Pairing in systems with population imbalance

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Outline

- The systems with population imbalance
- The general theory of pairing with population imbalance
- BEC-BCS crossover and bound states
- Numerical examples

Population imbalance

- Conventional BCS pairs particles on a Fermi surface with opposite momenta and spins: Cooper pair wace-function is invariant under time-reversal, i. e. simultaneous exchange of momentum and spin sign.
- In systems with population imbalance the pairing occurs between particles lying on different Fermi surfaces: The Cooper pair wave function is non-invariant under time reversal.

Some examples

- Metallic superconductors with paramagnetic impurities. The effect of impurities is to induce an average slitting of Fermi-levels of spin-up and spin-down electrons. This can be described by adding a Pauli paramagnetic term to the spectrum.
- Pairing in dilute Fermi gases. Experimentally realized in fermionic isotope of Lithium, ^{6}Li. The trap can be loaded with different numbers of atoms belonging to different hyperfine states. The pairing is among atoms in different hyperfine states.
- Neutron-proton pairing in nuclear systems, under isospin asymmetry. Such asymmetries arise naturally in for nuclei away from the valley of beta-stability. Most promising site of n-p pairing is dilute nuclear matter in supernova mantel.

Further examples

- Pairing in QCD matter (deconfined phase of nucleonic matter) at moderate densities. The pairing is among the u and d quarks with some color and flavor quantum numbers. Such matter can occur in the centers of massive stellar compact objects. The matter must be in β equilibrium with respect to the process

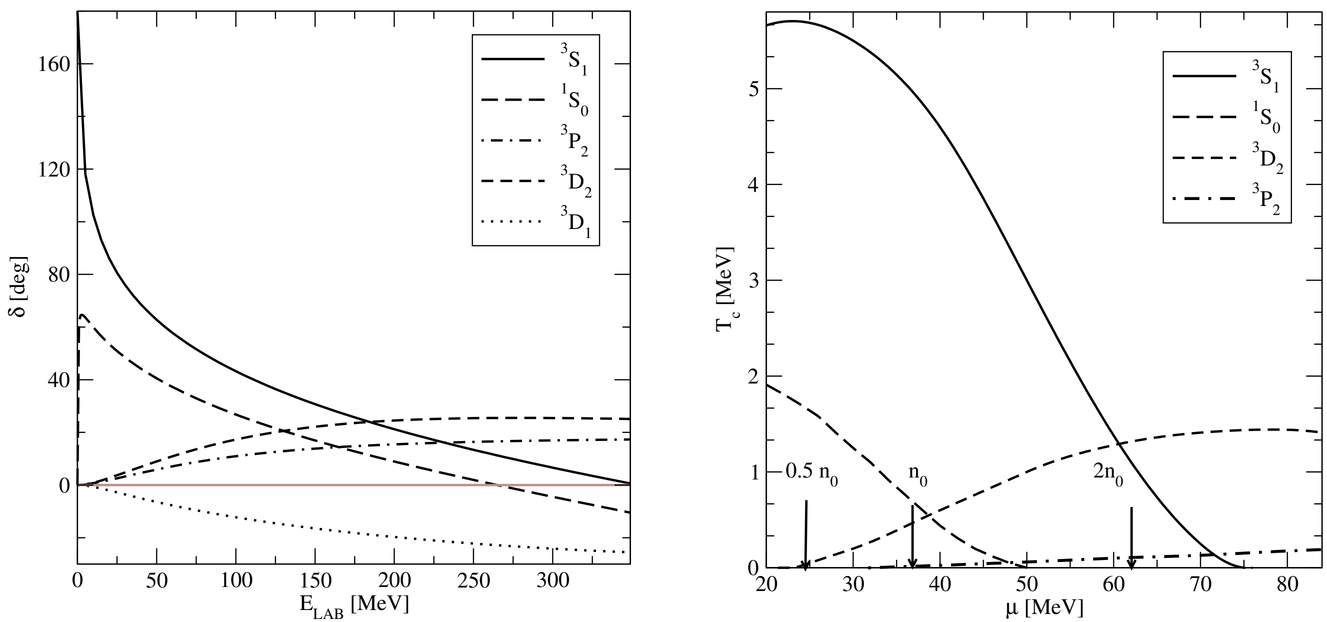
$$d \to u + e + \bar{\nu} \qquad u + e \to +d\nu$$

  This leads to asymmetry in the population of u and d quarks, with chemical potentials obeying $\mu_d = \mu_u + \mu_e$.

- *A wide class of systems, with characteristic energy scales differing by some 20 orders of magnitude, shear a common feature of pairing among imbalance populations.*

Nuclear systems

- Pairing interaction is due strong nuclear force, which is non-local and attractive at large distances, and strongly repulsive at short distances.



Dilute alkali gases

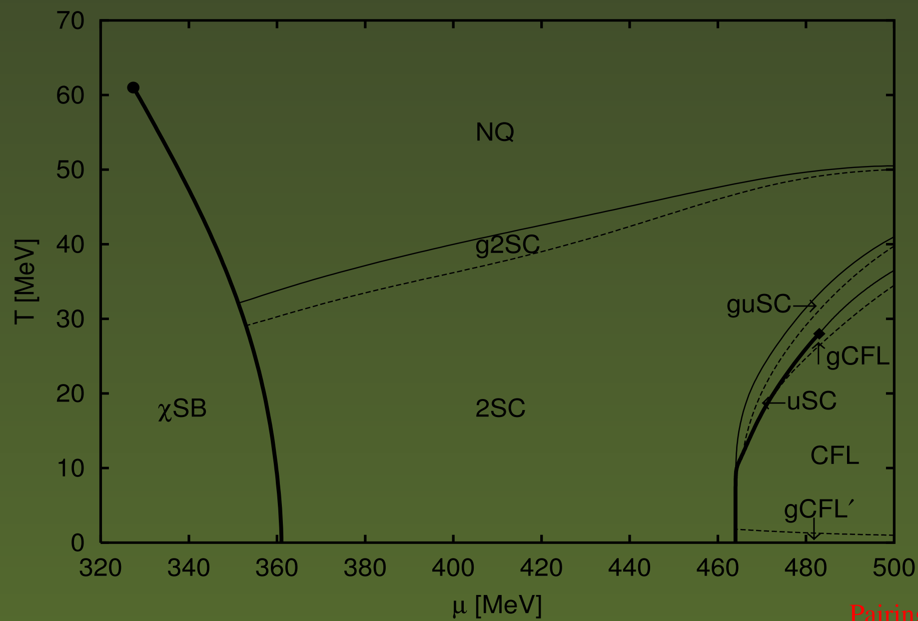
- The pairing interaction is essentially S wave and zero range (exception are dipolar gases).
- The Fermi-energies are of the order of mK; $T_c/T_F \sim 0.25$ high temperature superfluids!
- Interactions can be manipulated in the range $[-\infty, \infty]$

$$a_{\text{eff}} = \frac{a_S}{B - B_0} \quad \text{(Feshbach resonance mechanism)}$$

- Unitary limit $|a| \to \infty$ is universal
- Number of different species can be large: trapping 3 different species leads to the three-body physics (e.g. Efimov states).

Color superconductivity

- For densities relevant for compact objects, the interactions are non-perturbative. Effective models featuring contact interactions are available.
- Possible pairing channels span the 3 color $\times$ 3 flavor space; a typical ansatz $\Delta \propto \langle \psi^T(x) C\gamma_5\tau_2\lambda_2\psi(x)\rangle$.



BCS theory for systems with population imbalance

We consider a uniform gas of fermionic atoms in two hyperfine states (spins) labeled as $\uparrow$ and $\downarrow$. The Hamiltonian of the system is

$$\begin{aligned}\hat{H} &= \frac{1}{2m}\sum_{\alpha}\int d^3x \nabla\hat{\psi}^{\dagger}_{\alpha}(\mathbf{r})\nabla\hat{\psi}_{\alpha}(\mathbf{r}) \\ &- \sum_{\alpha\beta}\int d^3x\int d^3x'\hat{\psi}^{\dagger}_{\alpha}(\mathbf{r})\hat{\psi}^{\dagger}_{\beta}(\mathbf{r})V(\mathbf{r},\mathbf{r}')\hat{\psi}_{\beta}(\mathbf{r}')\hat{\psi}_{\alpha}(\mathbf{r})\end{aligned}$$

Define the GF of the system

$$i\hat{G}_{\alpha\beta}(x_1,x_2) \equiv i\begin{pmatrix} G_{\alpha\beta}(x_1,x_2) & F_{\alpha\beta}(x_1,x_2) \\ F^{\dagger}_{\alpha\beta}(x_1,x_2) & G^{\dagger}_{\alpha\beta}(x_1,x_2)\end{pmatrix}$$

DSE for fermions

$$\hat{G}_{\alpha}^{-1}(x_1)\hat{G}_{\alpha\beta}(x_1,x_2) = \hat{\mathbf{1}}\delta_{\alpha\beta}\delta(x_1-x_2) + i\sum_{\gamma}\int d^3x_3\,\hat{\Sigma}_{\alpha\gamma}(x_1,x_3)\hat{G}_{\gamma\beta}(x_3,x_2),$$

where $\hat{1}$ is a unit matrix, $G_{\alpha}^{-1}(x) \equiv i\partial/\partial t + \mathbf{\nabla}^2/2m_{\alpha} + \mu_{\alpha}$ and

$$\hat{G}_{\alpha}^{-1}(x) = \begin{pmatrix} G_{\alpha}^{-1}(x) & 0 \\ 0 & \left[G_{\alpha}^{-1}(x)\right]^* \end{pmatrix},$$

$$\hat{\Sigma}_{\alpha\beta}(x_1,x_2) \equiv \begin{pmatrix} \Sigma_{\alpha\beta}(x_1,x_2) & \Delta_{\alpha\beta}(x_1,x_2) \\ \Delta^{\dagger}_{\alpha\beta}(x_1,x_2) & \Sigma^{\dagger}_{\alpha\beta}(x_1,x_2) \end{pmatrix},$$

The self-energy matrix is defined as

$$\Delta_{\alpha\beta}(x_1, x_2) = \sum_{\gamma\kappa} \int \Gamma_{\alpha\beta\gamma\kappa}(x_1, x_2; x_3, x_4) F_{\gamma\kappa}(x_3, x_4) dx_3 dx_4.$$

The counterparts of the equations above are obtained by going over to the center of mass $X = (x_1 + x_2)/2$ and relative $x = x_1 - x_2$ coordinates in the two-point functions and carrying a Fourier transform with respect to the relative coordinates: $\hat{G}(x, X) \rightarrow \hat{G}(\omega, \vec{p}, \mathbf{R}, T)$, where $\omega, \vec{p}$ are the relative frequency and momentum, and $X \equiv (\mathbf{R}, T)$.

Quasiclassical functions

The Dyson equation for the quasiclassical functions is

$$\sum_\gamma \begin{pmatrix} \omega - \epsilon^+_{\alpha\gamma} & -\Delta_{\alpha\gamma} \\ -\Delta^\dagger_{\alpha\gamma} & \omega + \epsilon^-_{\alpha\gamma} \end{pmatrix} \begin{pmatrix} G_{\gamma\beta} & F_{\gamma\beta} \\ F^\dagger_{\gamma\beta} & G^\dagger_{\gamma\beta} \end{pmatrix} = \delta_{\alpha\beta}\hat{\mathbf{1}}, \tag{1}$$

where

$$\epsilon^\pm_{\alpha\beta} = \left(\mathbf{P}/2 \pm \vec{p}\right)^2/2m_\alpha - \mu_\alpha \pm \mathrm{Re}\,\Sigma_{\alpha\beta} - \mathrm{Im}\,\Sigma_{\alpha\beta}, \tag{2}$$

$\mathrm{Re}\,\Sigma_{\alpha\beta} \equiv (\Sigma_{\alpha\beta} - \Sigma^\dagger_{\alpha\beta})/2$, $\mathrm{Im}\,\Sigma_{\alpha\beta} \equiv (\Sigma_{\alpha\beta} + \Sigma^\dagger_{\alpha\beta})/2$;

Propagators and self-energies are functions of $\omega, \vec{p}$ and $\mathbf{P}$.

The quasiparticle excitation spectrum is determined in the standard fashion by finding the poles of the propagators in Eq. (**??**):

$$\omega_{\pm\pm} = \epsilon_A \pm \sqrt{\epsilon_S + \frac{1}{2}\mathrm{Tr}(\mathbf{\Delta}\mathbf{\Delta}^\dagger) \pm \frac{1}{2}\sqrt{[\mathrm{Tr}(\mathbf{\Delta}\mathbf{\Delta}^\dagger)]^2 - 4\mathrm{Det}\,(\mathbf{\Delta}\mathbf{\Delta}^\dagger)}}.$$

Here $\mathbf{\Delta} \equiv \Delta_{\alpha\beta}$, $\epsilon_S = (\epsilon^+ + \epsilon^-)/2$, and $\epsilon_A = (\epsilon^+ - \epsilon^-)/2$.
Approximation - equal spin pairing is unimportant -

$$\Delta_{\uparrow\uparrow}, \Delta_{\downarrow\downarrow} \ll \Delta_{\uparrow\downarrow}$$

The spectrum, in this case, simplifies to

$$\omega_{\uparrow\downarrow} = \epsilon_A \pm \sqrt{\epsilon_S^2 + |\Delta|^2},$$

The symmetric and asymmetric parts of the spectrum (which are even and odd with respect to the time-reversal symmetry) are defined as

$$\epsilon_S \equiv P^2/8m + p^2/2m + \mathrm{Re}\,\Sigma - \mu, \quad \epsilon_A \equiv \mathbf{P}\cdot\vec{p}/2m + \mathrm{Im}\,\Sigma - \delta\mu.$$

where

$$\mu = (\mu_\uparrow + \mu_\downarrow)/2 \quad \delta\mu = (\mu_\uparrow - \mu_\downarrow)/2$$

and

$$\mathrm{Re}\,\Sigma \equiv (\Sigma_{\uparrow\uparrow} - \Sigma^\dagger_{\downarrow\downarrow}) \qquad \mathrm{Im}\,\Sigma \equiv (\Sigma_{\uparrow\uparrow} + \Sigma^\dagger_{\downarrow\downarrow})/2$$

The limit $\epsilon_A \to 0$ corresponds to the BCS pairing in the spin symmetric matter. It is explicit now that the spectrum is twofold split due to three factors,

- the spin asymmetry ($\delta\mu \neq 0$)
- the finite-momentum of the Cooper pair ($\mathbf{P} \neq 0$)
- the finite lifetime of the quasiparticle ($\mathrm{Im}\Sigma \neq 0$).

The solution of the QC Dyson equation is

$$
\begin{aligned}
G_{\uparrow/\downarrow p} &= \frac{u_p^2}{\omega - \omega_{+/-} + i\eta} + \frac{v_p^2}{\omega - \omega_{-/+} + i\eta}, \\
F &= u_p v_p \left(\frac{1}{\omega - \omega_+ + i\eta} - \frac{1}{\omega - \omega_- + i\eta} \right),
\end{aligned}
$$

- The Bogolyubov amplitudes are

$$u_p^2 = \frac{1}{2} + \frac{\epsilon_S}{2\sqrt{\epsilon_S^2 + |\Delta|^2}}, \quad v_p^2 = \frac{1}{2} - \frac{\epsilon_S}{2\sqrt{\epsilon_S^2 + |\Delta|^2}}. \tag{3}$$

- BCS (Hartree) approximation: $\Gamma_{\alpha\beta\gamma\kappa} \to V_{\alpha\gamma}\delta_{\alpha\beta}\delta_{\gamma\kappa}$.
- Pairing mean field

$$\Delta(p, P) = \int \frac{dp' p'^2}{(2\pi)^2} \frac{V(p, p')\Delta(p', P)}{2\sqrt{\epsilon_S^2 + \Delta(p', P)^2}} \langle [f(\omega_+) - f(\omega_-)] \rangle$$

- Densities

$$\rho_{n/p}(P) = \int \frac{d^3 p}{(2\pi)^3} \left\{ u_p^2 f(\omega_\pm) + v_p^2 f(\omega_\mp) \right\}$$

Zero-range forces

We introduce a momentum renormalization scale Λ such that $\Delta \ll \varepsilon_\Lambda \ll \min[\varepsilon_{F\uparrow},\ \varepsilon_{F\downarrow}]$

$$\Delta^\dagger(p) = i \int U(\vec{p},\vec{p}')G_\downarrow^N(-p')\Delta^\dagger(p')G_\uparrow(p')\theta(\Lambda - |\vec{p}'|)\frac{d^4p'}{(2\pi)^4},$$

$$U(\vec{p},\vec{p}') = V(\vec{p},\vec{p}') + i \int V(\vec{p},\vec{p}'')G_\downarrow^N(-p'')G_\uparrow^N(p'')U(\vec{p}'',\vec{p}')\theta(|\vec{p}''| - \Lambda$$

- In the second equation the full propagator $G_\uparrow(p)$ was replaced by its counterpart in the unpaired state, $G_\uparrow^N(p)$.
- By construction, the on-shell integration is carried over momenta much larger than the Fermi momentum.
- The two equations decouple

Next write down the T matrix

$$T(\vec{p},\vec{p}') = V(\vec{p},\vec{p}') + i\int_0^{\infty} V(\vec{p},\vec{p}'')G^N_{\downarrow}(-p'')G^N_{\uparrow}(p'')T(\vec{p}'',\vec{p}')\frac{d^4p''}{(2\pi)^4}.$$

Combining Eqs. one finds a regular integral equation defining the pairing force

$$U(\vec{p},\vec{p}') = T(\vec{p},\vec{p}') - i\int_0^{\Lambda} U(\vec{p},\vec{p}'')G^N_{\downarrow}(-p'')G^N_{\uparrow}(p'')T(\vec{p}'',\vec{p}')\frac{d^4p''}{(2\pi)^4}.$$

In the dilute limit of interest, partial waves higher than the s-wave can be neglected, and the interaction is solely determined by the s wave scattering length $a_S < 0$, as $T(\vec{p},\vec{p}') = T_0 = 4\pi|a_S|/m$. The solution is straightforward:

The pairing interaction in the case of zero-range forces

$$U_0 = T_0 \left[1 - \int G^N_\downarrow(-p) G^N_\uparrow(p) \theta(\Lambda - |\vec{p}|) \frac{d^4p}{(2\pi)^4} \right]^{-1} .$$

For the zero-range interaction above the gap equation takes the form

$$\begin{aligned} \Delta^\dagger(\vec{p}) &= U_0 \int \mathrm{Im} F^\dagger(\vec{p}', \omega') f(\omega') \theta(\Lambda - |\vec{p}'|) \frac{d^3p' d\omega'}{(2\pi)^4} \\ &= \frac{U_0}{2} \int_0^\Lambda \frac{\Delta}{\sqrt{E_S^2 + \Delta^2}} \langle f(\omega_1) - f(\omega_2) \rangle \frac{p'^2 dp'}{(2\pi)^2}, \end{aligned}$$

where $\langle \ldots \rangle$ stands for the angle average and the second line follows by retaining only the pole part.

The normal self-energy in the T-matrix (ladder) approximation is defined as

$$\Sigma_\sigma(\vec{p}) \;=\; i\sum_{\sigma'}\int T_{\sigma\sigma'}(\vec{p},\vec{p}';\varepsilon_{\vec{p}}+\omega')G_{\sigma'}(p')\frac{d^4p'}{(2\pi)^4},$$

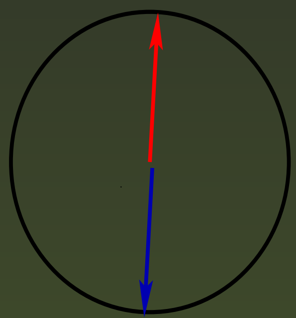
If the $T \simeq T_0$ the normal self-energy is momentum independent, purely real, and is given by

$$\Sigma_{\uparrow/\downarrow} = T_0\rho_{\downarrow/\uparrow}. \tag{6}$$

This constant shift in quasiparticle energy can be absorbed in the chemical potential $\mu^*_{\uparrow/\downarrow} = \mu_{\uparrow/\downarrow} - \Sigma_{\downarrow/\uparrow}$.

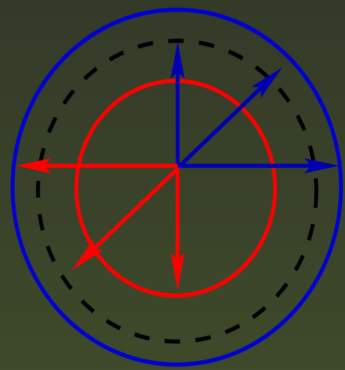
Realizations of superconducting phases with two species

BCS: $k = -k, \delta\mu = 0$



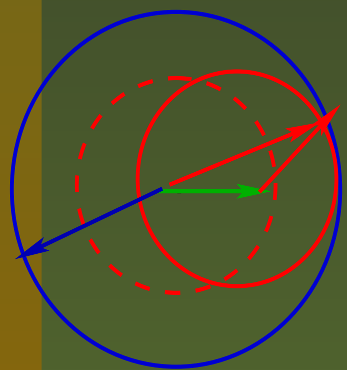
rotational/transl. symmety

ASYMMETRIC BCS: $k = -k, \quad \delta\mu \neq 0$



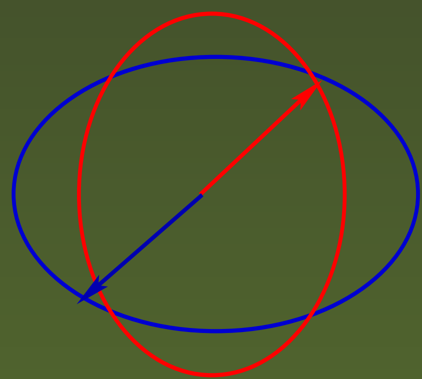
rotational/symmetry, time reversal broken

LOFF: $\quad k + P = -k', \quad \delta\mu \neq 0$



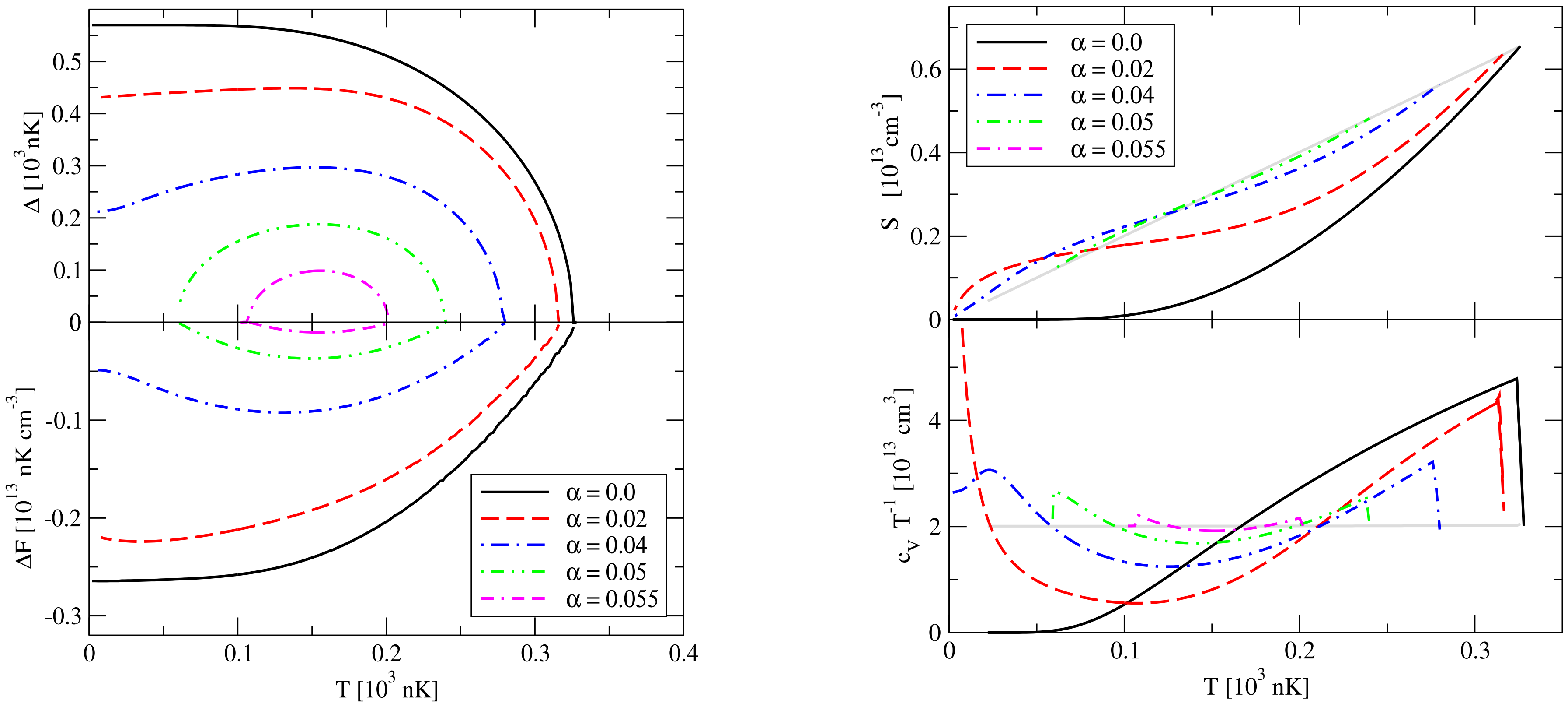
rotational/trans sym. broken

DFS phase: $k \sim k', \quad \delta\mu \neq 0$

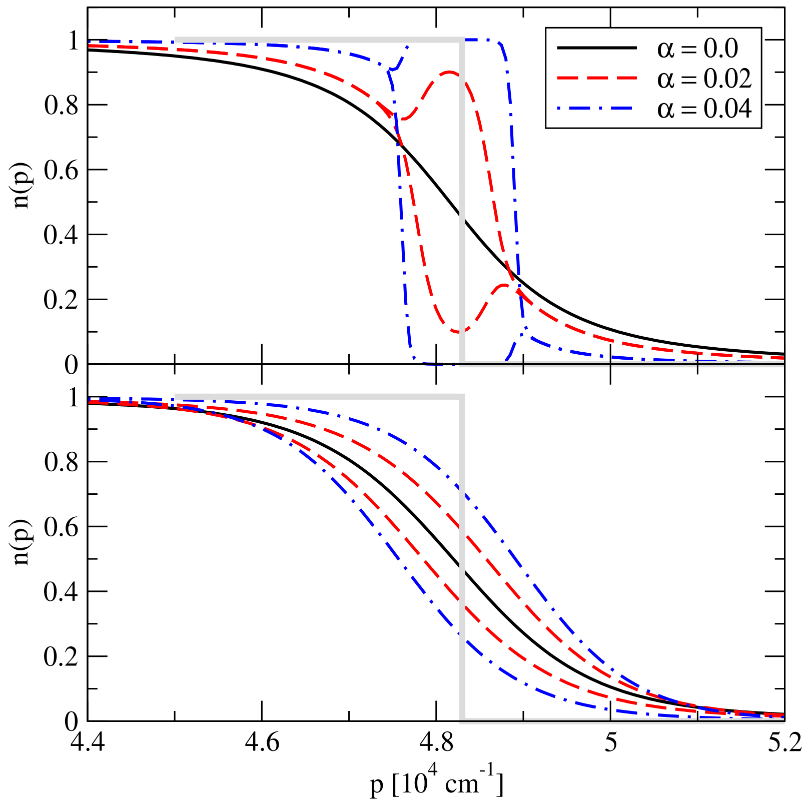


only rotational symmetry is broken to O(2)

Homogeneous imbalanced superfluids



Occupation numbers



- *the homogeneous phase is unstable in the weak coupling limit!*

Some analytical results

Consider the gap equation

$$1 = G \int \frac{d^3p}{(2\pi)^3} \frac{1 - 2f(E_p)}{2E_p}$$

The momentum integration is modified as follows

$$d^3p \to p^2 dp d\Omega \to p^2 dp d\sin\theta d\phi \to m p_F d(p^2/2m) d\Omega \to m p_F d\varepsilon_p d\Omega.$$

Where we assume that the system is degenerate and the momenta are restricted to the vicinity of the Fermi sphere. Then,

$$1 = G\nu(\mu) \int_0^\Lambda \frac{d\varepsilon_p}{2\sqrt{\varepsilon_p^2 + \Delta^2}} \tanh\left(\frac{\sqrt{\varepsilon_p^2 + \Delta^2}}{2T}\right), \quad \nu(p_F) = \frac{p_F m}{\pi^2 \hbar^3}$$

In the limit $T \to 0$, the $\tanh$ function is unity and

$$1 = \frac{G\nu(\mu)}{2} \int_0^{\Lambda} \frac{d\varepsilon_p}{\sqrt{\varepsilon_p^2 + \Delta^2}}$$

The integral is computed by defining a new variable ϕ

$$\varepsilon_p = \Delta \sinh \phi, \quad d\varepsilon_p = \Delta \cosh \phi d\phi,$$

then

$$1 = \frac{G\nu(\mu)}{2} \int_0^{\phi_0} \frac{\Delta \cosh \phi d\phi}{\Delta \sqrt{\sinh^2 \phi + 1}} = \frac{G\nu(\mu)}{2} \int_0^{\phi_0} d\phi = \frac{G\nu(\mu)}{2} \phi_0$$

where $\phi_0 = \operatorname{arcsinh}(\Lambda/\Delta)$.

Next we consider the case of weak coupling, i.e. $\Delta \ll \Lambda$ and expand the arcsinh function for large arguments

$$\lim_{x\to\infty} \operatorname{arcsinh} x = \ln(2x) + O(x^2),$$

so that we obtain

$$1 = \frac{G\nu(\mu)}{2} \ln\left(\frac{2\Lambda}{\Delta}\right)$$

By analogy

$$1 = \frac{G\nu(\mu)}{2} \int_0^\Lambda \frac{d\varepsilon_p}{\sqrt{\varepsilon_p^2 + \Delta^2}} \Big[1 - \theta(\sqrt{\varepsilon_p^2 + \Delta^2} + \delta\mu) - \theta(\sqrt{\varepsilon_p^2 + \Delta^2} - \delta\mu$$

$$\ln\frac{\Delta_0(0)}{\Delta_0} = \theta[\delta\mu - \Delta_0(0)]\mathrm{arcsinh}\frac{\sqrt{\delta\mu^2 - \Delta_0^2}}{\Delta_0}\,,$$
$$\ln\frac{\Delta_0(0)}{\delta\mu + \sqrt{\delta\mu^2 - \Delta_0^2}} = 0\,.$$

There are no solutions for $\delta\mu > \Delta_0(0)$. For $\delta\mu \leq \Delta_0(0)$ one has two solutions.

$$\begin{aligned} a) \quad \Delta_0 &= \Delta_0(0)\,, \\ b) \quad \left[\Delta_0\right]^2 &= 2\,\delta\mu\Delta_0(0) - \left[\Delta_0(0)\right]^2. \end{aligned}$$

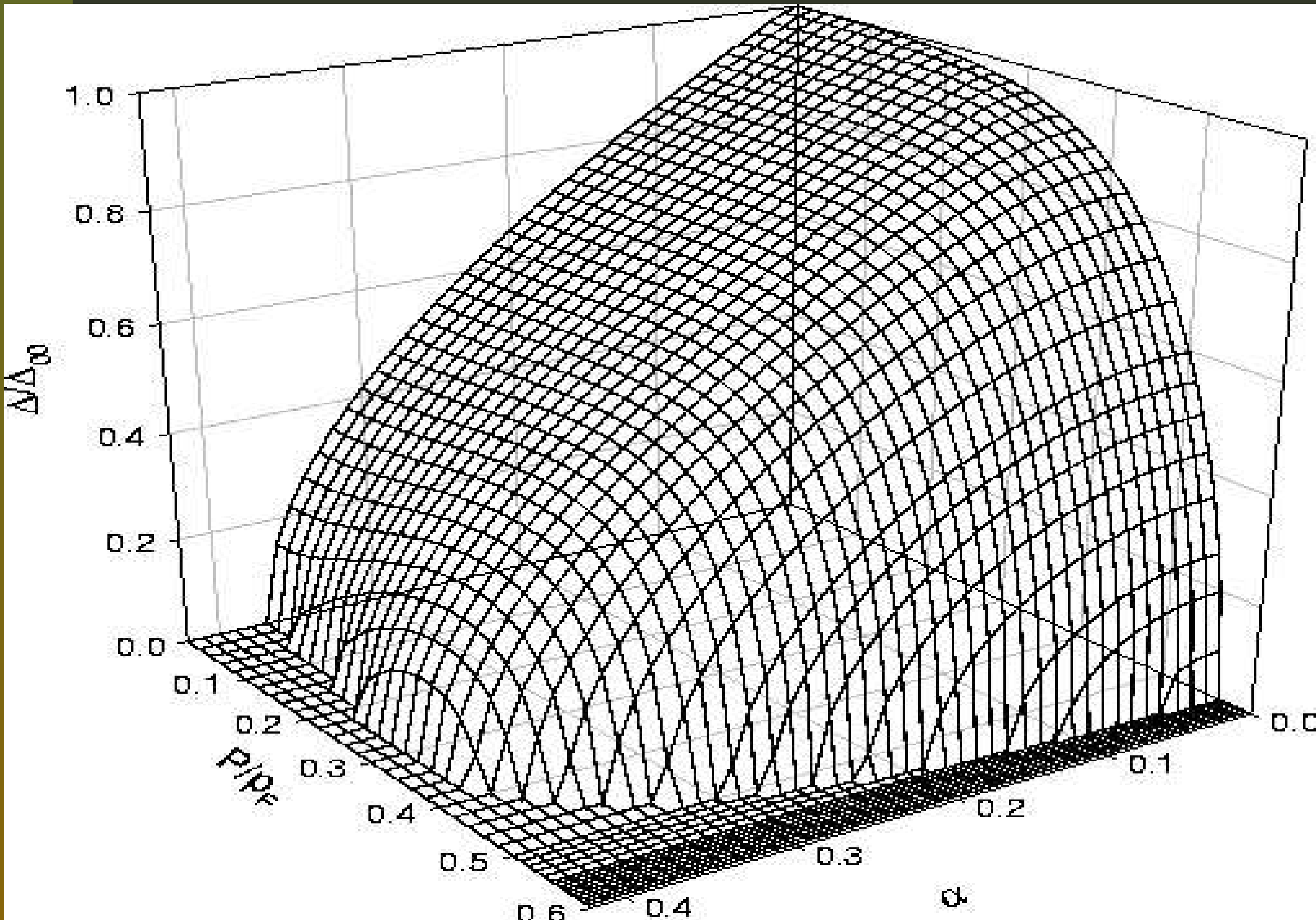
The gap $\Delta_0(\delta\mu)$ is independent of $\delta\mu$ in case a. The second solution is obtained by solving b).

- The first solution corresponds to a stable state
- since $\Delta_b < \Delta_a$ the second solution is unstable
- At the the Clogston transition point

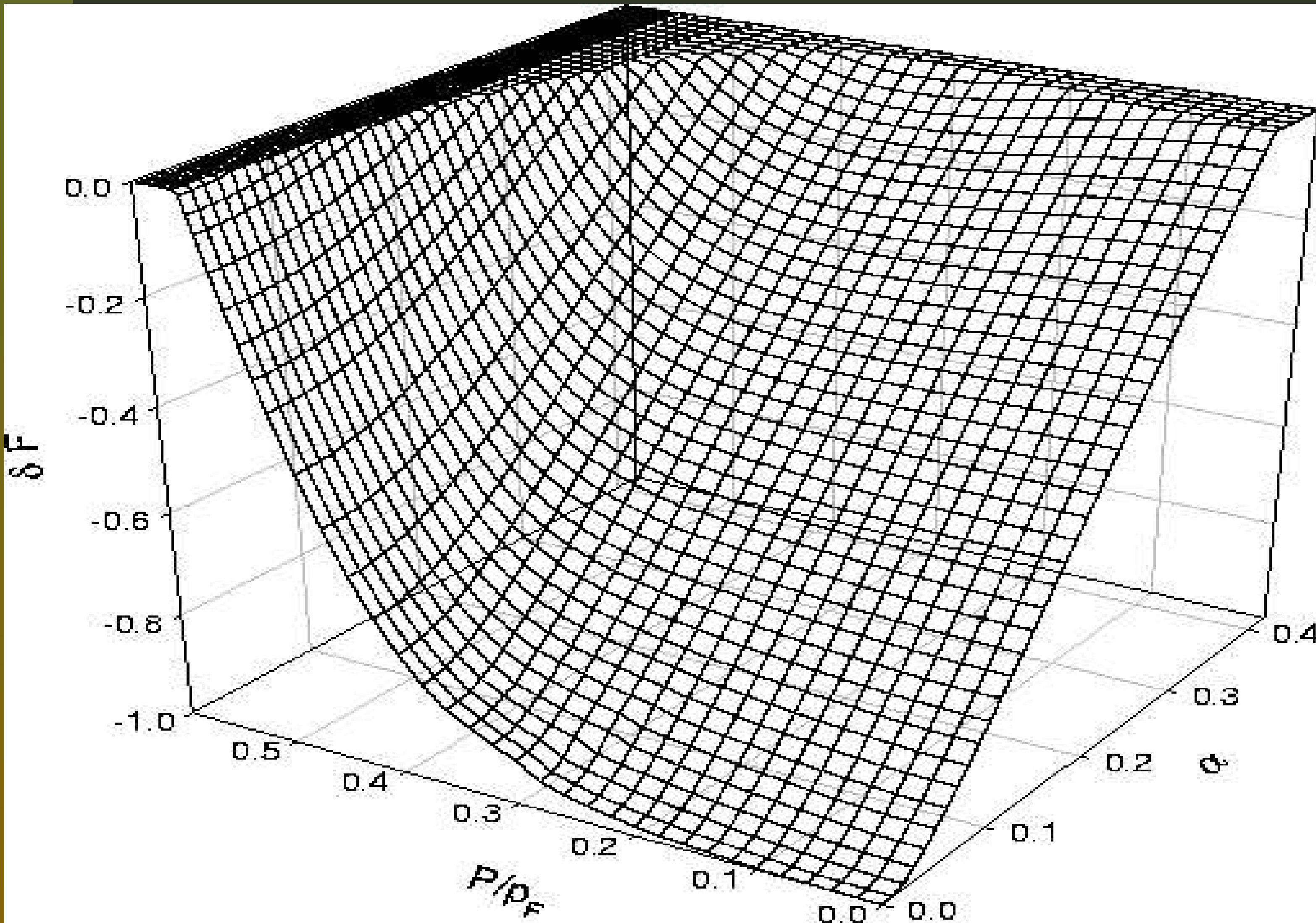
$$\delta\mu = \delta\mu_1 \equiv \frac{\Delta_0}{\sqrt{2}} ,$$

a transition from the superconducting ($\Delta_0 \neq 0$) to the normal phase ($\Delta_0 = 0$) occurs. This transition is first order, because for $\delta\mu < \delta\mu_1$ the gap does not depend on $\delta\mu$ and one passes abruptly to the normal phase.

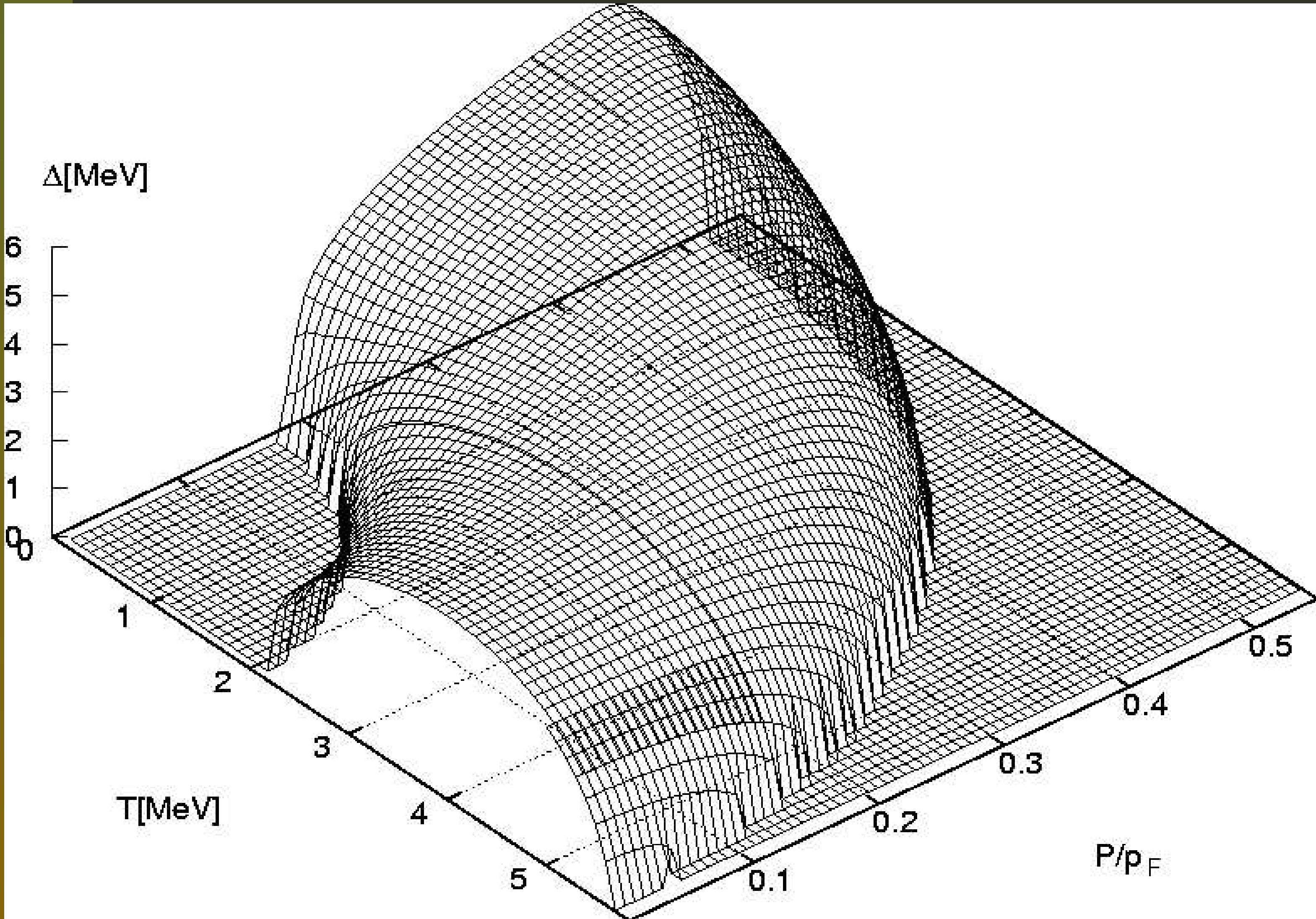
LOFF phase: $\delta\mu \neq 0$ $P \neq 0$



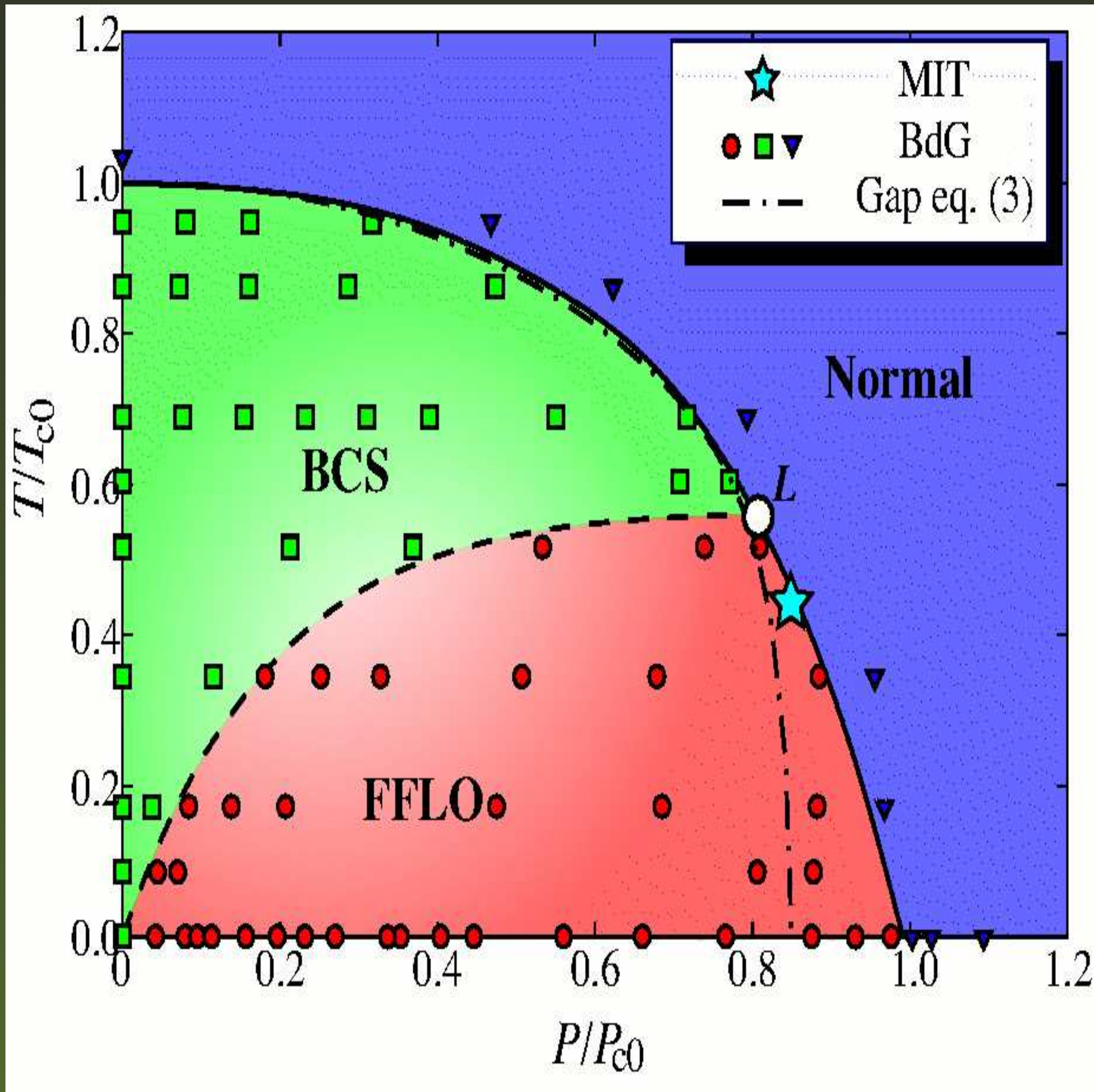
LOFF phases: free energy



LOFF phases: reentrance effect $P - T$ duality



BCS - LOFF phase transitions



BCS-BEC crossover

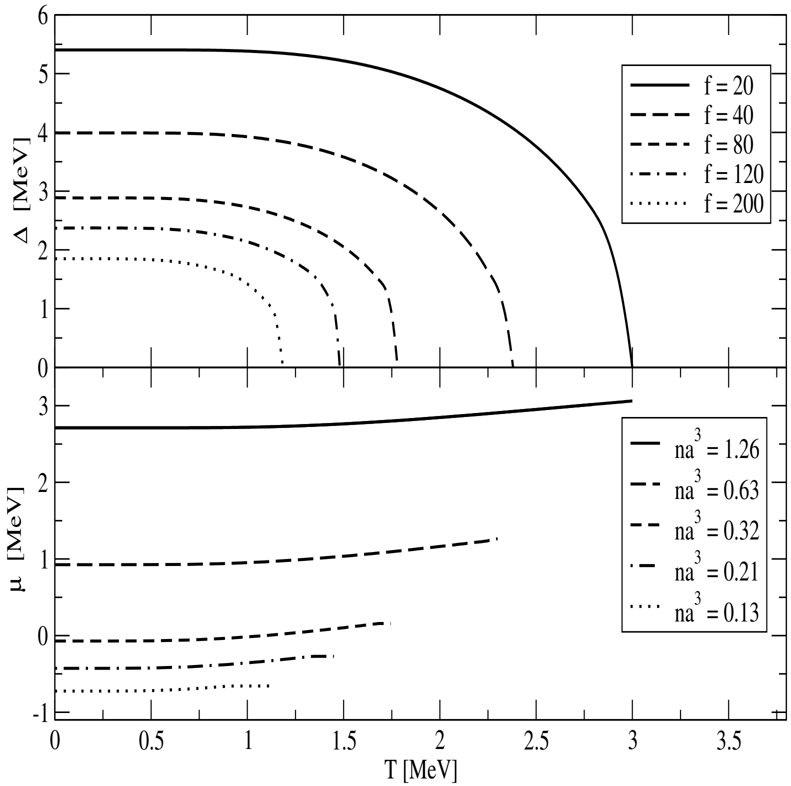
- Nozières-Schmitt-Rink conjecture: the BCS theory smoothly interpolates between the weak and strong couplings
- In the BEC limit the pair-wave function goes over to the Schrödinger equation

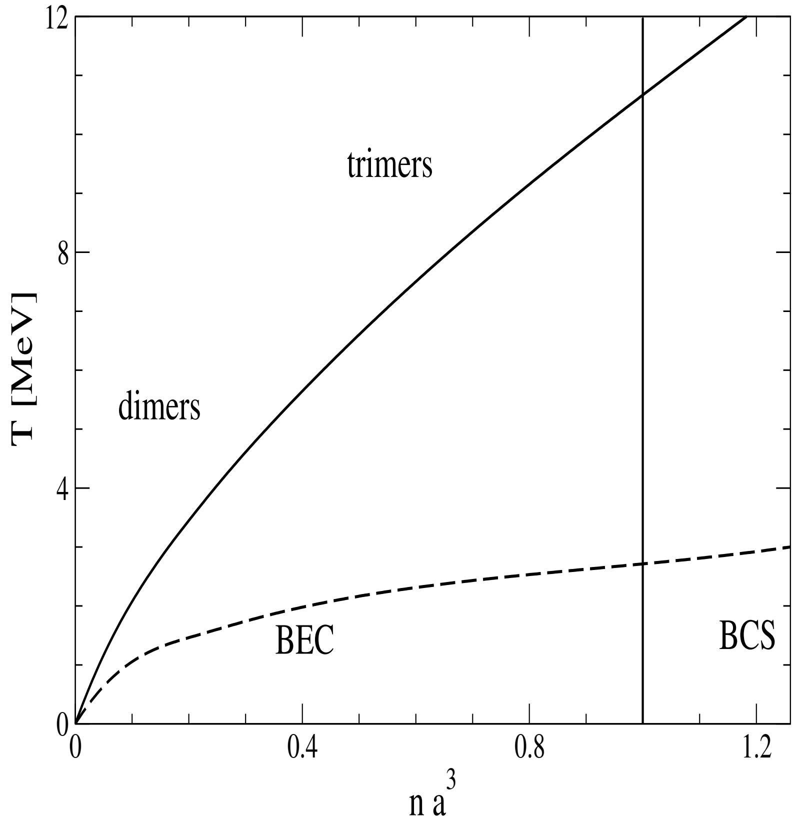
$$\psi(k) = \langle a^{\dagger}_{n,\vec{k}} a^{\dagger}_{p,-\vec{k}} \rangle = \frac{\Delta(k)}{2E_k}\left[1 - f(E_k^{+}) - f(E_k^{-})\right],$$

$$\frac{k^2}{m}\psi(k) + \left[1 - f(E_k^{+}) - f(E_k^{-})\right]\sum_{k'} V(k,k')\psi_{l'}(k') = 2\mu\,\psi(k)$$

- In unbalanced systems phases with broken space symmetries intervene.

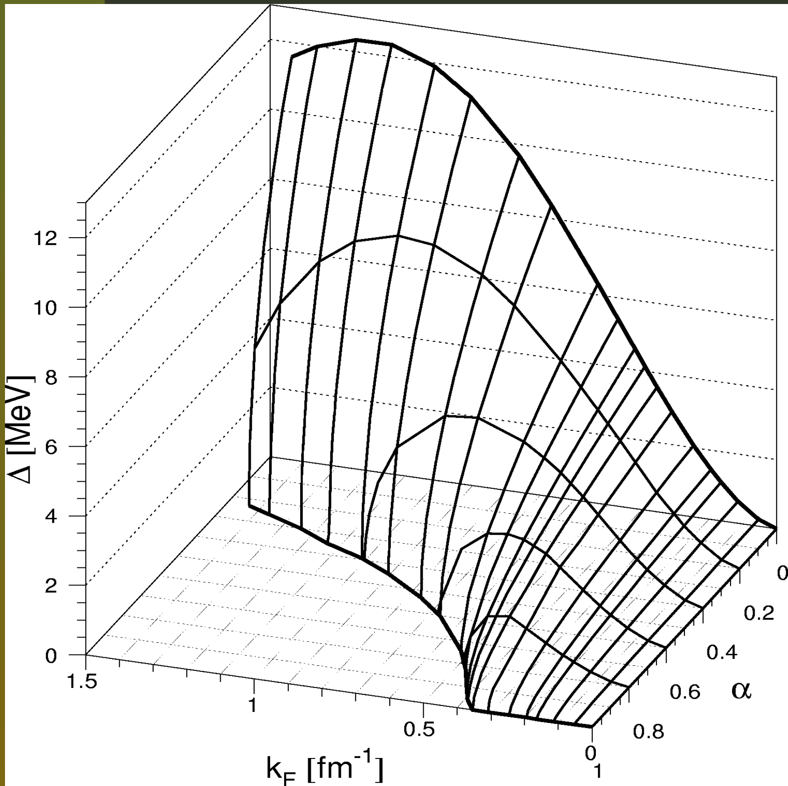
BCS-BEC crossover

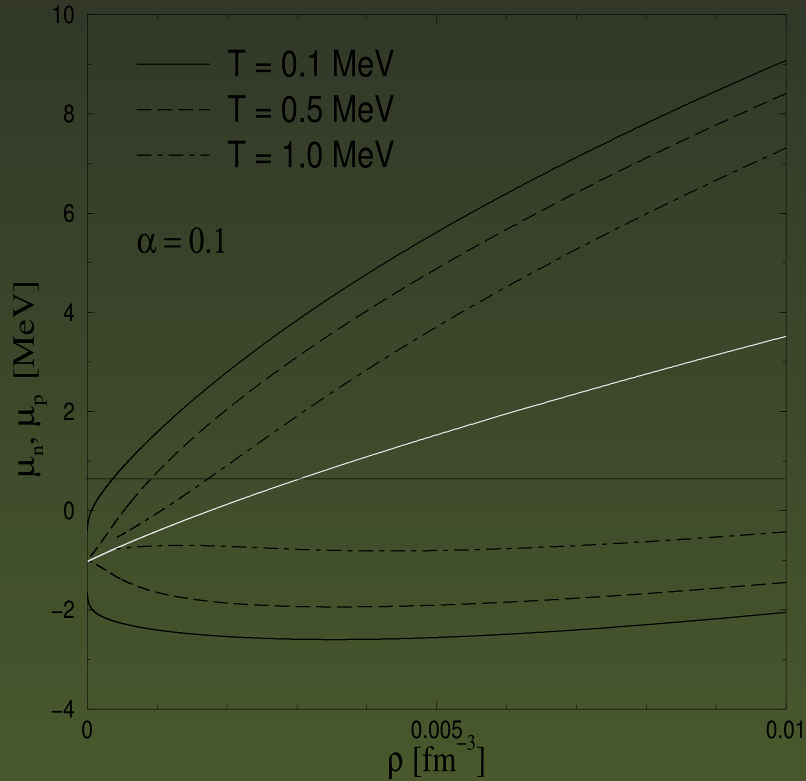




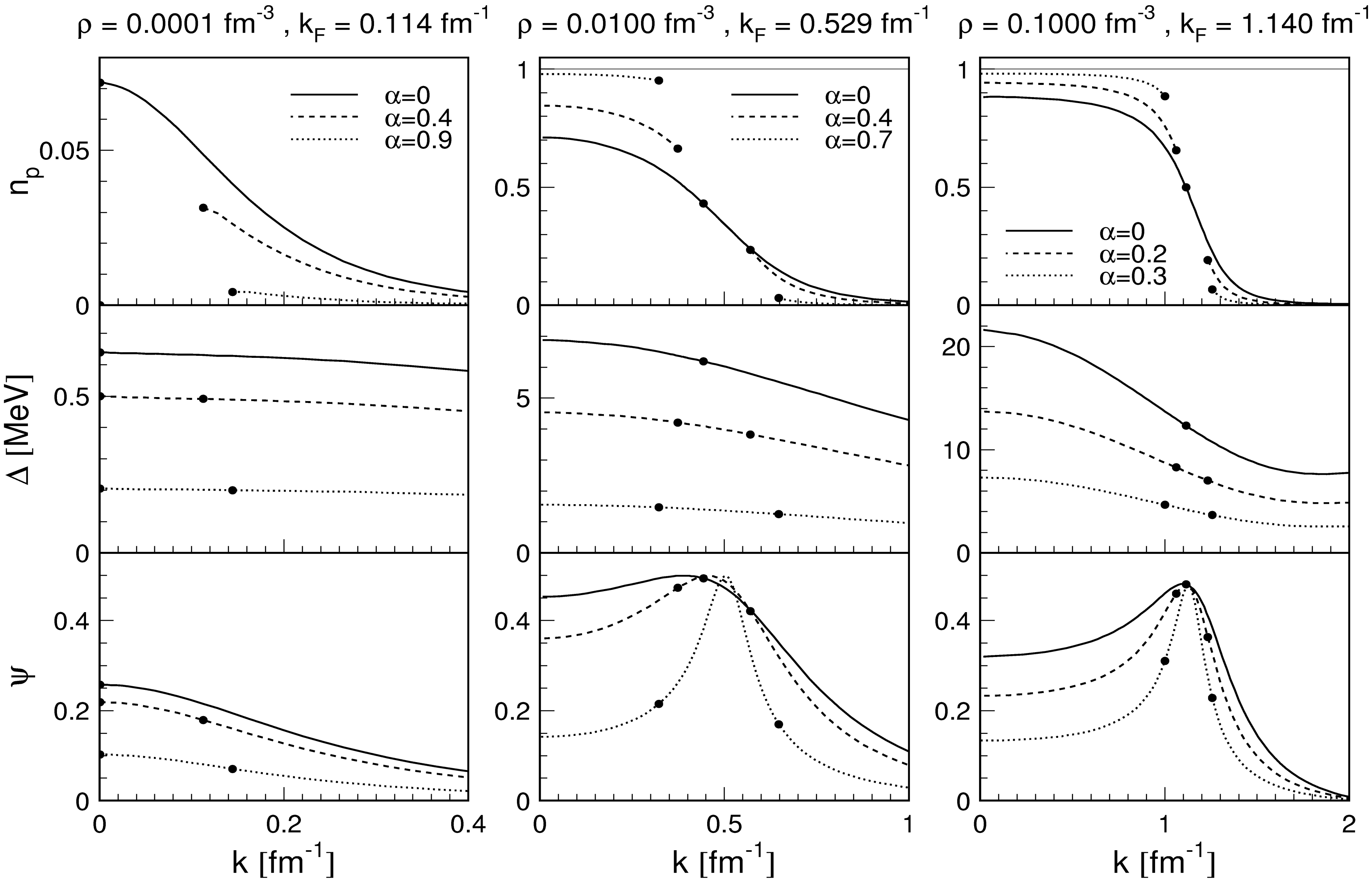
At the cross-over μ sign changes, coherence length $\xi \sim n^{1/3}$.

BCS-BEC crossover in asymmetric systems





Crossover continued



The three-body problem in background medium

- The three-body equation for the $\mathcal{T}$-matrix

$$\mathcal{T} = \mathcal{V} + \mathcal{V}\,\mathcal{G}\,\mathcal{V} = \mathcal{V} + \mathcal{V}\,\mathcal{G}_0\,\mathcal{T}, \qquad (7)$$

  where the interaction $\mathcal{V} = V_{12} + V_{23} + V_{13}$

- Reformulate the problem: $\mathcal{T} = \mathcal{T}^{(1)} + \mathcal{T}^{(2)} + \mathcal{T}^{(3)}$

$$\mathcal{T}^{(k)} = \mathcal{V}_{ij} + \mathcal{V}_{ij}\mathcal{G}_0\mathcal{T} \quad ijk = 123,\ 231,\ 312. \qquad (8)$$

  Define: $\mathcal{T}_{ij} = \mathcal{V}_{ij} + \mathcal{V}_{ij}\mathcal{G}_0\mathcal{T}_{ij}$ and eliminate the potentials

- Non-singular three-body equations (Bethe-Faddeev)

$$\mathcal{T}^{(k)} \;=\; \mathcal{T}_{ij} + \mathcal{T}_{ij}\mathcal{G}_0\left(\mathcal{T}^{(i)} + \mathcal{T}^{(j)}\right). \qquad (9)$$

Three-body propagator in background medium

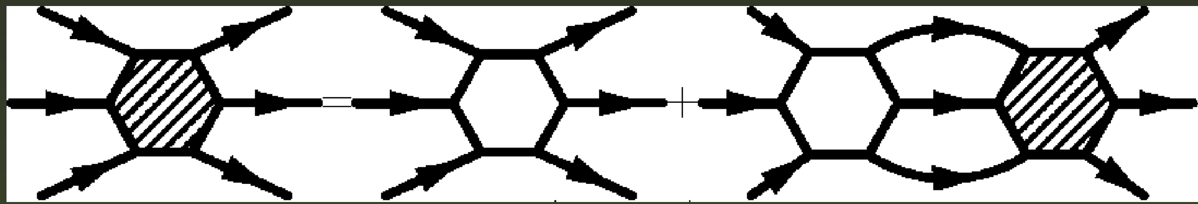
- The time structure of the three-body T-matrix

$$
\begin{aligned}
\mathcal{T}^{R\,(1)}(t,t') = {} & \mathcal{T}^{R}_{23}(t,t') \\
& + \int \Big[\mathcal{T}^{R\,(2)}(t,\bar{t}) + \mathcal{T}^{R\,(3)}(t,\bar{t}) \Big] \mathcal{G}^{R}_{0}(\bar{t},t'')\mathcal{T}^{R}_{23}(t'',t') d\bar{t}d
\end{aligned}
$$

- Possible particle-hole channels

$$
\mathcal{G}^{R}_{0}(t_1,t_2) \;=\; \theta(t_1-t_2) \begin{cases} G^{>}G^{>}G^{>}(t_1,t_2) - (> \leftrightarrow <) \\ G^{>}G^{>}G^{<}(t_1,t_2) - (> \leftrightarrow <) \\ G^{>}G^{<}G^{<}(t_1,,t_2) - (> \leftrightarrow < \\ G^{<}G^{<}G^{<}(t_1,t_2) - (> \leftrightarrow <) \end{cases}
$$

Particle-hole content of the T-matrix



- 3-particle – 3-hole scattering T-matrix

$$\mathcal{T}^{R\,(1)} = \mathcal{T}_{23}^{R} + \int \left[\mathcal{T}^{R\,(2)} + \mathcal{T}^{R\,(3)} \right] \frac{Q_3(\Omega')}{\Omega - \Omega' + i\eta} \mathcal{T}_{23}^{R}(\Omega') d\Omega$$

- 3-body Pauli-blocking: $\bar{f}_F = 1 - f_F$

$$Q_3(p_\alpha, p_\beta, p_\gamma) = \bar{f}_F(p_\alpha)\bar{f}_F(p_\beta)\bar{f}_F(p_\gamma) - f_F(p_\alpha)f_F(p_\beta)f_F$$

p_α are spanned in terms of Jacobi coordinates.

Bound states in background medium

- Bound state wave-function

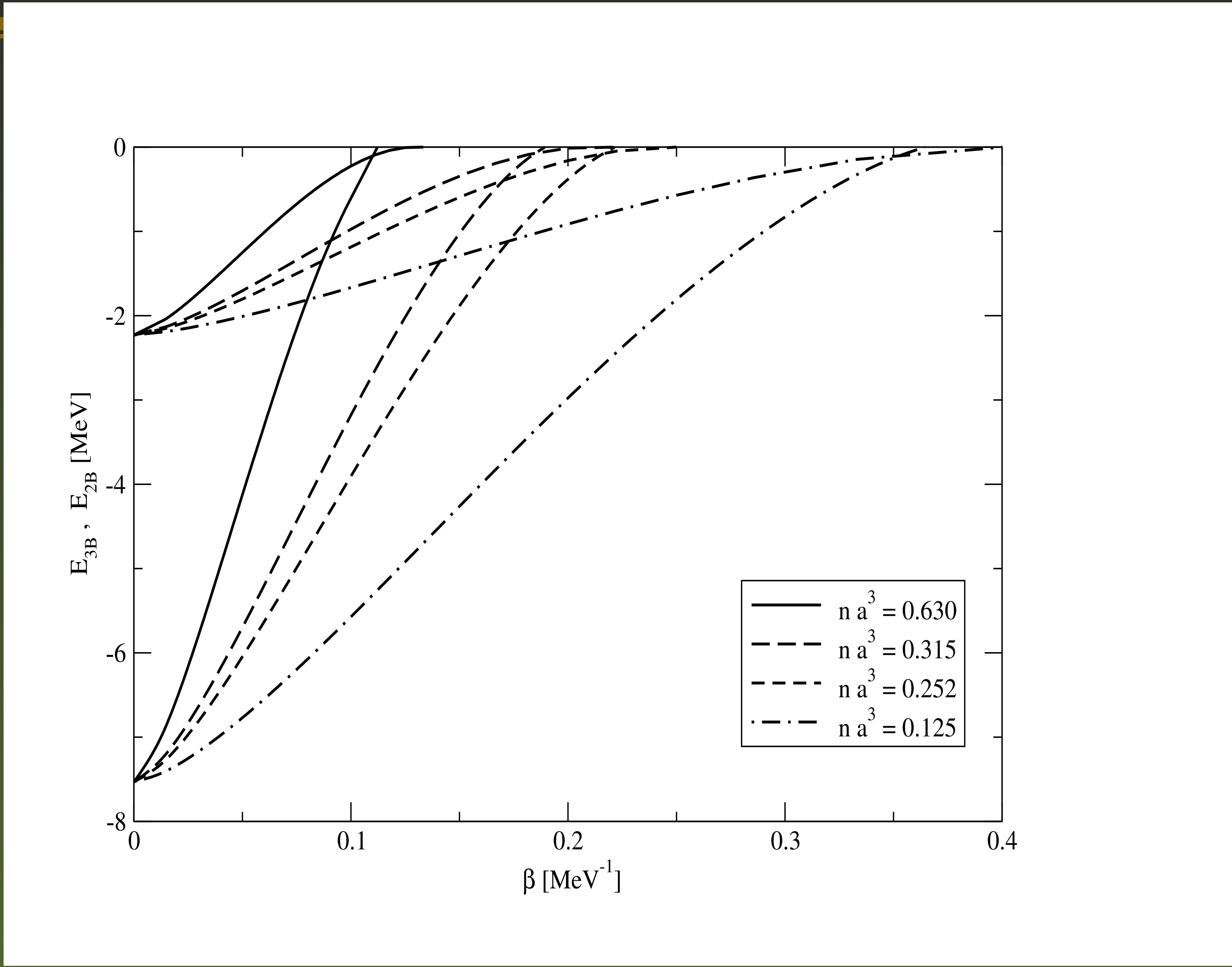
$$\Psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)}; \quad \psi^{(k)} = \mathcal{G}_0 T_{ij}(\psi^{(i)} + \psi^{(j)}) \tag{10}$$

- Need the channel T-matrix

$$T^R(\vec{p}, \vec{p}'; \boldsymbol{P}, E)$$
$$= V(\vec{p}, \vec{p}') + \int \frac{d\vec{p}''}{(2\pi)^3} V(\vec{p}, \vec{p}'') G_0^R(\vec{p}'', \boldsymbol{P}, E) T^R(\vec{p}''$$

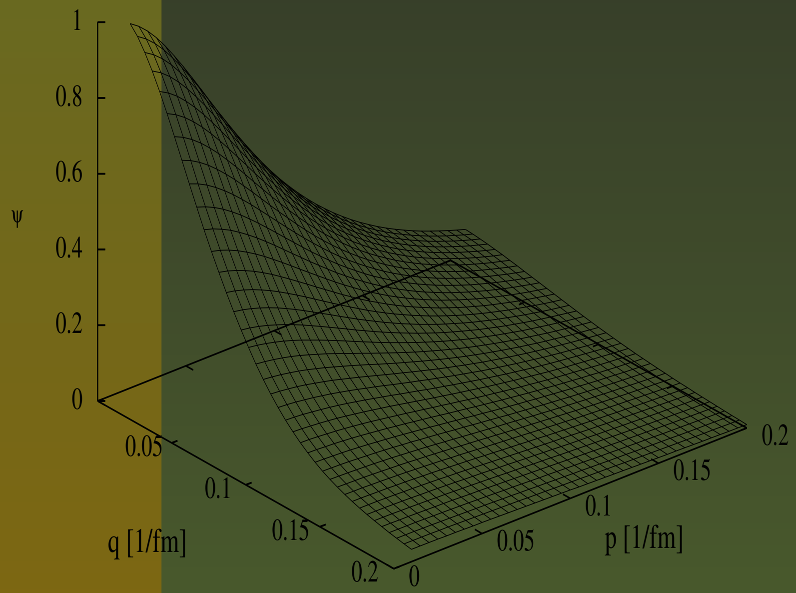
$$G_0^R(\vec{k}_1, \vec{k}_2, E) = \frac{Q_2(\vec{k}_1, \vec{k}_2)}{E - \epsilon(\vec{k}_1) - \epsilon(\vec{k}_2) + i\eta}, \tag{11}$$

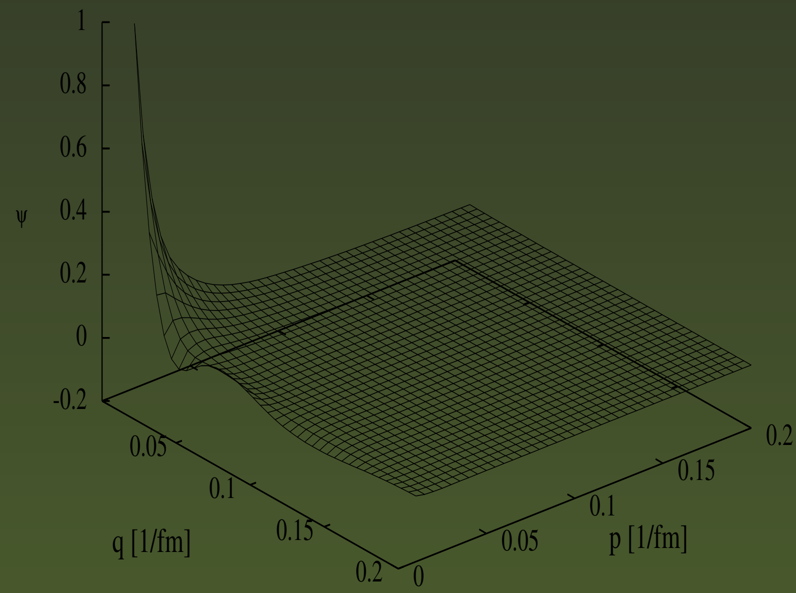
Temperature dependent binding energies of triton in nuclear matter



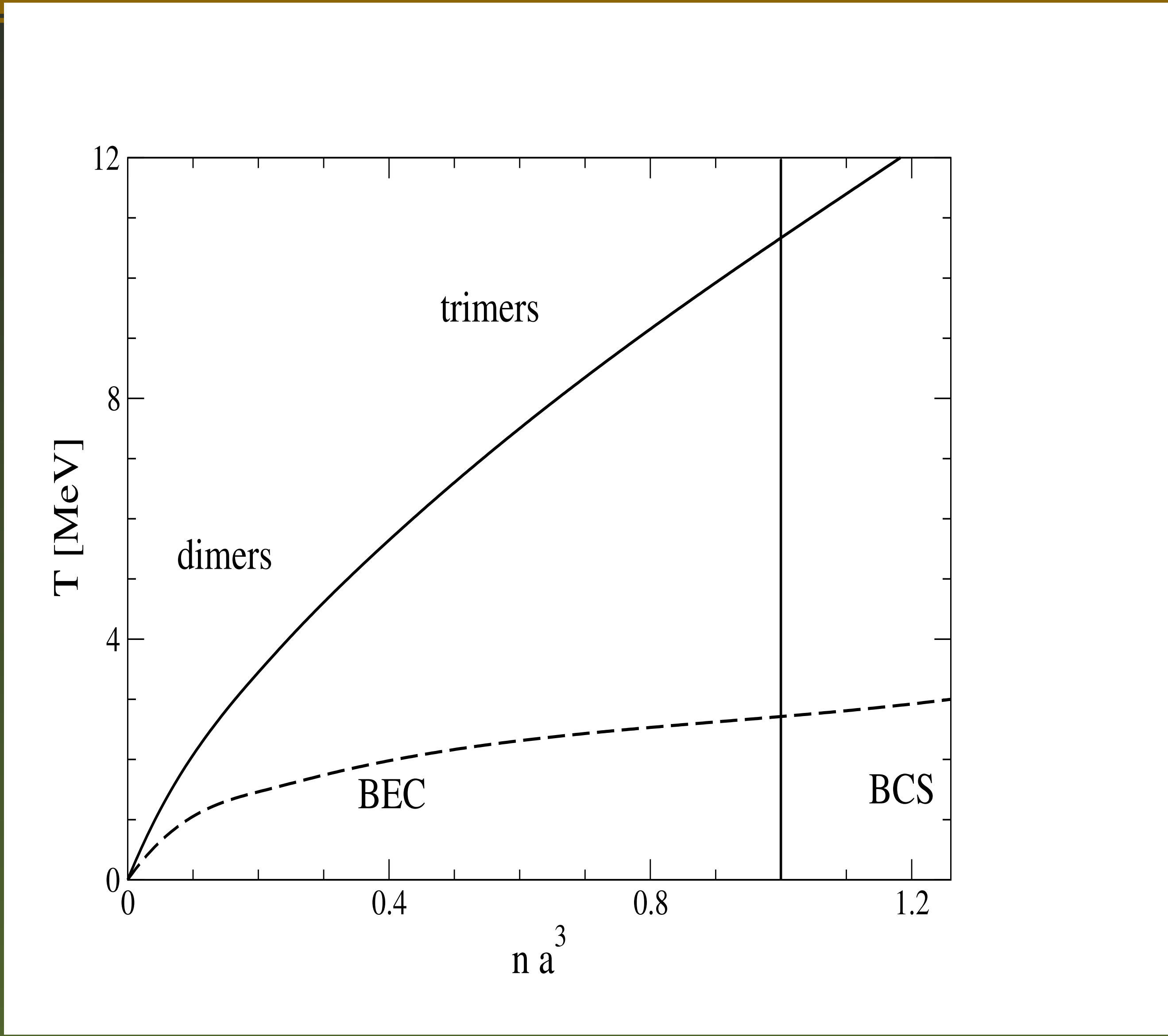
The ratio $\eta = E_{3B}(T)/E_{2B}(T)$ is independent of temperature.

The three-body wave-function



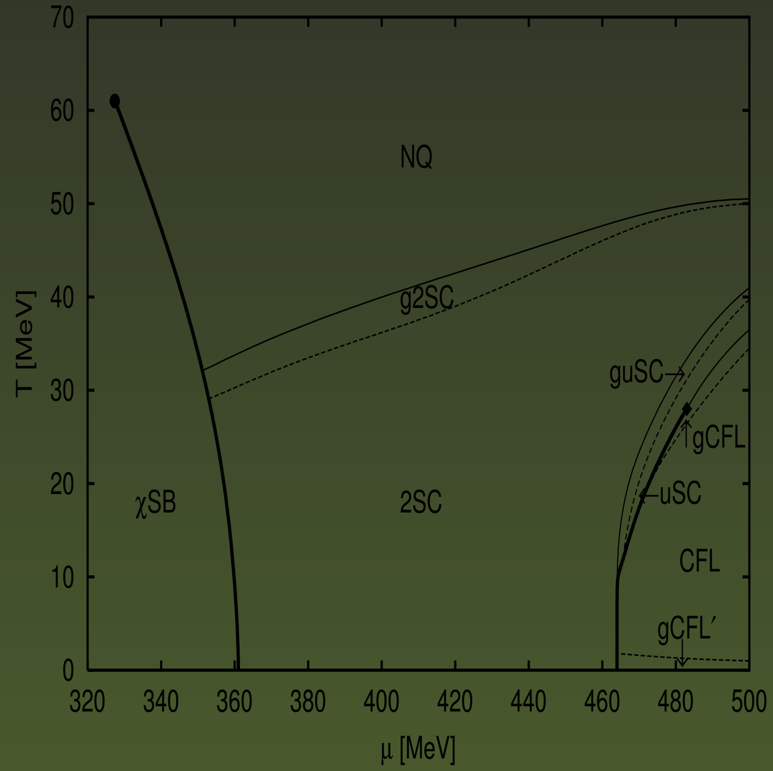


The phase diagram



The phase diagram could be much more complicated if the system supports bound states in the free

Color superconductivity



A schematic phase diagram of color superconducting matter

Color superconductivity

The Nambu-Jona-Lasinio (NJL) with two flavors $(N_f = 2)$ and three colors $(N_c = 3)$. Assume chiral symmetry. The Lagrangian density of the model is

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \bar{\psi}(x)(i\gamma^\mu\partial_\mu)\psi(x) \\ &+ G_1(\psi^T C\gamma_5\tau_2\lambda_A\psi(x))^\dagger(\psi^T C\gamma_5\tau_2\lambda_A\psi(x)),\end{aligned} \tag{12}$$

$C = i\gamma^2\gamma^0$, τ_2 Pauli matrix acting in the $\mathrm{SU}(2)_f$ flavor space, λ_A is the Gell-Mann matrix in the $\mathrm{SU}(3)_c$ color space. The coupling constant G_1 stands for the four-fermion contact interaction.

2SC superconductivity

The common Ansatz for the order parameter in the 2SC phase is

$$\Delta \propto \langle \psi^T(x) C\gamma_5 \tau_2 \lambda_2 \psi(x) \rangle,$$

The Ansatz for the order parameter [Eq. (**??**)] implies that the color $\mathrm{SU}(3)_c$ symmetry is reduced to $\mathrm{SU}(2)_c$ since only two of the quark colors are involved in the pairing while the third color remains unpaired. The gap equation and the partial densities are found from the thermodynamic potential density Ω:

$$\frac{\partial \Omega}{\partial \Delta} = 0, \qquad -\frac{\partial \Omega}{\partial \mu_f} = \rho_f;$$

the flavor index $f = u, d$ refers to up (u) and down (d) quarks.

Thermodynamic potential

$$\Omega(\beta\mu) = -\frac{1}{\beta}\sum_{\omega_n}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\mathrm{Tr}\ln\left[\beta\begin{pmatrix} S_{11}^{-1}(i\omega_n,\vec{p}) & S_{12}^{-1}(i\omega_n,\vec{p}) \\ S_{21}^{-1}(i\omega_n,\vec{p}) & S_{22}^{-1}(i\omega_n,\vec{p})\end{pmatrix}\right] + \frac{\Delta^2}{4G}$$

where β is the inverse temperature. The elements of the Nambu-Gor'kov matrix are 2×2 matrices defined as

$$diag\,[S_{11}^{-1}] = (\not p + \mu_u\gamma^0, \not p + \mu_d\gamma^0)$$
$$diag\,[S_{22}^{-1}] = (\not p - \mu_u\gamma^0, \not p - \mu_d\gamma^0)$$
$$diag\,[S_{12}^{-1}] = (\Delta\gamma_5\tau_2\lambda_2, \Delta\gamma_5\tau_2\lambda_2)$$
$$diag\,[S_{21}^{-1}] = (-\Delta^*\gamma_5\tau_2\lambda_2, -\Delta^*\gamma_5\tau_2\lambda_2)$$

with off-diagonal elements zero.

Thermodynamic potential

Carry out the traces in the spin, flavor and color spaces and the fermionic Matsubara summation over the frequencies ω_n

$$\begin{aligned}\Omega &= -2\int\frac{d^3p}{(2\pi)^3}\Bigg\{2p+\sum_{ij}\Bigg[\frac{1}{\beta}\log\left(1+e^{-\beta\xi_{ij}}\right)\\ &+ E_{ij}+\frac{2}{\beta}\log\left(1+e^{-\beta s_{ij}E_{ij}}\right)\Bigg]\Bigg\}+\frac{\Delta^2}{4G_1},\end{aligned}\tag{14}$$

$\xi_{\pm\pm} = (p\pm\mu)\pm\delta\mu$ $E_{\pm\pm} = \sqrt{(p\pm\mu)^2+|\Delta|^2}\pm\delta\mu$, where $\delta\mu = (\mu_u-\mu_d)/2$ and $\mu = (\mu_u+\mu_d)/2$ and μ_u and μ_d are the chemical potentials of u/d quarks.

Thermodynamic potential

The variations of the thermodynamic potential provide the gap equation

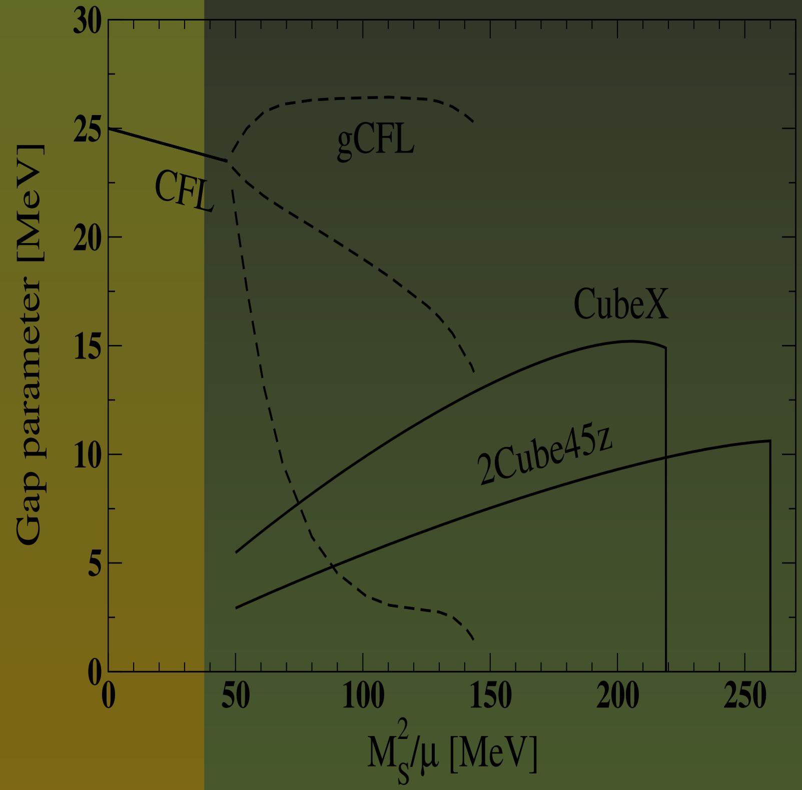
$$
\begin{aligned}
\Delta &= 8G_1 \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{\Delta}{E_{+-} + E_{++}} \left[\tanh\left(\frac{\beta E_{++}}{2}\right) + \tanh\left(\frac{\beta E_{+-}}{2}\right) \right] \right. \\
&\qquad \left. + \frac{\Delta}{E_{-+} + E_{--}} \left[\tanh\left(\frac{\beta E_{-+}}{2}\right) + \tanh\left(\frac{\beta E_{--}}{2}\right) \right] \right\}
\end{aligned}
$$

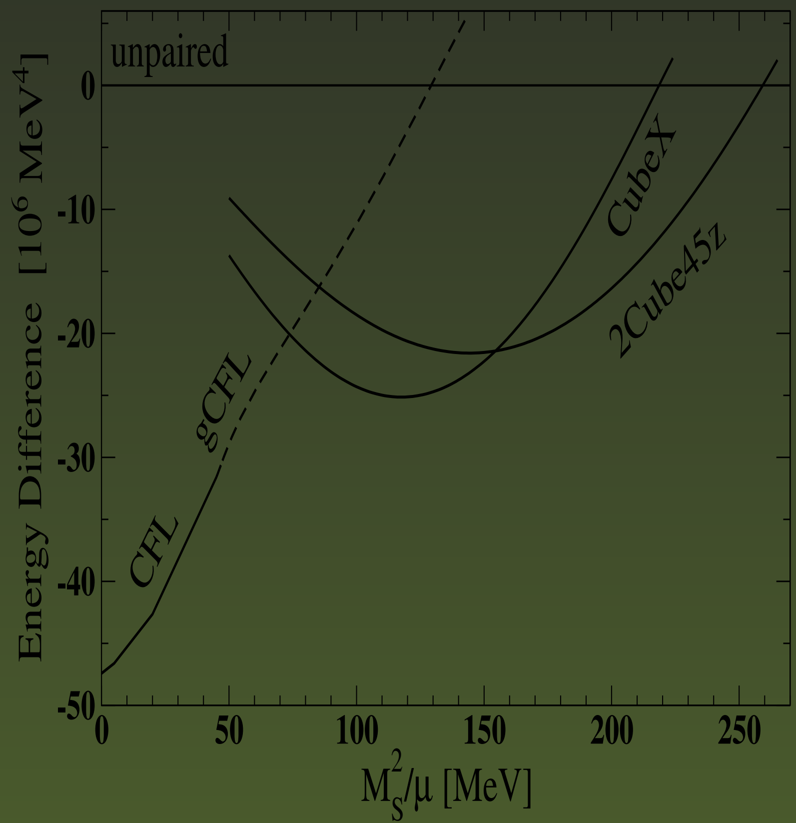
Thermodynamic potential

and the partial densities of the up/down quarks

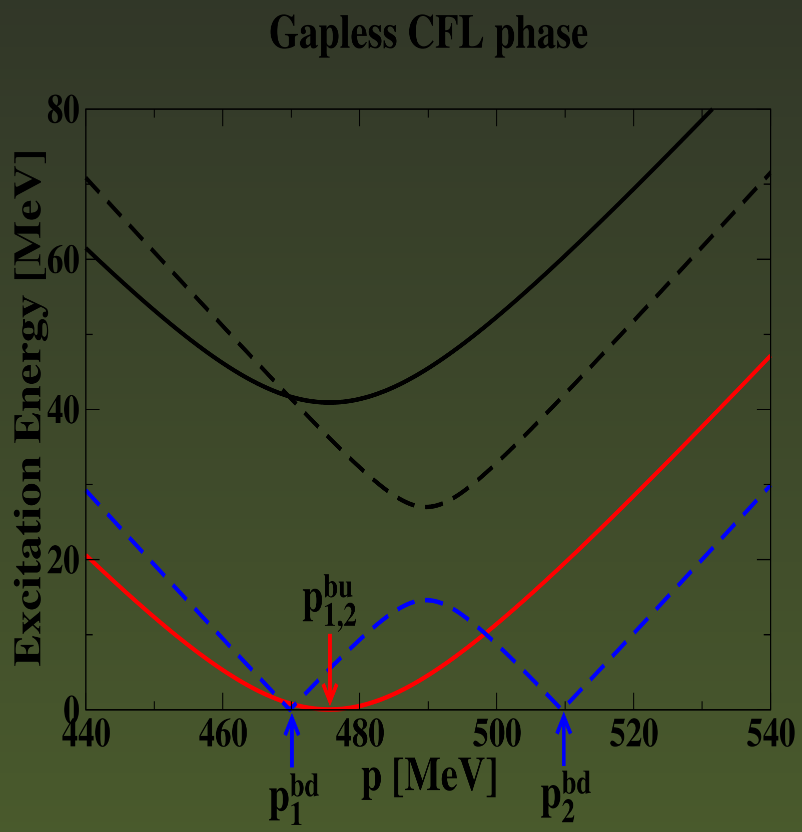
$$
\begin{aligned}
\rho_{u/d} \quad = \quad & \int \frac{d^3p}{(2\pi)^3} \Bigg\{ 2f(\xi_{-\mp}) - 2f(\xi_{+\pm}) \\
& \mp \left[1 \pm \frac{\xi_{--} + \xi_{-+}}{E_{--} + E_{-+}} \right] \tanh\left(\frac{\beta E_{--}}{2} \right) \\
& \mp \left[1 \mp \frac{\xi_{+-} + \xi_{++}}{E_{+-} + E_{++}} \right] \tanh\left(\frac{\beta E_{+-}}{2} \right) \\
& \pm \left[1 \mp \frac{\xi_{--} + \xi_{-+}}{E_{--} + E_{-+}} \right] \tanh\left(\frac{\beta E_{-+}}{2} \right) \\
& \pm \left[1 \pm \frac{\xi_{+-} + \xi_{++}}{E_{+-} + E_{++}} \right] \tanh\left(\frac{\beta E_{++}}{2} \right) \Bigg\},
\end{aligned}
\tag{15}
$$

Solutions





The pairing gaps and the thermodynamic potential in the color superconducting phases.



Excitation spectrum of quarks in gapless superconducting phases.