LEMMA 3.3 (Hamilton's Trick **[44]**). *Let $u : M \times (0, T) \to \mathbb{R}$ be a C^1 function such that for every time t, there exists a value $\delta > 0$ and a compact subset $K \subset M \setminus \partial M$ such that at every time $t' \in (t - \delta, t + \delta)$ the maximum $u_{\max}(t') = \max_{p \in M} u(p, t')$ is attained at least at one point of K.*
Then, $u_{\max}$ is a locally Lipschitz function in $(0, T)$ and at every differentiability time $t \in (0, T)$ we have

$$\frac{du_{\max}(t)}{dt} = \frac{\partial u(p, t)}{\partial t}$$

where $p \in M$ is any inner point such that $u(\cdot, t)$ gets its maximum at p.

PROOF. Fixing $t \in (0, T)$, we have $\delta > 0$ and K as in the hypotheses, hence on $K \times (t - \delta, t + \delta)$ the function u is Lipschitz with some constant C. Consider a value $0 < \varepsilon < \delta$, then we have

$$u_{\max}(t + \varepsilon) = u(q, t + \varepsilon) \leq u(q, t) + \varepsilon C \leq u_{\max}(t) + \varepsilon C \,,$$

for some $q \in K$, hence,

$$\frac{u_{\max}(t + \varepsilon) - u_{\max}(t)}{\varepsilon} \leq C \,.$$

Analogously,

$$u_{\max}(t) = u(p, t) \leq u(p, t + \varepsilon) + \varepsilon C \leq u_{\max}(t + \varepsilon) + \varepsilon C \,,$$

for some $p \in K$, hence,

$$\frac{u_{\max}(t) - u_{\max}(t + \varepsilon)}{\varepsilon} \leq C \,.$$

With the same argument, considering $-\delta < \varepsilon < 0$, we conclude that $u_{\max}$ is a locally Lipschitz function in $(0, T)$, hence differentiable at almost every time.
Suppose that t is one of such times, let p be a point in the nonempty set $\{p \in M \setminus \partial M \,|\, u(p, t) = u_{\max}(t)\}$.
By Lagrange's Theorem, for every $0 < \varepsilon < \delta$, $u(p, t + \varepsilon) = u(p, t) + \varepsilon \frac{\partial u(p, \xi)}{\partial t}$ for some ξ, hence

$$u_{\max}(t + \varepsilon) \geq u(p, t + \varepsilon) = u_{\max}(t) + \varepsilon \frac{\partial u(p, \xi)}{\partial t} \,,$$

which implies, if we choose $\varepsilon > 0$,

$$\frac{u_{\max}(t + \varepsilon) - u_{\max}(t)}{\varepsilon} \geq \frac{\partial u(p, \xi)}{\partial t} \,.$$

Sending ε to zero, we get $u'_{\max}(t) \geq \frac{\partial u(p, t)}{\partial t}$.
If instead we choose $-\delta < \varepsilon < 0$,

$$\frac{u_{\max}(t + \varepsilon) - u_{\max}(t)}{\varepsilon} \leq \frac{\partial u(p, \xi)}{\partial t}$$

and when $\varepsilon \to 0$, we have $u'_{\max}(t) \leq \frac{\partial u(p, t)}{\partial t}$. Thus, we are done. $\square$