmic, Δ-friendly resurgence monomials, but they fail to ensure convergence in Object Synthesis: there is generically divergence+resurgence in the u-parameters

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This specific resurgence ("synthesis resurgence") is of an unusually complex sort. It obeys the so-called Drawbridge Equation **BE***

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* **BE*** involves "invariants of invariants".

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* **BE*** gives necessary+sufficient conditions for the success (i.e. convergence) of "twistless" synthesis (with $c=0$)

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* with **BE*** (unlike with BE) the action of alien derivations on the test functions doesn't reduce in a straightforward manner to the action of ordinary differential operators. It involves an intricate mixture of integration//differentiation.

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* yet the corresponding effective alien algebra (i.e. the quotient of ALIEN by the annihilator ideal) is isomorphic to an algebra of ordinary differential operators. (Hence the name of Drawbridge Equation).

NB. This tendency for effective alien algebras to behave like ordinary differential algebras is extremely robust: it survives even when the direct equivalence fails.

Coequational resurgence. Heuristics I.

Example of singular ODE in $\underline{t}$ with singular parameter $\underline{\varepsilon}$.

$$(*) \quad \varepsilon t^2 \partial_t y(t,\varepsilon) + y(t,\varepsilon) = \varphi(t, y, \varepsilon) \in \mathbb{C}\{t,y,\varepsilon\}$$

$$\{t \sim 0 ; \varepsilon \sim 0\} \longrightarrow \{z = 1/t \sim \infty ; x = 1/\varepsilon \sim \infty\}$$

critical variable z ; critical covariable x.

$$(**) \quad (\partial_z + \omega x)\, y(z,x) = b(z) \in \mathbb{C}\{z^{-1}\}$$

One should expect far-going affinities between $\underline{\text{eq}}^{l}$ and $\underline{\text{coeq}}^{l}$. divergence//resurgence, since in special cases y depends on the sole product εt or $x z$. But there are also bound to be differences. Indeed:

Eq^l: Borel

$$\begin{cases} z\text{-plane} \longrightarrow \zeta\text{-plane} \\ z^{-n} \longrightarrow \zeta^{n-1}/(n-1)! \\ \partial_z \longrightarrow -\zeta \end{cases}$$

$$y(z,x) \longrightarrow {}^{\wedge}y(\zeta,x) \equiv (-\zeta + x\omega)^{-1}\, \hat{b}(\zeta)$$

Coeq^l: Borel

$$\begin{cases} x\text{-plane} \longrightarrow \xi\text{-plane} \\ x^{-n} \longrightarrow \xi^{n-1}/(n-1)! \\ x \longrightarrow \partial_\xi \end{cases}$$

$$y(z,x) \longrightarrow y^{\wedge}(z,\xi) \equiv \frac{1}{\omega}\, b\!\left(z - \frac{\xi}{\omega}\right)$$

Eq^l: ${}^{\wedge}y$ has "rigid" singular points : $\zeta = \omega x$ ($\omega \in \mathbb{Z}$ typically)

${}^{\wedge}y$ has "flexible" residues : $-\hat{b}(\omega x)$

Coeq^l: $y^{\wedge}$ has "flexible" singular points : $\xi = \omega(z - v_i)$

$y^{\wedge}$ has "rigid" residues : e.g. simple poles.

Coequational resurgence. Heuristics II.

Time-indep^t Schröd. Eq. in dim 1. $\boxed{\hbar^2 \partial_q^2 \Psi = W(q)\Psi}$ *

$$W(q) = q^\nu + \alpha_1 q^{\nu-1} + \dots + \alpha_\nu \;;\; E = -\alpha_\nu = \text{energy} \;;\; \hbar = \frac{2}{x} = \text{Planck's Cst.}$$

$$\Psi(z,x) = C_+(x)\, e^{\frac{xz}{2}} \sqrt{q'(z)}\, \varphi_+(z,x) + C_-(x)\, e^{-\frac{xz}{2}} \sqrt{q'(z)}\, \varphi_-(z,x)$$

with $z = z(q) = \int_0^q \sqrt{W(q')}\, dq'$; $H(z) = \frac{1}{2} q''(z)/q'(z)$

$$\boxed{\partial_z^2 \varphi_\pm \pm x \partial_z \varphi_\pm = \left[H^2(z) - H'(z)\right] \varphi_\mp} \quad \begin{matrix} \in \mathbb{C}[[z^{-\frac{2}{2+\nu}}]] \\ \in \mathbb{C}[[x^{-1}]] \end{matrix}$$

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z - resurgence:

BE$_1$ $\boxed{\overset{(z)}{\Delta}_{\pm x_i} \varphi_\pm = S_i(x)\, \varphi_\mp}$ $\left\{ \begin{array}{l} i = 1, \dots, \nu+2 \\ S_i(x) \text{ entire fct of } x \\ x_i \text{ over } x \text{ in } \mathbb{C}. \end{array} \right.$

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x - resurgence

BE$_2$ $\boxed{\overset{(x)}{\Delta}_{\pm z \pm \omega_j} \varphi_\pm = P_{j\pm}(x)\, \varphi_\mp}$ $(j = 1, \dots, \nu)$

with $P_{j\pm}(x)$ rational functions of $V_1(x), \dots, V_\nu(x)$ with $\prod_{j=1}^{\nu} V_j(x) = 1$

BE$_3$
$$\boxed{\begin{array}{ll} \overset{(x)}{\Delta}_{n(\omega_i - \omega_j)} V_k = 0 & (\text{if } k \neq i, j) \\ \overset{(x)}{\Delta}_{\pm n\omega_i \mp n\omega_j} V_i = \pm \frac{1}{n} \left[- \frac{V_{i+1} V_{i+2} \dots V_{j-1}}{V_{j+1} V_{j+2} \dots V_{i-1}} \right]^n & (i \neq j) \end{array}}$$

and $\omega_i = \int_{\gamma_i} \sqrt{W(q')}\, dq'$

In **BE$_1$** : rigid Ω (x_i over x) but flexible resurgence

In **BE$_2$**, **BE$_3$** : flexible Ω ($\omega_i = \omega_i(W)$) but rigid resurgence.

Coequational resurgence. Heuristics III.

($\mathbf{z}$ = variable $\sim \infty$; $\mathbf{x}$ = covariable $\sim \infty$)

A singular ODE whose coefficients (for simplicity) are:
(i) holomorphic at infinity
(ii) meromorphic in the large with simple poles $\frac{\beta_i}{z-\alpha_i}$
leads to solutions $y(z,x)$ expandable in series of elementary monomials $\mathcal{W}^{\bullet}(z)$ defined by the induction:

$$\left[\partial_z + x\omega_1 + \dots + x\omega_r\right] \mathcal{W}^{\binom{x\omega_1 \dots x\omega_r}{\alpha_1 \dots \alpha_r}}(z) \equiv -\mathcal{W}^{\binom{x\omega_1 \dots x\omega_{r-1}}{\alpha_1 \dots \alpha_{r-1}}} \cdot \frac{1}{z-\alpha_i}$$

These expansions provide a full description of the resurgence and differential properties in the variable $\mathbf{z}$.

To go over to the covariable $\mathbf{x}$, the challenge is to rewrite the $\mathcal{W}^{\bullet}(z)$ in terms of the familiar $\mathcal{V}^{\bullet}(x)$ so as to make explicit the resurgence and differential properties in $\mathbf{x}$:

$$\mathcal{W}^{\binom{x\omega_1 \dots x\omega_r}{\alpha_1 \dots \alpha_r}}(z) = \sum \gamma(\cdot)\, \mathcal{V}^{\binom{u_1 \dots u_r}{v_1 \dots v_r}}(x)$$

This rewriting will involve two completely different sets of parameters:
(*) the $u_i := \omega_i$ and their sums $u_i + u_{i+1} + \dots + u_j$
(*) the $v_i := z - \alpha_i$ and their differences $v_i - v_j$.

Both will constantly interact under a set of specific operations (the so-called "flexions") and under preservation of $\sum u_i v_i$ and $\sum du_i \wedge dv_i$

Bimoulds

$$M^\bullet = \left\{ M^{w_1,\dots,w_r} = M^{\binom{u_1 \;\dots\; u_r}{v_1 \;\dots\; v_r}} \right\}$$

For fixed length $\underline{\underline{r}}$: $\begin{cases} \text{"short" notation } (w_1,\dots,w_r) \\ \text{"long" notation } ([w_0], w_1,\dots,w_r) \end{cases}$

with the condition $u_0 + \dots + u_r \equiv 0$ and ...
the dual condition that all v_j be defined up to a common additive constant

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Basic unary operations: swap // pus // push.

* **swap:** $(\mathrm{swap}\, A)^{\binom{u_1 \dots u_r}{v_1 \dots v_r}} = A^{\binom{v_r - v_{r-1} \;\dots\; v_2 - v_1,\; v_1}{u_{12\dots r} \;\dots\; u_{12},\; u_1}}$

* **pus:** circular permutation on the "short" notation
* **push:** circular permutation on the "long" notation

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swap: involution ; pus: order $\underline{\underline{r}}$; push: order $r+1$

Basic symmetries:

$A^\bullet$ bialternal : $A^\bullet$ and swap $A^\bullet$ alternal

$A^\bullet$ bisymmetral : $A^\bullet$ and swap $A^\bullet$ symmetral.

Prop: $\{ A^\bullet \text{ bialternal} \} \Longleftrightarrow$

$$\left\{ \begin{array}{l} A^\bullet \text{ \underline{even} (globally in } w) \\ A^\bullet \text{ \underline{\underline{push}}-invariant} \\ A^\bullet \text{ and swap } A^\bullet \text{ \underline{\underline{pus}}-variant (i.e. with zero \underline{\underline{pus}}-average)} \end{array} \right\}$$

The scramble transform.

15

scramble: $A^\bullet \longrightarrow \mathrm{scram}\,A^\bullet$ (turns bimould//mould into bimould)

with
$$\begin{cases} (\mathrm{scram}\,A)^{w} = \sum_{1\le i\le r(w)!!} \epsilon_i\, A^{w^i} \\ \epsilon_i \in \{1,-1\} \;;\; \text{sequence } w^i \text{ flexion of sequence } w \end{cases}$$

$$(\mathrm{scram}\,A)^{\binom{u_1}{v_1}} := A^{\binom{u_1}{v_1}}$$

$$(\mathrm{scram}\,A)^{\binom{u_1,u_2}{v_1,v_2}} := A^{\binom{u_1,\;u_2}{v_1,\;v_2}} + A^{\binom{u_{12}\;,\;u_1}{v_2\;,\;v_1-v_2}} - A^{\binom{u_{12}\;,\;u_2}{v_1\;,\;v_2-v_1}}$$

$$(\mathrm{scram}\,A)^{\binom{u_1,u_2,u_3}{v_1,v_2,v_3}} := +A^{\binom{u_1\;,\;u_2\;,\;u_3}{v_1\;,\;v_2\;,\;v_3}}$$
$$+A^{\binom{u_{12}\;,\;u_1\;,\;u_3}{v_2\;,\;v_1-v_2\;,\;v_3}} - A^{\binom{u_{123}\;,\;u_{23}\;,\;u_2}{v_3\;,\;v_3-v_1\;,\;v_2-v_3}} + \dots$$
(15 terms altogether)

Remark: when applied to a mould $A^\bullet$, the scramble is defined as above, but with $u_i'\,v_i'$ in place of $\binom{u_i'}{v_i'}$ on the right hand side.

Prop 1: The scramble turns a symmetral//alternal $A^\bullet$ into a symmetral//alternal $\mathrm{scram}\,A^\bullet$

Prop 2: The scramble turns a symmetral//alternal $A^\bullet$ that is even in each w_i into a bisymmetral//bialternal $\mathrm{scram}\,A^\bullet$

Remark: There exist scores of alternative definitions//characterisations for the scramble transform. But underlying all these, there is an important prelie product: **preari**, leading to a Lie bracket **ari** and a group law **gari**.
ari//gari preserves alternal//symmetral and bialternal//bisymmetral

Coequational monomials // monics.

Prop: $$\boxed{\mathcal{W}^{\bullet}(z) \equiv \mathrm{scram}\, \mathcal{V}^{\bullet}(x)} \qquad (*)$$

More precisely: $\mathcal{W}^{\left(\cdots\ x\,\omega_i\ \cdots \atop \cdots\ \ \alpha_i\ \ \cdots\right)}(z) = \mathrm{scram}\, \mathcal{V}^{\left(\cdots u_i \cdots \atop \cdots v_i \cdots\right)}(x)$ with $\begin{cases} u_i = \omega_i \\ v_i = z - \alpha_i \end{cases}$

The variable z moves up; the covariable x moves down.

Proof: Plug $\left\{\partial_{v_i} \mathcal{V}^{v_1,\dots,v_r}(x) = \dots\right\}$ into $(*)$ to get $\left\{\partial_z \mathcal{W}^{\bullet}(z) = \dots\right\}$

Complete system of coequational monomials // monics.

$$\begin{cases} \mathcal{S}^{\bullet}(x) := \mathrm{scram}\, \mathcal{V}^{\bullet}(x) \in \text{\underline{symmetral}} \\ \mathcal{T}^{\bullet}(x) := \mathcal{S}^{\bullet}(x) \times I^{\bullet} \times \mathrm{inv}\, \mathcal{S}^{\bullet}(x) \in \text{\underline{alternal}} \end{cases} \Bigg\| \begin{matrix} \text{monomials} \\ (\text{"flexible"}) \end{matrix}$$

$$\begin{cases} \mathrm{tes}^{\bullet} := \mathrm{scram}\, \mathrm{V}^{\bullet} \in \text{\underline{alternal}} \\ \mathrm{tez}^{\bullet} := \text{similar} \in \text{\underline{alternal}} \end{cases} \Bigg\| \begin{matrix} \text{monics} \\ (\text{"rigid", in fact: locally const}) \end{matrix}$$

Overall closure under alien derivation.

$$(**)\quad \Delta_\omega \mathcal{S}^{\bullet}(x) = \left(\mathrm{tes}^{\bullet} \circ_\omega \mathcal{T}^{\bullet}(x)\right) \times \mathcal{S}^{\bullet}(x)$$

$$(**)\quad \Delta_\omega \mathcal{T}^{\bullet}(x) = \mathrm{tez}^{\bullet} \circ_\omega \mathcal{T}^{\bullet}(z)$$

nonlinear in $\mathcal{T}^{\bullet}$

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equational	monomials // monics.	coequational
$\mathcal{V}^{\bullet}(z) // \mathrm{V}^{\bullet}$	flexible // rigid	$\mathcal{S}^{\bullet}(x), \mathcal{T}^{\bullet}(x) // \mathrm{tes}^{\bullet}, \mathrm{tez}^{\bullet}$
$\mathcal{V}^{\bullet}(z) // \mathrm{V}^{\bullet}$	symmetral // alternal	$\mathcal{S}^{\bullet}(x) // \mathcal{T}^{\bullet}(x), \mathrm{tes}^{\bullet}, \mathrm{tez}^{\bullet}$
nothing	regular // singular	$\mathcal{S}^{\bullet}, \mathcal{T}^{\bullet}, \mathrm{tes}^{\bullet}, \mathrm{tez}^{\bullet} // \mathcal{S}^{\bullet}, \mathcal{C}^{\bullet}, \mathrm{tes}^{\bullet}, \mathrm{tez}^{\bullet}$

The tesselation coefficients : definitions

Definition 1: $\mathbf{tes}^{\bullet} := \mathbf{scram}\, \mathbf{V}^{\bullet}$

with $\mathbf{V}^{\omega_1,\dots,\omega_r} := \int_0^{\omega_1+\dots\omega_r} \frac{dt_{r-1}}{\omega_1+\dots+\omega_r-t_{r-1}} \cdots \int_0^{t_3} \frac{dt_2}{\omega_1+\omega_2-t_2} \int_0^{t_2} \frac{dt_1}{\omega_1-t_1}$

Thus: $\mathbf{tes}^{w_1} \equiv 1$; $\mathbf{tes}^{w_1,w_2} \equiv \mathbf{V}^{u_1 v_1, u_2 v_2} + \mathbf{V}^{u_{12} v_2, u_1(v_1-v_2)} - \mathbf{V}^{u_{12} v_1, u_2(v_2-v_1)}$

$\Longrightarrow$ local constancy ; global non constancy ; jump rules .

Definition 2: $\boxed{\mathbf{D}_{i,j}\, \mathbf{tes}^{\mathbf{w}} = \mathbf{tes}^{\mathbf{w}^*}\, \mathbf{tes}^{\mathbf{w}^{**}}}$ $\begin{cases} i-j \not\equiv 0,1 \\ i,j \in \mathbb{Z}_{r+1} \end{cases}$

crossing of the hypersurface $\Im(H_{i,j}(\mathbf{w})) = 0$

with $H_{i,j}(\mathbf{w}) = \Big(\sum_{circ(j\le q<i)} u_q (v_q - v_j)\Big) \Big/ \Big(\sum_{circ(i\le q<j)} u_q (v_q - v_j)\Big)$

Definition 3: "elementary" in the sense of involving only (± 1)

Elementary induction formula:

$$\text{tes}^{\mathbf{w}} = \sum_{0\le n\le r(\mathbf{w})} \text{push}^n \sum_{\mathbf{w}'\mathbf{w}''=\mathbf{w}} \text{sig}^{\mathbf{w}'\mathbf{w}''}\, \text{tes}^{\mathbf{w}^*}\, \text{tes}^{\mathbf{w}^{**}}$$

The notations are as follows.
We fix $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and set $\Re_\theta : z \in \mathbb{C} \mapsto \Re(e^{i\theta} z) \in \mathbb{R}$. Then we define:

$$f_{\mathbf{w}}^{\mathbf{w}'} := <\mathbf{u}', \mathbf{v}'><\mathbf{u}, \mathbf{v}>^{-1} \quad , \quad g_{\mathbf{w}}^{\mathbf{w}'} := <\mathbf{u}', \Re_\theta \mathbf{v}'><\mathbf{u}, \Re_\theta \mathbf{v}>^{-1}$$
$$f_{\mathbf{w}}^{\mathbf{w}''} := <\mathbf{u}'', \mathbf{v}''><\mathbf{u}, \mathbf{v}>^{-1} \quad , \quad g_{\mathbf{w}}^{\mathbf{w}''} := <\mathbf{u}'', \Re_\theta \mathbf{v}''><\mathbf{u}, \Re_\theta \mathbf{v}>^{-1}$$

From these scalars we construct the crucial sign factor *sig* which takes its values in $\{-1, 0, 1\}$. Here, the abbreviation *si*(.) stands for *sign*($\Im$(.)).

$$\begin{aligned} \text{sig}^{\mathbf{w}',\mathbf{w}''} = \text{sig}_\theta^{\mathbf{w}',\mathbf{w}''} := \frac{1}{8} &\Big(\text{si}(f_{\mathbf{w}}^{\mathbf{w}'} - f_{\mathbf{w}}^{\mathbf{w}''}) - \text{si}(g_{\mathbf{w}}^{\mathbf{w}'} - g_{\mathbf{w}}^{\mathbf{w}''})\Big) \times \\ &\Big(1 + \text{si}(f_{\mathbf{w}}^{\mathbf{w}'}/g_{\mathbf{w}}^{\mathbf{w}'})\ \text{si}(f_{\mathbf{w}}^{\mathbf{w}'} - g_{\mathbf{w}}^{\mathbf{w}'})\Big) \times \\ &\Big(1 + \text{si}(f_{\mathbf{w}}^{\mathbf{w}''}/g_{\mathbf{w}}^{\mathbf{w}''})\ \text{si}(f_{\mathbf{w}}^{\mathbf{w}''} - g_{\mathbf{w}}^{\mathbf{w}''})\Big) \end{aligned}$$

Lastly, the pair $(\mathbf{w}^*, \mathbf{w}^{**})$ is constructed from the pair $(\mathbf{w}', \mathbf{w}'')$ according to:

$$\mathbf{u}^* := \mathbf{u}' \quad , \quad \mathbf{v}^* := \mathbf{v}'\ <\mathbf{u},\mathbf{v}>^{-1}\ \Im g_{\mathbf{w}}^{\mathbf{w}'} \ -\ \Re_\theta \mathbf{v}'\ <\mathbf{u}, \Re_\theta \mathbf{v}>^{-1}\ \Im g_{\mathbf{w}}^{\mathbf{w}'}$$
$$\mathbf{u}^{**} := \mathbf{u}'' \quad , \quad \mathbf{v}^{**} := \mathbf{v}''\ <\mathbf{u},\mathbf{v}>^{-1}\ \Im g_{\mathbf{w}}^{\mathbf{w}''} \ -\ \Re_\theta \mathbf{v}''\ <\mathbf{u}, \Re_\theta \mathbf{v}>^{-1}\ \Im g_{\mathbf{w}}^{\mathbf{w}''}$$

 The tesselation coefficients: **10 key properties**.

P_1 $\mathbf{tes}^{\binom{u_1 \ldots u_r}{v_1 \ldots v_r}} \equiv \mathbf{tes}^{\binom{a u_1 \ldots a u_r}{b v_1 \ldots b v_r}}$ $\forall a,b \in \mathbb{C}^2$ || $\mathbf{tes}^\bullet$ lives on the bi-projective $\mathbb{C}^{2r}$

P_2 the bimould $\mathbf{tes}^\bullet$ is bialternal (i.e. alt. and with alt. swappee)

P_3 in fact $\mathbf{swap}\ \mathbf{tes}^\bullet = \mathbf{tes}^\bullet$

P_4 $\mathbf{tes}^\bullet$ is <u>push</u>-invariant

P_5 $\mathbf{tes}^\bullet$ is <u>pus</u>-variant (i.e. of zero <u>pus</u>-average)

P_6 $\mathbf{tes}^\bullet$ is locally constant, with values in $\{-1, 0, 1\}$

P_7 for random sequences $\boldsymbol{w}$ of fixed length $\underline{r}$ (large) the probability of getting nonzero values for $\mathbf{tes}^{\boldsymbol{w}}$ is >0 but exceedingly small.

P_8 for $\underline{r}$ fixed, the three constancy domains S_-, S_0, S_+ are path-connected.

P_9 for $\underline{r}$ fixed, the hypersurfaces $\mathfrak{I}(H_{ij}(w)) \equiv 0$ <u>limit</u> but do not <u>separate</u> the sets S_-, S_0, S_+

P_{10} $\mathbf{tes}^{\boldsymbol{w}} = 0$ whenever $\boldsymbol{w}$ is "semi-real", i.e. when one of its two component sequences $\boldsymbol{u}$ or $\boldsymbol{v}$ is real

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Coequational resurgence: second // third B.E.

Equation: $$\boxed{\partial_z\, y = x \cdot y + \sum a_n(z)\, y^{n+1}}$$

$$\begin{cases} z = \text{variable} \\ x = \text{covariable} \end{cases} \qquad \begin{cases} n \in \mathbb{N}^* : \text{"unilateral case"}. \\ n \in \mathbb{N}^* \cup \{-1\} : \text{"bilateral case"}. \end{cases}$$

Formal integral: $y = y(z, x, u) \nearrow \mathbb{C}[[z^{-1}]] \otimes \mathbb{C}\{u\, e^{xz}\} \otimes \mathrm{Param}(x)$

$\qquad\qquad\qquad\qquad\qquad\quad\;\; \searrow \mathbb{C}[[x^{-1}]] \otimes \mathbb{C}\{u\, e^{xz}\} \otimes \mathrm{Param}(z)$

Equational resurgence:

$$\mathrm{BE}_1 \quad {}^{(z)}\!\Delta_\omega\, y = \mathbb{A}_\omega\, y \qquad \begin{cases} \omega \in \Omega^1 = \{-x, x, 2x, 3x, 4x \ldots\} \\ \mathbb{A}_\omega = A_\omega(x)\, u^{n+1}\partial_u \\ A_\omega(x) = \underline{\text{entire function}} \text{ of } x \end{cases}$$

Coequational resurgence:

$$\mathrm{BE}_2 \quad {}^{(x)}\!\Delta_\omega\, y = \mathbb{P}_{\omega(x)}\, y \qquad \begin{cases} \omega \in \Omega^2 = \left\{ \begin{matrix} n(z-\alpha_i), \\ n = -1, 1, 2 \ldots \end{matrix} \right\} \\ \mathbb{P}_\omega = P_{\omega(x)}\, u^{n+1}\partial_u \\ P_\omega(x) \text{ } \underline{\text{resurgent}} \text{ in } x \end{cases}$$

$$\mathrm{BE}_3 \quad \left[{}^{(x)}\!\Delta_\omega, \mathbb{P}_{\omega_0}\right] = \begin{cases} \displaystyle\sum_{\omega+\omega_0=\omega_1+\omega_2} [\mathbb{P}_{\omega_1}, \mathbb{P}_{\omega_2}] & \text{in the unilateral case} \\ \displaystyle\sum_{\substack{\omega+\omega_0= \\ \omega_1+\omega_2+\ldots+\omega_r}} \mathrm{ten}(\mathbb{P}_{\omega_1}, \ldots, \mathbb{P}_{\omega_r}) & \text{in the bilateral case.} \end{cases}$$

with $\omega \in \Omega^3 = \{ n(\alpha_i - \alpha_j)\, ;\ n = -1, 1, 2, \ldots \}$

$\mathbb{Z}$ BE_3 is nonlinear (<u>bi</u>- or <u>multi</u>-linear) even for linear eq. // syst.

$\mathbb{Z}$ Link between $\mathbb{A}_\omega(x)$ and $\mathbb{P}_\omega(x)$. "Autark" functions.

 Coequational ODE. (20)

At the monomial level: Combining the definition:

$$(*) \qquad \mathcal{W}^{\bullet}(z) \equiv \mathrm{scram}\ \mathcal{V}^{\bullet}(x)$$

with the rule:

$$(**) \qquad \boxed{(\partial_x + \eta_1 + \dots + \eta_r)\, \mathcal{V}^{\eta_1,\dots,\eta_r}_{(x)} = -\mathcal{V}^{\eta_1,\dots,\eta_{r-1}}_{(x)} \frac{1}{x}}$$

we get:

$$(***) \qquad (\partial_x + \langle\bullet\rangle)\, \mathrm{scram}\ \mathcal{V}^{\bullet}(x) = \mathrm{preari}(\mathrm{scram}\ \mathcal{V}^{\bullet}_{(x)}, I^{\bullet})$$

with $\langle w \rangle := \sum u_i v_i$; $I^{w_1} \equiv 1$; $I^{w_1,\dots,w_r} \equiv 0$ if $r \neq 1$.

At the global level: Let $\mathbf{S}$ be a singular differential system, with variable z and covariable x. Let y be its solution, but viewed as a function of the covariable x, with the corresponding "invariants" $\mathbb{P}_\omega$. Plugging (***) into the explicit x-expansion of y and $\mathbb{P}_\omega$, we get a differential "cosystem" ${}^{co}\mathbf{S}$, which completely describes the differential properties (in x) of y and $\mathbb{P}_\omega$. The "cosystem" ${}^{co}\mathbf{S}$ crucially involves the tesselation coefficients.

$\mathbf{Z}$ This applies equally to linear or nonlinear systems S. However, even when the original system S is linear or affine, the co-system ${}^{co}S$ is usually (highly) nonlinear!

Memento.

21

Equational resurgence (in the variable $z \sim \infty$)

The basic Bridge Equation **BE~BE$_1$** solves most problems pertaining to Singular local Objects, such as:

* Description * Iteration * Resummation * Conjugation

* Transcendence (via the "display")

* "Analysis" (explicitly, via the invariants and the expansion in terms of monomials)

* "Synthesis" (explicitly and canonically, via the spherical monomials)

The closely related "Drawbridge Equation" **BE$_1^*$** yields all possible obstructions to "twistless" synthesis ($c=0$)

Coequational resurgence (in the covariable $x \sim \infty$)

BE$_2$ and **BE$_3$** provide an exhaustive description of the loosely "dual" coequational resurgence, with the locally constant tesselation coefficients accounting for the "rigidity" and nonlinearity of coeq. resurgence.

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* On the arithmetical side, the eq.//coeq. duality finds its reflection in the phenomenon of arithmetical dimorphy

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* On the functional side, the eq.//coeq. duality explains the dual nature of the holomorphic invariants, which are simultaneously entire and resurgent functions of x, with "finiteness" or "closure" properties almost reminiscent of algebra

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* "Natural" versus "random" or "generic".

⇒ divergent power series of natural origin tend to be resurgent

⇒ entire functions of natural origin tend to be autark.

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* Inadequacy of differential Galois theory.