In the general case, repeating this argument, unfortunately, one could obtain a union of hyperplanes, or even more disturbing, integer multiplicity hyperplanes. Hence, a possible flat limit.

REMARK 4.32. Another line to produce a homothetic blow up limit, is to apply, instead of Stone's argument, White's Theorem 4.24, excluding the presence of singularities in the case $\Sigma = 1$ (recall Definition 4.7). As the set of reachable points $\mathcal{S}$ is compact, if $\Sigma > 1$ there must exists a point $x_0 = \widehat{p}$ such that $\Theta(p) > 1$, otherwise, by a covering argument, White's Theorem implies that the curvature is bounded as $t \nearrow T$.

REMARK 4.33. Finally, we can also obtain a homothetic limit by rescaling the hypersurfaces around *moving* points as follows. Rescaling the *maximal* monotonicity formula (4.6) around the points x_t which are the maximum point which realize $\sigma(t)$ in Definition 4.7, that is,

$$\sigma(t) = \max_{x_0 \in \mathbb{R}^{n+1}} \int_M \frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\, d\mu_t = \int_M \frac{e^{-\frac{|x-x_t|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\, d\mu_t\,,$$

where now the rescaled hypersurfaces with associated measures $\widetilde{\mu}_s$ are given by

$$\widetilde{\varphi}(q,s) = \frac{\varphi(q,t(s)) - x_t}{\sqrt{2(T-t(s))}} \qquad s = s(t) = -\frac{1}{2}\log(T-t)\,,$$

we get

$$\frac{d}{ds}\int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_s = -\int_M e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s \leq 0\,.$$

It follows that, integrating this formula as before, we get

$$\sigma(0) - \Sigma = \int_{-\frac{1}{2}\log T}^{+\infty} \int_M e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s\, ds < +\infty\,,$$

and with the same argument we can produce a homothetic limit hypersurface $\widetilde{M}_\infty$ such that

$$\int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}}\, d\widetilde{\mathcal{H}}^n = \Sigma \geq 1\,.$$

Since when $\Sigma = 1$ the curvature is bounded, the limit hypersurface $\widetilde{M}_\infty$ cannot be a single hyperplane for the origin. If the initial hypersurface was embedded, this limit also cannot be flat.

4. Embedded Hypersurfaces with Nonnegative Mean Curvature

If the compact initial hypersurface is embedded and has $\mathrm{H} \geq 0$ (this condition is often called *mean convexity*) or at some time the evolving hypersurface achieve it, then the analysis of the previous section can be pushed forward, since we have a new condition that all the possible limits of rescaled hypersurfaces have to satisfy. Under such hypothesis, Problem 4.28 and consequently Problem 4.31 have a satisfying solution.

Actually, in this class, *every* singular point is a special singular point and it is indeed possible to classify all the embedded hypersurfaces in $\mathbb{R}^{n+1}$ such that $\mathrm{H}+\langle x\,|\,\nu\rangle=0$ and $\mathrm{H}\geq 0$, arising from a rescaling around a singularity point, see **[53, 65]** or **[90]**.

We recall here that in this case, after a positive time $\varepsilon>0$, there exist a constant $\alpha>0$ such that $\alpha|\mathrm{A}|\leq\mathrm{H}\leq n|\mathrm{A}|$ everywhere on M for every time $t\geq\varepsilon$, Corollary 3.22.
Hence, for every $t\in[\varepsilon,T)$ we have

$$\frac{\alpha}{\sqrt{2(T-t)}}\leq\max_{p\in M}\mathrm{H}(p,t)\leq\frac{C}{\sqrt{2(T-t)}}\,.$$

PROPOSITION 4.34 (Huisken **[53, 65]**). *Let $M\subset\mathbb{R}^{n+1}$ be a mean convex, smooth, embedded hypersurface in $\mathbb{R}^{n+1}$ such that $\mathrm{H}+\langle x\,|\,\nu\rangle=0$ at every $x\in M$ and there exists a constant C such that $|\mathrm{A}|,|\nabla\mathrm{A}|\leq C$ and $\mathcal{H}^n(M\cap B_R)\leq Ce^R$, for every ball of radius $R>0$ in $\mathbb{R}^{n+1}$. Then, up to rotation in $\mathbb{R}^{n+1}$, M must be one of only $(n+1)$ possible hypersurfaces, namely, either a hyperplane for the origin, or the sphere $\mathbb{S}^n(\sqrt{n})$ or one of the cylinders $\mathbb{S}^m(\sqrt{m})\times\mathbb{R}^{n-m}$.*

In the special one–dimensional case the only embedded smooth curves in $\mathbb{R}^2$ satisfying the structural equation $k+\langle x\,|\,\nu\rangle=0$ are the lines through the origin and the unit circle.

PROOF. Let us assume that M is connected. If the theorem is true in this case, it it easy to see that it is not possible to have a nonconnected embedded hypersurface satisfying the hypotheses. Indeed, any connected component has to belong to the list of the statement and every two hypersurfaces in such list either coincide or have some intersections.

We deal separately with the case $n=1$.
Fixing a reference point on a curve γ we have an arclength parameter s which gives a unit tangent vector field and a unit normal vector field ν which is the rotation of $\pi/2$ in $\mathbb{R}^2$ of the vector τ. Then, it follows that $k=\langle\partial_s\tau\,|\,\nu\rangle$.
The relation $k=-\langle\gamma\,|\,\nu\rangle$ implies $k_s=k\langle\gamma\,|\,\tau\rangle$. Suppose that at some point $k=0$, then also $k_s=0$ at the same point, hence, by the uniqueness theorem applied to this ODE for the curvature k we can conclude that k is identically zero and we are dealing with a line L, which then, as $\langle x\,|\,\nu\rangle=0$ for every $x\in L$, it must pass for the origin of $\mathbb{R}^2$.
So we suppose that k is always nonzero and possibly reversing the orientation of the curve we can also assume that $k>0$ at every point, that is, the curve is strictly convex.
Computing the derivative of $|\gamma|^2$,

$$\partial_s|\gamma|^2=2\langle\gamma\,|\,\tau\rangle=2k_s/k=2\partial_s\log k$$

we get $k=Ce^{|\gamma|^2/2}$ for some constant $C>0$, so if k is bounded above and below away from zero, the curve is also bounded in $\mathbb{R}^2$, hence, it is compact being embedded. As a consequence, it is closed.
We consider now a new coordinate $\theta=\arccos\langle e_1\,|\,\nu\rangle$, this can be done globally as we know that the curve is convex.
Then, we have $\partial_s\theta=k$ and

$$k_\theta=k_s/k=\langle\gamma\,|\,\tau\rangle\qquad k_{\theta\theta}=\frac{\partial_s k_\theta}{k}=\frac{1+k\langle\gamma\,|\,\nu\rangle}{k}=\frac{1}{k}-k\,.$$

Multiplying both sides for $2k_\theta$ we get $\partial_\theta[k_\theta^2 + k^2 - \log k^2] = 0$, that is, the quantity $k_\theta^2 + k^2 - \log k^2$ is equal to some constant E along all the curve. Notice that such quantity E cannot be less than 1, moreover, if $E = 1$ we have k constant equal to one and the curve must be the unit circle.
For other values of $E > 1$ it is easy to see, as the function, $x - \log x$ is convex, that k must be bounded above and below away from zero, hence, by what we said before the curve is a simple closed curve.
We look now at the critical points of the curvature k, they must be isolated (hence finite) and non degenerate ($k_{\theta\theta} \neq 0$), otherwise the ODE $k_{\theta\theta} = \frac{1}{k} - k$ implies that k_θ is identically zero, k is constant and again we are dealing with the unit circle.
Suppose now that k_- and k_+ are a pair of consecutive critical values of k, hence the two distinct positive zeroes of the function $E + \log k^2 - k^2$ when $E > 1$.
We have that the change $\Delta\theta$ in the angle θ along the piece of curve from the points corresponding to k_- and k_+ on γ is given by the integral

$$I(E) = \int_{k_-}^{k_+} \frac{dk}{\sqrt{E - k^2 + \log k^2}}\,.$$

As the four vertex theorem **[72, 77]** says that there are at least four critical points of k on the curve, there must be at least four pieces like the one above, hence, the total change in the angle θ along the curve must be at least $4I(E)$.
As the curve γ is simple, the total change must be 2π, so we have $4I(E) \leq 2\pi$, that is,

$$I(E) = \int_{k_-}^{k_+} \frac{dk}{\sqrt{E - k^2 + \log k^2}} \leq \pi/2\,.$$

The analysis of Abresch and Langer in **[1]** (see also the work of Epstein and Weinstein **[31]**) shows that $I(E)$ is always strictly larger than $\pi/2$ for every $E > 1$, which is a contradiction and γ must be a circle.

Actually, Abresch and Langer (and also Epstein and Weinstein) classify *all* the closed curves in $\mathbb{R}^2$ satisfying the structural equation $k + \langle \gamma \,|\, \nu \rangle = 0$.
We remark that, like for other results, the one–dimensional case does not follows from the general one below. Moreover, even if the study of the integral $I(E)$ is done with elementary tools, the proof of the inequality $I(E) > \pi/2$ is quite involved making definitely nontrivial this classification result even for simple closed curves (we underline that the n–dimensional generalization, Problem 4.35, is open).

Suppose now that $n \geq 2$.
By covariant differentiation of the equation $\mathrm{H} + \langle x \,|\, \nu \rangle = 0$ in an orthonormal frame $\{e_1, \dots, e_n\}$ on M we get

$$\nabla_i \mathrm{H} = \langle x \,|\, e_k \rangle h_{ik}$$
$$\nabla_i \nabla_j \mathrm{H} = h_{ij} + \langle x \,|\, \nu \rangle h_{ik} h_{jk} + \langle x \,|\, e_k \rangle \nabla_i h_{jk} = h_{ij} - \mathrm{H} h_{ik} h_{jk} + \langle x \,|\, e_k \rangle \nabla_k h_{ij} \tag{4.14}$$

where we used Codazzi and Gauss–Weingarten equations.
Contracting now with g^{ij} and h^{ij} respectively we have

$$\Delta \mathrm{H} = \mathrm{H} - \mathrm{H}|\mathrm{A}|^2 + \langle x \,|\, e_k \rangle \nabla_k \mathrm{H} = \mathrm{H}(1 - |\mathrm{A}|^2) + \langle x \,|\, \nabla \mathrm{H} \rangle \tag{4.15}$$

$$h^{ij}\nabla_i\nabla_j\mathrm{H} = |\mathrm{A}|^2 - \mathrm{H}\mathrm{tr}(\mathrm{A}^3) + \langle x\,|\,e_k\rangle\nabla_k|\mathrm{A}|^2/2$$

which implies, by Simon's identity (2.3),

$$\Delta|\mathrm{A}|^2 = 2|\mathrm{A}|^2(1-|\mathrm{A}|^2) + 2|\nabla\mathrm{A}|^2 + \langle x\,|\,\nabla|\mathrm{A}|^2\rangle\,. \tag{4.16}$$

From equation (4.15) and the strong maximum principle for elliptic equations we see that, since M satisfies $\mathrm{H} \geq 0$ by assumption and $\Delta\mathrm{H} \leq \mathrm{H} + \langle x\,|\,\nabla\mathrm{H}\rangle$, we must either have that $\mathrm{H} \equiv 0$ or $\mathrm{H} > 0$ on all M.
Of these two possibilities the situation that $\mathrm{H} \equiv 0$ is easily handled: as x is tangent vector field on M, by the equation $\langle x\,|\,\nu\rangle = 0$, there is a solution of the ODE $\gamma'(t) = x(\gamma(t)) = \gamma(t)$ in M for $t \in \mathbb{R}$, but the solution is simply the line passing by x and the origin in $\mathbb{R}^{n+1}$, so M has to be a cone in $\mathbb{R}^{n+1}$. Being M smooth, the only possibility is that M is a hyperplane through the origin of $\mathbb{R}^{n+1}$.
Therefore we may assume henceforth, as we do, that the mean curvature satisfies the strict inequality $\mathrm{H} > 0$ everywhere (so that division by H and $|\mathrm{A}|$ is allowed).
Now let $R > 0$ and define η to be the inward unit conormal to $M \cap B_R(0)$ along $\partial\big(M \cap B_R(0)\big)$, which is a smooth boundary for almost every $R > 0$ (by Sard's theorem). Then, supposing that R is a *regular value* for the function $|x|$ on M, from equation (4.15) and the divergence theorem, we obtain

$$\begin{aligned}
\varepsilon_R &= -\int_{\partial(M\cap B_R(0))} |\mathrm{A}|\langle\nabla\mathrm{H}\,|\,\eta\rangle e^{-R^2/2}\,d\mathcal{H}^{n-1} \qquad (4.17)\\
&= \int_{M\cap B_R(0)} |\mathrm{A}|\Delta\mathrm{H}e^{-|x|^2/2} + \langle\nabla(|\mathrm{A}|e^{-|x|^2/2})\,|\,\nabla\mathrm{H}\rangle\,d\mathcal{H}^n\\
&= \int_{M\cap B_R(0)} |\mathrm{A}|\mathrm{H}(1-|\mathrm{A}|^2)e^{-|x|^2/2} + |\mathrm{A}|\langle x\,|\,\nabla\mathrm{H}\rangle e^{-|x|^2/2}\,d\mathcal{H}^n\\
&\quad + \int_{M\cap B_R(0)} \frac{1}{2|\mathrm{A}|}\langle\nabla|\mathrm{A}|^2\,|\,\nabla\mathrm{H}\rangle e^{-|x|^2/2} - |\mathrm{A}|\langle x\,|\,\nabla\mathrm{H}\rangle e^{-|x|^2/2}\,d\mathcal{H}^n\\
&= \int_{M\cap B_R(0)} \left(|\mathrm{A}|\mathrm{H}(1-|\mathrm{A}|^2) + \frac{1}{2|\mathrm{A}|}\langle\nabla|\mathrm{A}|^2\,|\,\nabla\mathrm{H}\rangle\right)e^{-|x|^2/2}\,d\mathcal{H}^n\,.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
\delta_R = & -\int_{\partial(M\cap B_R(0))} \frac{\mathrm{H}}{|\mathrm{A}|}\langle\nabla|\mathrm{A}|^2\,|\,\eta\rangle e^{-R^2/2}\,d\mathcal{H}^{n-1} \qquad (4.18)\\
= & \int_{M\cap B_R(0)} \frac{\mathrm{H}}{|\mathrm{A}|}\Delta|\mathrm{A}|^2 e^{-|x|^2/2} + \Big\langle\nabla\Big(\frac{\mathrm{H}}{|\mathrm{A}|}|e^{-|x|^2/2}\Big)\,\Big|\,\nabla|\mathrm{A}|^2\Big\rangle\,d\mathcal{H}^n\\
= & \int_{M\cap B_R(0)} 2|\mathrm{A}|\mathrm{H}(1-|\mathrm{A}|^2)e^{-|x|^2/2} + \frac{2\mathrm{H}|\nabla\mathrm{A}|^2}{|\mathrm{A}|}e^{-|x|^2/2} + \frac{\mathrm{H}}{|\mathrm{A}|}\langle x\,|\,\nabla|\mathrm{A}|^2\rangle e^{-|x|^2/2}\,d\mathcal{H}^n\\
& + \int_{M\cap B_R(0)} \frac{\langle\nabla\mathrm{H}\,|\,\nabla|\mathrm{A}|^2\rangle}{|\mathrm{A}|}e^{-|x|^2/2} - \frac{\mathrm{H}|\nabla|\mathrm{A}|^2|^2}{2|\mathrm{A}|^3}e^{-|x|^2/2} - \frac{\mathrm{H}}{|\mathrm{A}|}\langle x\,|\,\nabla|\mathrm{A}|^2\rangle e^{-|x|^2/2}\,d\mathcal{H}^n\\
= & \int_{M\cap B_R(0)} \Big(2|\mathrm{A}|\mathrm{H}(1-|\mathrm{A}|^2) + \frac{2\mathrm{H}|\nabla\mathrm{A}|^2}{|\mathrm{A}|} + \frac{\langle\nabla\mathrm{H}\,|\,\nabla|\mathrm{A}|^2\rangle}{|\mathrm{A}|} - \frac{\mathrm{H}|\nabla|\mathrm{A}|^2|^2}{2|\mathrm{A}|^3}\Big)\,e^{-|x|^2/2}\,d\mathcal{H}^n\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sigma_R = 2\delta_R - 4\varepsilon_R = & \int_{M\cap B_R(0)} \Big(\frac{4\mathrm{H}|\nabla\mathrm{A}|^2}{|\mathrm{A}|} - \frac{\mathrm{H}|\nabla|\mathrm{A}|^2|^2}{|\mathrm{A}|^3}\Big)\,e^{-|x|^2/2}\,d\mathcal{H}^n \qquad (4.19)\\
= & \int_{M\cap B_R(0)} (4|\mathrm{A}|^2|\nabla\mathrm{A}|^2 - |\nabla|\mathrm{A}|^2|^2)\frac{\mathrm{H}}{|\mathrm{A}|^3}e^{-|x|^2/2}\,d\mathcal{H}^n\,.
\end{aligned}
$$

As we have $4|\mathrm{A}|^2|\nabla\mathrm{A}|^2 \geq |\nabla|\mathrm{A}|^2|^2$, this quantity σ_R is nonnegative and nondecreasing in R.
If now we show that $\liminf_{R\to+\infty}\sigma_R = 0$ (on the set of regular values) we can conclude that at every point of M,

$$
4|\mathrm{A}|^2|\nabla\mathrm{A}|^2 = |\nabla|\mathrm{A}|^2|^2\,. \qquad (4.20)
$$

We have,

$$
\begin{aligned}
|\sigma_R| = & \Big|-2\int_{\partial(M\cap B_R(0))} \frac{\mathrm{H}}{|\mathrm{A}|}\langle\nabla|\mathrm{A}|^2\,|\,\eta\rangle e^{-R^2/2}\,d\mathcal{H}^{n-1} + 4\int_{\partial(M\cap B_R(0))} |\mathrm{A}|\langle\nabla\mathrm{H}\,|\,\eta\rangle e^{-R^2/2}\,d\mathcal{H}^{n-1}\Big|\\
\leq & \,4e^{-R^2/2}\int_{\partial(M\cap B_R(0))} \frac{\mathrm{H}}{|\mathrm{A}|}|\nabla|\mathrm{A}|^2| + |\mathrm{A}||\nabla\mathrm{H}|\,d\mathcal{H}^{n-1}\\
\leq & \,8e^{-R^2/2}\int_{\partial(M\cap B_R(0))} \mathrm{H}|\nabla\mathrm{A}| + |\mathrm{A}||\nabla\mathrm{H}|\,d\mathcal{H}^{n-1}\\
\leq & \,Ce^{-R^2/2}\mathcal{H}^{n-1}(\partial(M\cap B_R(0)))\,,
\end{aligned}
$$

by the estimates on A and $\nabla\mathrm{A}$ in the hypotheses.
Now, suppose that definitely on the set of regular values in $\mathbb{R}^+$ we have

$$
\mathcal{H}^{n-1}(\partial(M\cap B_R(0))) \geq \delta Re^{R^2/4}
$$

for some constant $\delta > 0$, for every $R > r_1$. Setting x^M to be the projection of the vector x on the tangent space to M, as the function $R \mapsto \mathcal{H}^n(M\cap B_R(0))$ is monotone and continuous from the left and actually continuous at every regular value of $|x|$ on M, we can differentiate it almost everywhere in $\mathbb{R}^+$ and we have (by the coarea formula, see **[32]** or **[82]**),

$$\begin{aligned}\mathcal{H}^n(M\cap B_R(0))-\mathcal{H}^n(M\cap B_r(0)) &\geq \int_r^R \frac{d}{d\xi}\mathcal{H}^n(M\cap B_\xi(0))\,d\xi\\ &\geq \int_r^R\int_{\partial(M\cap B_\xi(0))}|\nabla^M|x||^{-1}\,d\mathcal{H}^{n-1}\,d\xi\\ &= \int_r^R\int_{\partial(M\cap B_\xi(0))}|x|/|x^M|\,d\mathcal{H}^{n-1}\,d\xi\\ &\geq \int_r^R\int_{\partial(M\cap B_\xi(0))}d\mathcal{H}^{n-1}\,d\xi\,,\end{aligned}$$

where the derivative in the integral is taken only at the points where it exists and $\nabla^M|x|$ denotes the projection of the gradient of the function $|x|$ on the tangent space to M. Hence, if r is larger than r_1,

$$\begin{aligned}\mathcal{H}^n(M\cap B_R(0))-\mathcal{H}^n(M\cap B_r(0)) &\geq \int_r^R\int_{\partial(M\cap B_\xi(0))}d\mathcal{H}^{n-1}\,d\xi\\ &\geq \delta\int_r^R \xi e^{\xi^2/4}\,d\xi\\ &= 2\delta(e^{R^2/4}-e^{r^2/4})\end{aligned}$$

so if R goes to $+\infty$, the quantity $\mathcal{H}^n(M\cap B_R(0))e^{-R}$ diverges, contradicting the hypotheses in the statement. Hence, the $\liminf$ on the set of regular values as R goes to $+\infty$ of the quantity $e^{-R^2/4}\mathcal{H}^{n-1}(\partial(M\cap B_R(0)))$ has to be zero. It follows the same for σ_R and equation (4.20) holds.

Making explicit such equation, by the equality condition in the Cauchy–Schwartz inequality, it immediately follows that, fixing k, at every point there exists a constant c_k such that

$$\nabla_k h_{ij} = c_k h_{ij}$$

for every i,j. Tracing with the metric and with h^{ij}, we get $\nabla_k \mathrm{H} = c_k\mathrm{H}$ and $\nabla_k|\mathrm{A}|^2 = 2c_k|\mathrm{A}|^2$, hence $c_k = \nabla_k \log \mathrm{H}$ and $\nabla_k \log|\mathrm{A}|^2 = 2c_k = 2\nabla_k \log \mathrm{H}$.
This implies that locally $|\mathrm{A}| = \alpha\mathrm{H}$ for some constant $\alpha > 0$, by connectedness, this relation has to hold globally on M.
Suppose now that at a point $|\nabla\mathrm{H}| \neq 0$, then, $\nabla_k h_{ij} = c_k h_{ij} = \frac{\nabla_k \mathrm{H}}{\mathrm{H}} h_{ij}$ which is a symmetric 3–tensor by Codazzi equations, hence, $\nabla_k \mathrm{H} h_{ij} = \nabla_j \mathrm{H} h_{ik}$. Computing then in normal coordinates, with an orthonormal basis $\{e_1 \ldots, e_n\}$ such that $e_1 = \nabla\mathrm{H}/|\nabla\mathrm{H}|$ we have

$$0 = |\nabla_k \mathrm{H} h_{ij} - \nabla_j \mathrm{H} h_{ik}|^2 = 2|\nabla\mathrm{H}|^2\left(|\mathrm{A}|^2 - \sum_{i=1}^n h_{1i}^2\right).$$

Hence, $|A|^2 = \sum_i^n h_{1i}^2$ then

$$|A|^2 = h_{11}^2 + 2\sum_{i=2}^n h_{1i}^2 + \sum_{i,j\neq 1}^n h_{ij} = |A|^2 \sum_{i=2}^n h_{1i}^2 + \sum_{i,j\neq 1}^n h_{ij}$$

so $h_{ij} = 0$ unless $i = j = 1$. This means that A has rank one.
Thus, we have two possible (non mutually excluding) situations at every point of M: either A has rank one or $\nabla H = 0$.

If the kernel of A is empty everywhere, A must have rank at least two, as we assumed $n \geq 2$, then we have $\nabla H = 0$ which implies $\nabla A = 0$ and, by equation (4.14) $h_{ij} = H h_{ik} h_{kj}$. This means that all the eigenvalues of A are 0 or $1/H$. As the kernel is empty, $A = Hg/n$, precisely $H = \sqrt{n}$ and $A = g/\sqrt{n}$. Then, the hypersurface M has to be the sphere $\mathbb{S}^n(\sqrt{n})$.
Indeed, computing

$$\Delta|x|^2 = 2n + 2\langle x \,|\, \Delta x\rangle = 2n + 2H\langle x \,|\, \nu\rangle = 2n - 2H^2 = 0\,,$$

by the structural equation $H + \langle x \,|\, \nu\rangle = 0$, being $|x|^2$ a harmonic function on M, looking at the point of M of minimum distance from the origin, by the strong maximum principle for elliptic equations, it must be constant on M.

We suppose now that the kernel of A is not empty at some point $p \in M$, then let $v_1(p), \dots, v_{n-m}(p) \in \mathbb{R}^{n+1}$ be a family of unit orthonormal tangent vectors spanning such $(n-m)$–dimensional kernel, that is $h_{ij} v_k^j = 0$. Then the geodesic $\gamma(s)$ in M from p with initial velocity $v_k(p)$ satisfies

$$\nabla_s(h_{ij}\gamma_s^j) = H^{-1}\langle \nabla H \,|\, \gamma_s\rangle h_{ij}\gamma_s^j$$

hence, by Gronwall's lemma, it holds $h_{ij}(\gamma(s))\gamma_s^j(s) = 0$ for every s.
Being γ a geodesic in M, the normal to the curve in $\mathbb{R}^{n+1}$ is the normal to M, then setting k to be the curvature of γ in $\mathbb{R}^{n+1}$, we have

$$k = \left\langle \nu \,\middle|\, \frac{d}{ds}\gamma_s \right\rangle = h_{ij}\gamma_s^i\gamma_s^j = 0\,,$$

thus γ is a line in $\mathbb{R}^{n+1}$.
Hence, all the $(n-m)$–dimensional affine subspace $p + S(p) \subset \mathbb{R}^{n+1}$ is contained in M, where we set $S(p) = \langle v_1(p), \dots, v_m(p)\rangle \subset \mathbb{R}^{n+1}$.
Let now $\sigma(s)$ a geodesic from p to another point q, parametrized in arclength, and extend by parallel transport the vectors v_k along σ,

$$\nabla_s(h_{ij}v_k^j) = \langle \nabla H \,|\, \sigma_s\rangle h_{ij}v_k^j$$

and again by Gronwall's lemma, $h_{ij}v_k^j(s) =$ for every s, in particular $v_k(q)$ is contained in the kernel of A at $q \in M$. This argument clearly shows that the kernel of $S(p)$ of A at p has constant dimension $(n-m)$ with $0 < m < n$ (as A is never zero) and all the affine $(n-m)$–dimensional subspaces $p + S(p) \subset \mathbb{R}^{n+1}$ are all contained in M.

Moreover, as $h_{ij}v_k^j = 0$ along such geodesic, looking at things in $\mathbb{R}^{n+1}$, denoting with D the covariant derivative in $\mathbb{R}^{n+1}$, we have

$$D_s v_k = \nabla_s v_k + h_{ij} v_k^j \sigma_s^i \nu = 0$$

so the subspaces $S(p)$ are all a common $(n-m)$–dimensional vector subspace that we denote with S and $M = M + S \subset \mathbb{R}^{n+1}$.
By Sard's theorem, there exist a vector $y \in S$ such that $N = M \cap (y + S^\perp)$ is a smooth m–dimensional submanifold of $\mathbb{R}^{n+1}$, then as $M = M + S$, it is easy to see that $M = N \times S$, but this means that $L = S^\perp \cap M$ is a smooth m–dimensional submanifold of $S^\perp = \mathbb{R}^{m+1}$ with $M = L \times S$.
Moreover, as S is in the tangent space to every point of L, the normal ν to M at a point of L stays in $S^\perp = \mathbb{R}^{m+1}$ so it coincides with the normal to L in $S^\perp = \mathbb{R}^{m+1}$, then a simple computation shows that the mean curvature of M at the points of L is equal to the mean curvature of L as a hypersurface in $S^\perp = \mathbb{R}^{m+1}$. This shows that L is a hypersurface in $\mathbb{R}^{m+1}$ satisfying the relative structural equation. Finally, as, by construction, the second fundamental form of L has empty kernel, by the previous discussion, $L = \mathbb{S}^m(\sqrt{m})$ and $M = \mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$ and we are done.

□

OPEN PROBLEM 4.35. Without the assumption $\mathrm{H} > 0$ this result is not true, an example is the Angenent torus **[13]**. It is an open question if there exists a smooth embedding of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ such that $\mathrm{H} + \langle x \,|\, \nu \rangle = 0$, different by the unit sphere.

COROLLARY 4.36. *Every limit hypersurface obtained by rescaling around a type I singularity point of the motion by mean curvature of a compact, embedded initial hypersurface with* $\mathrm{H} \geq 0$, *up to rotation in* $\mathbb{R}^{n+1}$, M *must be either a hyperplane for the origin, or the sphere* $\mathbb{S}^n(\sqrt{n})$ *or one of the cylinders* $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$.

We discuss now what are the possible values of the limit heat density function, following Stone **[89]**. As the value of $\Theta(p)$ is the Huisken's functional on any limit of rescaled hypersurfaces, and these latter are "finite", we have that the possible values are 1 in the case of a hyperplane and

$$\Theta^{n,m} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}} e^{-\frac{|x|^2}{2}} \, d\mathcal{H}^n$$

for $m \in \{1, \dots, n\}$.
A straightforward computation gives for $m > 0$

$$\Theta^{n,m} = \left(\frac{m}{2\pi e}\right)^{m/2} \omega_m$$

where ω_m denotes the volume of the unit m–sphere. Notice that $\Theta^{n,m}$ does not depend on n, so we can simply write $\Theta^m = \Theta^{n,m}$.

LEMMA 4.37 (Stone **[89]**). *The values of* Θ^m *are all distinct and larger than* 1 *for* $m > 0$. *Indeed the numbers* $\{\Theta^m : m = 1, 2, \dots\}$ *form a strictly decreasing sequence in* m, *with* $\Theta^m \searrow \sqrt{2}$ *as* $m \to \infty$.