The Structure of Turbulence in Newtonian and Viscoelastic flows: Polymers and Drag Reduction Lectures 1/2

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Structures of the mechanics of complex bodies

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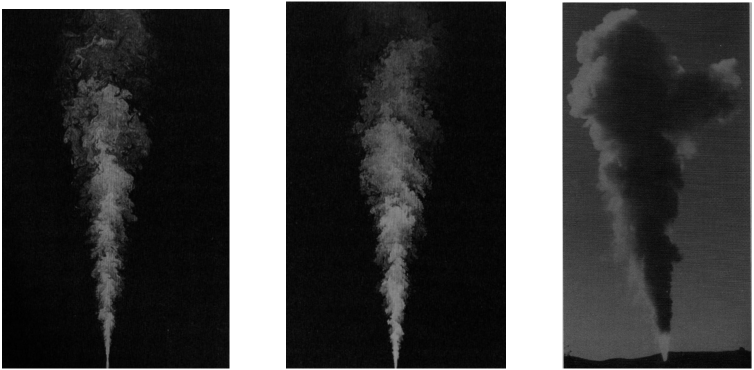
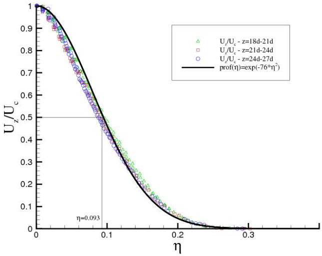
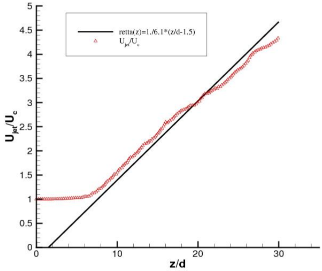
High Reynolds number jets



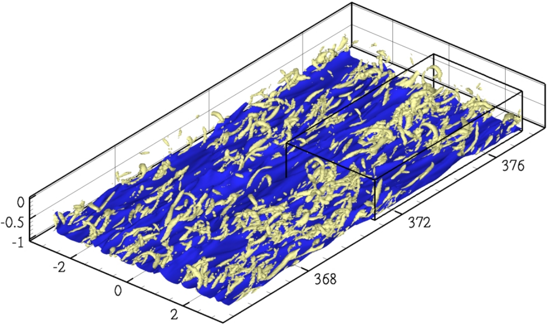
Da sinistra: $Re = 5000$; $Re = 20000$; $Re \simeq 2 \cdot 10^8$

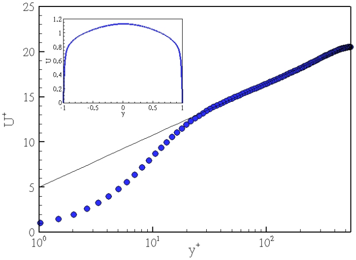
High Reynolds number jets



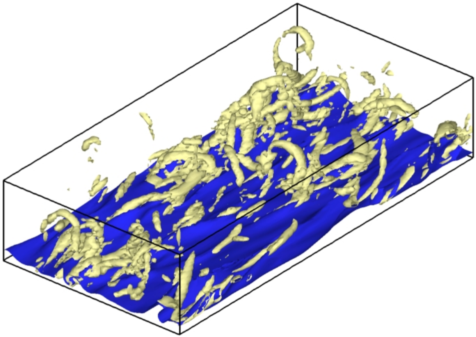


Wall bounded flows





Wall bounded flows I



Introduction

Certain polymers reduce the drag significantly wrt Newtonian

(a few ppm yield $DR = (D_0 - D_p)/D_0 \simeq 70\%$)

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Introduction

Rheological properties: viscoelasticity, shear thinning

In laminar flow a drag-reducing polymer solution (small concentration) is indistinguishable from a pure Newtonian fluid

Drag-reduction due to the polymer/turbulence interaction

Outline

- Thu 17:00/18:00 Turbulence basics
- Fri 16:00/17:00 Turbulence & Walls
- Sat 09:00/10:00 Drag-reducing polymers
- Sat 17:00/18:00 Polymers & Turbulence

Turbulence basics

Given a body of dimension L moving with speed V through a Newtonian incompressible (isocoric) flow with viscosity μ and mass density ρ, the dissipation rate of mechanical energy per unit volume

$$\epsilon = 2\mu S : S$$

($S = 1/2(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the deformation rate) may be expected to scale as

$$\epsilon \propto \mu V^2 L^{-2}.$$

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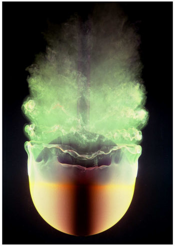
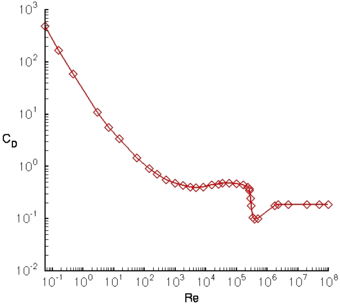
In fact the correct answer at large Reynolds number $Re = \mu VL/\rho$ is

$$\epsilon \propto \rho V^3/L.$$

Dissipative anomaly

$Re = 2. \times 10^4$ $\qquad$ $Re = 5. \times 10^5$





Drag & Energy budget

$$D = \frac{1}{2}\rho V_b^2 C_D L_b^2$$

$$V_b D = \int_{\mathcal{B}} \epsilon \, dV \propto \epsilon \, L_b^3$$

$$\lim_{Re \to \infty} C_D(Re) = \tilde{C}_D$$

Dissipation independent of viscosity and set by the large scales

$$\epsilon_\infty \propto \frac{\rho V_b^3}{2 L_b} \tilde{C}_D$$

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The energy cascade

- Viscosity ineffective at large scales
- Energy injected at large scales moves towards small scales
- Eventually viscous dissipation is activated at small scales
- An inertial range sets in where the energy flux is constant

Homogeneous - "isotropic" Navier Stokes turbulence

Assume the Navier Stokes equation for the solenoidal velocity field $\mathbf{u}(\mathbf{x}, t; \omega)$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} + \mathbf{f}$$

in a periodic box $\mathcal{D}$, with initial data

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$$

subject to a large-scale Gaussian stochastic forcing $\mathbf{f}(\mathbf{x}, t; \omega)$

$$\langle \mathbf{f}(\mathbf{x}, t) \otimes \mathbf{f}(\mathbf{x} + \mathbf{r}, t + \tau) \rangle = C_f(\mathbf{r}) \delta(\tau)$$

Reynolds decomposition and turbulent kinetic energy

Define an average of q over different realizations of the process

$$\langle q(\mathbf{x}, t)\rangle = \lim_{N \to \mathrm{big}} \frac{1}{N} \sum_{k=1}^{N} q(\mathbf{x}, t, \omega_k)\,,$$

and its fluctuation (Reynolds decomposition)

$$q'(\mathbf{x}, t, \omega) = q(\mathbf{x}, t, \omega) - \langle q(\mathbf{x}, t)\rangle$$

The averaged kinetic energy density (energy per unit mass)

$$K(\mathbf{x}, t, \omega) = \frac{1}{2}\mathbf{u}(\mathbf{x}, t, \omega) \cdot \mathbf{u}(\mathbf{x}, t, \omega)$$

with Reynolds decomposition is

$$\langle K(\mathbf{x}, t)\rangle = \frac{1}{2}\langle \mathbf{u}(\mathbf{x}, t\rangle \cdot \langle \mathbf{u}(\mathbf{x}, t)\rangle + \frac{1}{2}\langle \mathbf{u}'(\mathbf{x}, t\rangle \cdot \langle \mathbf{u}'(\mathbf{x}, t)\rangle = K_M(\mathbf{x}, t) + K_T(\mathbf{x}, t)$$

The equation for mean field and fluctuations

Averaging NS yields the equation for the mean field $\langle \mathbf{u} \rangle$:

$$\frac{\partial \langle \mathbf{u} \rangle}{\partial t} + \langle \mathbf{u} \rangle \cdot \nabla \langle \mathbf{u} \rangle = -\nabla \langle p \rangle + \nu \Delta \langle \mathbf{u} \rangle + \langle \mathbf{f} \rangle - \nabla \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle$$

- Equation not closed: Reynolds stresses $\mathbf{\Sigma}_R = -\langle \mathbf{u}' \otimes \mathbf{u}' \rangle$.

The equation for the fluctuation is then

$$\frac{\partial \mathbf{u}'}{\partial t} + \langle \mathbf{u} \rangle \cdot \nabla \mathbf{u}' = -\nabla p' + \nu \Delta \mathbf{u}' + \mathbf{f}' \\ -\nabla \cdot \left[\mathbf{u}' \otimes \mathbf{u}' - \langle \mathbf{u}' \otimes \mathbf{u}' \rangle + \langle \mathbf{u} \rangle \otimes \mathbf{u}' \right]$$

The equation for the TKE

The turbulent kinetic energy $K_T(\mathbf{x}, t) = 1/2\langle \mathbf{u}' \cdot \mathbf{u}' \rangle$ obeys an evolution equation derived from NS

$$\frac{\partial K_T}{\partial t} + \nabla \cdot (\langle \mathbf{u} \rangle K_T) = \nabla \cdot \phi_T + \pi_T - \epsilon_T + \langle \mathbf{f}' \cdot \mathbf{u}' \rangle$$

- $\phi_T = -\langle p'\mathbf{u}' + 1/2 u'^2 \mathbf{u}' - \mathbf{u}' \cdot \mathbf{\Sigma}' \rangle$ is the spatial flux of TKE ($\mathbf{\Sigma}' = \nu(\nabla \mathbf{u}' + \nabla \mathbf{u}'^T)$)
- $\pi_T = \langle \mathbf{u}' \otimes \mathbf{u}' \rangle : \nabla \langle \mathbf{u} \rangle$ is the production rate of TKE (conservative exchange from mean flow to fluctuations)
- $\epsilon_T = \langle \mathbf{\Sigma}' : \left(\nabla \mathbf{u}' + \nabla \mathbf{u}'^T \right) /2 \rangle$ is the dissipation rate of TKE (semi-positive definite and zero only for rigid fluctuations)

The equation for the TKE (homogeneous-isotropic)

The turbulent kinetic energy $K_T(\mathbf{x}, t) = 1/2\langle \mathbf{u}' \cdot \mathbf{u}' \rangle$ obeys an evolution equation derived from NS

$$\frac{\partial K_T}{\partial t} = -\epsilon_T + \langle \mathbf{f}' \cdot \mathbf{u}' \rangle$$

- $\langle \mathbf{u} \rangle = 0$ by suitable choice of reference frame
- $\epsilon_T = \langle \mathbf{\Sigma}' : \left(\nabla \mathbf{u}' + \nabla \mathbf{u}'^T \right) /2 \rangle$ is the dissipation rate of TKE (semi-positive definite and zero only for rigid fluctuations)
- Statistically stationary forcing and dissipation anomaly => the power extracted from the forcing is viscosity independent (provided the Reynolds number is large enough!)

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Warning on dissipation anomaly

- It is hard to predict the existence of cat's whisker from Schroedinger equation ...
  Once you observe a cat however ... the whisker tells you something about the equation ...

  Similarly, dissipation anomaly is a phenomenology suggested by experimental observation. It is used as an additional axiom to learn more about turbulence.

- It should be taken by no means to imply the power extracted from the external source if determined by dissipation.

- It means instead: The large scales which couple to the forcing are viscosity-independent.

Implication of dissipation anomaly

- $\epsilon_T = 2\nu\langle \mathbf{S}' : \mathbf{S}' \rangle$ independent of ν implies

$$|\nabla \mathbf{u}'| \propto \frac{1}{\sqrt{\nu}}$$

  i.e. the field becomes singular as the viscosity is reduced (the Reynolds number increases).

- Since $\nabla \cdot \mathbf{u}' = 0$, $|\nabla \mathbf{u}'| \propto |\boldsymbol{\omega}'|$, with $\boldsymbol{\omega}' = \nabla \times \mathbf{u}'$ the vorticity fluctuation, one has

$$|\boldsymbol{\omega}| \propto \frac{1}{\sqrt{\nu}}$$

  (in h.i.t. $\epsilon_T = 2\nu\langle \boldsymbol{\omega}' \cdot \boldsymbol{\omega}' \rangle$, with $\Omega_T = 1/2\langle \boldsymbol{\omega}' \cdot \boldsymbol{\omega}' \rangle$ the turbulent enstrophy.)

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Vorticity transport equation and enstrophy

- By taking the curl of NS one ends up with the transport equation for the fluctuating vorticty $\partial \boldsymbol{\omega}'/\partial t + \langle \mathbf{u} \rangle \cdot \nabla \boldsymbol{\omega}' = \ldots$
- The equation for the turbulent enstrophy is then

$$\frac{\partial \Omega_T}{\partial t} + \mathbf{u} \cdot \nabla \Omega_T + \nabla \cdot \langle \frac{{\omega'}^2}{2} \mathbf{u}' \rangle + \langle \boldsymbol{\omega}' \otimes \mathbf{u}' \rangle : \nabla \langle \boldsymbol{\omega} \rangle =$$
$$\langle \boldsymbol{\omega}' \otimes \langle \boldsymbol{\omega} \rangle : \nabla \mathbf{u}' \rangle + \langle \boldsymbol{\omega}' \otimes \boldsymbol{\omega}' : \nabla \mathbf{u}' \rangle + \langle \boldsymbol{\omega}' \otimes \boldsymbol{\omega}' \rangle : \nabla \langle \mathbf{u} \rangle$$
$$+ \nu \Delta \Omega_T - \nu \langle \nabla \boldsymbol{\omega}' : \nabla \boldsymbol{\omega}' \rangle + \langle \boldsymbol{\omega}' \cdot \nabla \times \mathbf{f}' \rangle$$

- In h.i.t. ($\langle \mathbf{u} \rangle = \langle \boldsymbol{\omega} \rangle = 0$) it reduces to

$$\frac{\partial \Omega_T}{\partial t} = \langle \boldsymbol{\omega}' \otimes \boldsymbol{\omega}' : \nabla \mathbf{u}' \rangle - \nu \langle \nabla \boldsymbol{\omega}' : \nabla \boldsymbol{\omega}' \rangle$$

(where the fluctuating force is conservative, $\nabla \times \mathbf{f}' = 0$).

- In stat. steady h.i.t. enstrophy production is positive, $\langle \boldsymbol{\omega}' \otimes \boldsymbol{\omega}' : \nabla \mathbf{u}' \rangle = \nu \langle \nabla \boldsymbol{\omega}' : \nabla \boldsymbol{\omega}' \rangle > 0$.

Correlation tensor

One defines a (single-time) correlation tensor

$$\mathbf{C}(\mathbf{x},\mathbf{r},t) = \langle \mathbf{u}'(\mathbf{x}+\mathbf{r},t) \otimes \mathbf{u}'(\mathbf{x},t) \rangle \qquad C_{\alpha\,\beta} = \langle u'_\alpha(\mathbf{x}+\mathbf{r},t) u'_\beta(\mathbf{x},t) \rangle$$

typically with finite correlation length L_0,

$$\mathbf{C}(\mathbf{x},\mathbf{r},t) \simeq 0 \qquad\qquad |\mathbf{r}| > L_0 \; .$$

As a consequence, extensive quantities *do not fluctuate* in the limit of a large domain

Example: Fluctuation intensity in the tke

Define the turbulent kinetic energy of a certain realization $\mathbf{u}(\mathbf{x}, t, \omega)$

$$\mathrm{K}_T(t,\omega) = \int_{\mathcal{D}} K_T(\mathbf{x}, t, \omega) dV_x \ .$$

Its fluctuation is

$$\mathrm{K}'_T(t,\omega) = \mathrm{K}_T(t,\omega) - \langle \mathrm{K}_T(t) \rangle$$

and we have

$$\langle \mathrm{K}'_T(t)^2 \rangle = \sum_{r\,s} \int_{\mathcal{D}_r} \int_{\mathcal{D}_s} \langle K'_T(\mathbf{x}, t, \omega) K'_T(\mathbf{y}, t, \omega) \rangle dV_x dV_y$$

$$\simeq n \langle \left[\int_{D_1} K'_T(\mathbf{x}, t) dV_x \right]^2 \rangle \, ; \qquad \frac{\sqrt{\langle \mathrm{K}'_T(t)^2 \rangle}}{\langle K_T(t) \rangle} \simeq 1/\sqrt{n}$$

Example: Drag force exerted on a body

Define the drag force of a certain realization $\mathbf{u}(\mathbf{x}, t, \omega)$

$$\mathrm{D}(t,\omega) = \hat{\mathbf{U}}_\infty \cdot \int_{\partial\mathcal{B}} \mathbf{T}(\mathbf{x}, t, \omega) \cdot \mathbf{n}(\mathbf{x}) dS_x \ .$$

Its fluctuation is

$$\mathrm{D}'(t,\omega) = \mathrm{D}(t,\omega) - \langle \mathrm{D}(t) \rangle$$

and we have

$$\sqrt{\langle \mathrm{D}'(t)^2 \rangle} \simeq \frac{1}{\sqrt{n}} \langle D(t) \rangle$$

Kármán-Howarth equation

Homogeneous, isotropic, stationary turbulence

1. Take the Navier Stokes equations for $\mathbf{u}'_1 = \mathbf{u}'(\mathbf{x}_1)$ at $\mathbf{x}_1$
2. Scalar multiply by $\mathbf{u}'_2 = \mathbf{u}'(\mathbf{x}_2)$, $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{r}$
3. Repeat steps 1-2 by exchanging points $\mathbf{x}_1$ and $\mathbf{x}_2$
4. Add the two equations up
   Let's introduce the change of variables $(\mathbf{x}_1, \mathbf{x}_2) \rightarrow (\mathbf{x}_c, \mathbf{r})$, with $\mathbf{x}_c = 1/2\,(\mathbf{x}_1 + \mathbf{x}_2)$, $\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1$.
   Two-point averages depend only on separation $\mathbf{r}$ ($\nabla_c = 0$),

$$\nabla_{2/1} = \pm\nabla_r \qquad\qquad \frac{\partial}{\partial x^\alpha_{2/1}} = \pm\frac{\partial}{\partial r^\alpha}$$

Terms involving pressure vanish by "integration by parts"
$\langle \mathbf{u}_2 \cdot \nabla_1 p'_1 \rangle = \nabla_1 \cdot \langle \mathbf{u}_2 p'_1 \rangle = -\nabla_r \cdot \langle \mathbf{u}_2 p'_1 \rangle = -p'_1 \nabla_2 \cdot \mathbf{u}'_2 = 0$

Kármán-Howarth equation

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$$\nabla_{2/1} = \pm \nabla_r \qquad \frac{\partial}{\partial x^\alpha_{2/1}} = \pm \frac{\partial}{\partial r^\alpha}$$

Terms arising from the convective part take the form
$\langle \mathbf{u}'_2 \cdot \nabla_1 \cdot \mathbf{u}'_1 \otimes \mathbf{u}'_1 \rangle = \nabla_1 \cdot \langle \mathbf{u}'_1 \cdot \mathbf{u}'_2\, \mathbf{u}'_1 \rangle = -\nabla_r \cdot \langle \mathbf{u}'_1 \cdot \mathbf{u}'_2\, \mathbf{u}'_1 \rangle$

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Kármán-Howarth equation

Homogeneous, isotropic, stationary turbulence

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2. Scalar multiply by $\mathbf{u}'_2 = \mathbf{u}'(\mathbf{x}_2)$, $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{r}$
3. Repeat steps 1-2 by exchanging points $\mathbf{x}_1$ and $\mathbf{x}_2$
4. Add the two equations up

On account of homogeneity and solenoidality one gets the Kármán-Howarth (KH) equation ($\delta\mathbf{u}' = \mathbf{u}'_2 - \mathbf{u}'_1$)

$$\frac{\partial \langle \mathbf{u}'_1 \cdot \mathbf{u}'_2 \rangle}{\partial t} + \nabla_r \cdot \langle \left(\mathbf{u}'_1 \cdot \mathbf{u}'_2 \right) \, \delta\mathbf{u}' \rangle = \langle \mathbf{f}'_1 \cdot \mathbf{u}'_2 + \mathbf{f}'_2 \cdot \mathbf{u}'_1 \rangle + 2\nu \Delta_r \langle \mathbf{u}'_1 \cdot \mathbf{u}'_2 \rangle$$

The energy spectrum

The trace of the correlation tensor
$C(\mathbf{r}) = \mathrm{tr}\,[\mathbf{C}(\mathbf{r})] = \langle \mathbf{u}'_1 \cdot \mathbf{u}'_2 \rangle$
is the object appearing in the Kármán-Howarth equation.

- Its Fourier transform defines the energy spectrum

$$E(\mathbf{k}, t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} C(\mathbf{r}) e^{-\jmath \mathbf{k} \cdot \mathbf{r}} dV_r = \mathcal{F}[C]$$

Then

$$K_T(t) = \frac{1}{2} \int E(\mathbf{k}, t) dV_k \qquad\qquad \epsilon_T = \frac{1}{2} \nu \int k^2 E(\mathbf{k}, t) dV_k$$

The energy flux in wave-number space

The energy flux in the sapce of scales is defined as

$$\nabla_k \cdot \phi_k = \jmath \mathbf{k} \cdot \mathbf{T}(\mathbf{k}, t)$$

with

$$\mathbf{T}(\mathbf{k}, t) = \frac{1}{(2\pi)^3} \int \langle \mathbf{u}_1' \cdot \mathbf{u}_2' \delta \mathbf{u}' \rangle e^{-\jmath \mathbf{k} \cdot \mathbf{r}} dV_r$$

The spectral view

The Fourier transform of the Kármán -Howarth equation is then

$$\frac{\partial E(\mathbf{k},t)}{\partial t} - \nabla_k \cdot \phi_k(\mathbf{k},t) = -\nu k^2 E(\mathbf{k},t) + F(\mathbf{k},t)$$

$(F = \mathcal{F}[\langle \mathbf{u}_1' \cdot \mathbf{f}_2' + \mathbf{u}_2' \cdot \mathbf{f}_1' \rangle])$.

Since $\phi_k(\mathbf{k}) = \phi(k)\hat{\mathbf{k}}$ ($\hat{\mathbf{k}} = \mathbf{k}/k$), with the spherical averages

$$\bar{E}(k) = \frac{1}{4\pi}\int_\Omega k^2 E(\mathbf{k}) d\Omega \qquad \bar{\Phi}_k(k) = \frac{1}{4\pi}\int_\Omega k^2 \phi_k(\mathbf{k}) d\Omega\,,$$

the equation for the spectrum reads

$$\frac{\partial \bar{E}(k,t)}{\partial t} = \frac{\partial \bar{\Phi}_k(k,t)}{\partial k} - 2\nu k^2 \bar{E}(k,t) + \bar{F}(k,t)$$

Spectral balance in the low-wn (forcing) range $k \leq k_F$

Assume $\bar{F}(k,t)$ of compact support in k-space, i.e. $\bar{F}(k,t) \equiv 0$ for $k > k_F$.

In the low wave-number band $0 \leq k \leq k_F$ integrate to get

$$\frac{d}{dt}\int_0^{k_F} \bar{E}(k,t) = \bar{\Phi}_k(k_F,t) - 2\nu \int_0^{k_F} k^2 \bar{E}(k,t)dk + \int_0^{k_F} \bar{F}(k,t)dk$$

In a statistically steady state, decreasing ν one ends up with

$$\bar{\Phi}_k(k_F) = -\int_0^{k_F} \bar{F}(k)dk$$

We have an energy flux across k_F from low to high wavenumbers $\bar{\Phi}_k(k_F)$ which exactly removes the power injected by the forcing

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Spectral balance in the high-wn (dissipative) range $k_F << k_D \leq k$

Now $\nu k^2 \bar{E}(k,t) \equiv 0$ for $k < K_D(\nu)$.
In the high wave-number band $k_D(\nu) \leq k < +\infty$ integrate to get

$$\frac{d}{dt}\int_{k_D}^{+\infty} \bar{E}(k,t) = -\bar{\Phi}_k(k_D,t) - 2\nu \int_{k_D}^{+\infty} k^2 \bar{E}(k,t)dk$$

In a statistically steady state, one ends up with

$$\bar{\Phi}_k(k_D) = -2\nu \int_{k_D}^{+\infty} k^2 \bar{E}(k,t)dk$$

We have an energy flux across k_D entering the high wn-band which feeds the scales where energy dissipation takes place

Spectral balance in the intermediate (inertial) range $k_F << k << k_D$

Now $\nu k^2 \bar{E}(k,t) = \bar{F}(k,t) \equiv 0$ for $k_F << k << K_D(\nu)$.
In the inertial band $k_F << k << k_D(\nu)$ integrate to get

$$\frac{d}{dt}\int_{k_1}^{k_2} \bar{E}(k,t) = \bar{\Phi}_k(k_2,t) - \bar{\Phi}_k(k_1,t)$$

In a statistically steady state, one ends up with

$$\bar{\Phi}_k(k_2) = \bar{\Phi}_k(k_1) = \bar{\Phi}_k(k_F) = \bar{\Phi}_k(k_D)$$

Constant energy flux across the inertial range $\bar{\Phi}_k(k) = \bar{\Phi}_0$.

Energy transfer through the inertial range

The energy balance, $\epsilon_T = \langle \mathbf{u}' \cdot \mathbf{f}' \rangle$, leads to the conclusion that

$$\bar{\Phi}_0 \equiv 2\epsilon_T = 2\langle \mathbf{u}' \cdot \mathbf{f}' \rangle$$

Dimensional analisys provides an estimate for the viscous scale,

$$\eta = \left(\frac{\nu^3}{\epsilon_T} \right)^{1/4}$$

$$\bar{E}(k) = h(k, \nu, \epsilon_T ; k_F) \qquad => \qquad \frac{\bar{E}}{\epsilon_T k^{-5/3}} = h^*(k\eta, \frac{k_F}{k})$$

- When $k >> k_F$, $E/(\epsilon_T k^{-5/3}) \simeq h^*(k\eta, 0) = h_0^*(k\eta)$
- In the inertial range, $k\eta << 1$, $h_0^*(k\eta) \simeq h_0^*(0) = C_K$

$$\bar{E}(k) \simeq C_k \epsilon_T k^{-5/3}$$

From KH to Kolmogorov

Homogeneous, isotropic, stationary turbulence

1. Define $\delta \mathbf{u} = \mathbf{u}_2 - \mathbf{u}_1$, $\delta u^2 = \delta \mathbf{u} \cdot \delta \mathbf{u}$
2. In the KH equation express (homogeneity)
   - $\nabla_r \cdot \langle \mathbf{u}_1 \cdot \mathbf{u}_2 \delta \mathbf{u} \rangle = 1/2 \nabla_r \cdot \langle \delta u^2 \delta \mathbf{u} \rangle$
   - $\langle \mathbf{u}_1 \cdot \mathbf{u}_2 \rangle = -1/2 \langle \delta u^2 \rangle + \langle u^2 \rangle$
   - $\langle \mathbf{f}_2 \cdot \mathbf{u}_1 + \mathbf{f}_1 \cdot \mathbf{u}_2 \rangle = -\langle \delta \mathbf{f} \cdot \delta \mathbf{u} \rangle + 2 \langle \mathbf{f} \cdot \mathbf{u} \rangle$
3. Use the turbulent kinetic energy equation $\langle \mathbf{f} \cdot \mathbf{u} \rangle = \epsilon_T$

It follows a form of the Kolmogorov equation (see e.g. Frish)

$$\frac{\partial \langle \delta u^2 \rangle}{\partial t} + \nabla_r \cdot \underbrace{\langle \delta u^2 \delta \mathbf{u} \rangle} = -4 \langle \epsilon_T \rangle + 2 \langle \delta \mathbf{f} \cdot \delta \mathbf{u} \rangle + 2 \nu \Delta_r \langle \delta u^2 \rangle$$

Flusso $\mathbf{\Phi}_r$ nello spazio delle scale

Kolmogorov equation

In isotropic conditions ($\hat{\mathbf{r}} = \mathbf{r}/r$)

$$\boldsymbol{\Phi}_r = \Phi_r \hat{\mathbf{r}}, \qquad \Phi_r = \delta u^2 \delta u_r$$

$$\nabla_r \cdot \boldsymbol{\Phi} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \Phi_r\right) \qquad \frac{1}{r^2}\frac{d}{dr}\left(r^2 \Phi_r\right) \simeq -4\langle \epsilon \rangle$$

$$\Phi_r = -\frac{4}{3}\langle \epsilon \rangle r + \text{corrections} + \text{unsteadiness}$$

- Small scale corrections due to diffusion by viscosity
- Large scale corrections due to velocity-forcing correlation
- Under isotropy we also have $\langle \delta u^2 \delta u_r \rangle = 5/3 \langle \delta u_r^3 \rangle$ $(\delta u_r \equiv \delta u_\mathrm{L})$

One finally has

$$\langle \delta u_L^3 \rangle = -\frac{4}{5}\epsilon_T r + 6\nu \frac{d\langle \delta u_L^2 \rangle}{dr}$$

Consequences of Kolmogorov equation

- In the inertial range $r >> \eta$

$$\langle \delta u_L^3 \rangle = -\frac{4}{5}\epsilon_T r$$

- In the dissipative range $r \simeq \eta$

$$\nu \frac{d\langle \delta u_L^2 \rangle}{dr} = \frac{2}{15}\epsilon_T r \qquad => \qquad \langle \delta u_L^2 \rangle = \frac{1}{15\nu}\epsilon_T r^2$$

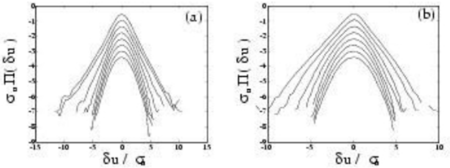
- In the inertial range $\delta u_L \propto \epsilon_T^{1/3} r^{1/3}$
- In the dissipative range $\delta u_L \propto \epsilon_T^{1/2} r$
- Order of magnitude estimate for the gradients

$$|\nabla \mathbf{u}'| \propto \sqrt{\frac{\epsilon_T}{15\nu}} = \mathcal{O}(1/\sqrt{\nu})$$

Kolmogorov '41 Theory

For isotropic ensembles the pdf of velocity increments

$$\delta u(r) = [\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})] \cdot \frac{\mathbf{r}}{r}$$



is invariant under rotations

For its characteristic function

$$\hat{p}(s,r) = \frac{1}{2\pi} \int p(\delta u, r) e^{-\jmath s\, \delta u} d\,(\delta u)$$

we have

$$\hat{p}(s,r) = \sum_k \frac{(-\jmath s)^k}{k!} \langle (\delta u)^k \rangle$$

K'41 Theory

The structure functions depend on ϵ and ν and dimensional analysis yields

$$\langle [\delta u(r)]^k \rangle = S_k(r; \epsilon, \nu) = C_k f_k(r/\eta) \, (\epsilon r)^{k/3}$$

where the Kolmogorov scale is

$$\eta = \left(\frac{\rho \nu^3}{\epsilon} \right)^{1/4}$$

In the Inertial Range ($r/\eta \to \infty$, $f_k \to 1$)

$$S_k(r) \propto \epsilon^{k/3} r^{k/3}$$

Intermittency

In fact dissipation is a highly intermittent field $\epsilon = \epsilon(\mathbf{x}, t)$.
In a ball $\mathcal{B}_r$ of radius r the spatial average dissipation

$$\epsilon_r = \frac{3}{4\pi r^3} \int_{\mathcal{B}_r} \epsilon(\mathbf{x}, t) d^3\mathbf{x}$$

is itself a stochastic variable, and its moments manifest scaling laws of the form

$$\langle (\epsilon_r)^k \rangle \propto r^{\tau(k)}$$

The flatness

$$F_4(r) = \frac{\langle (\epsilon_r)^4 \rangle}{\langle (\epsilon_r)^2 \rangle^2}$$

increases decreasing the scale in the inertial range ($\lim_{r\to 0} \lim_{Re\to\infty} F_4(r, Re) = \infty$).

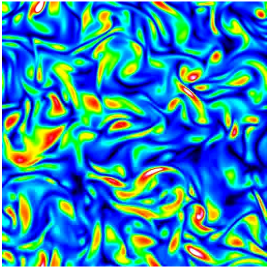
Intermittency corrections and K'62

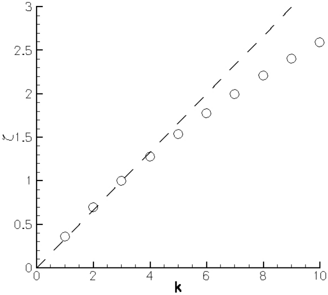
The intermittency of the dissipation field affects the structure functions

$$\langle (\delta u)^k \rangle \propto \langle (\epsilon_r)^{k/3} \rangle r^{k/3} \propto r^{\zeta(k)}$$

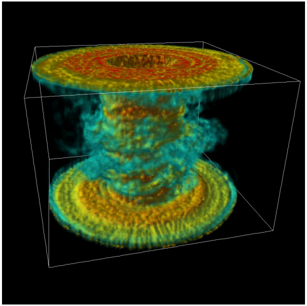
where the scaling exponents include intermittency corrections

$$\zeta(k) = k/3 + \tau(k/3)$$





Multifractal formalism



In a stat. homogeneous Hölder continous field $|\mathbf{u}(y) - \mathbf{u}(x)| \leq \not{c}|\mathbf{y} - \mathbf{x}|^{h(\mathbf{x})}$ the pdf $p(h)$ is $\mathbf{x}$-independent.

The exponent h is found on a set $\mathcal{S}(h)$ of dimension $D(h)$.

The probability a ball $\mathcal{B}_r$ intersects $\mathcal{S}(h)$ is $P \propto r^{3-D(h)}$

The scaling $\delta u^k \propto r^{kh_0}$, $h_0 \in [h, h+dh]$, occurs with probability $p(h) r^{3-D(h)} dh$

$$\langle (\delta u)^k \rangle \propto \int p(h) r^{kh+3-D(h)} dh$$

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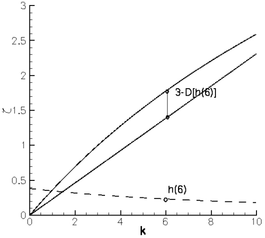
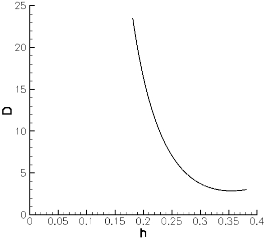
Multifractal formalism II





$$\langle (\delta u)^k \rangle \propto r^{\zeta(k)}$$

$$\zeta(k) = \min_h \left[kh + 3 - D(h) \right]$$

$$h = \tilde{h}(k) : \qquad \frac{d}{dh}\left[kh + 3 - D(h) \right] = 0$$

$$\zeta(k) = k\tilde{h}(k) + 3 - D\left[\tilde{h}(k)\right]$$

$$\frac{d\zeta(k)}{dk} = \tilde{h}(k)$$

$$D\left[\tilde{h}(k)\right] = k\frac{d\zeta(k)}{dk} - \zeta(k) + 3$$

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