The functional determinant

- It is constructed out of the spectrum of the Laplacian (or some more general operator)
- It has interesting relations with conformal geometry and sharp inequalities (M-T, Onofri, Beckner, etc...)
- It has interest in string theory (related to the weight of integration over all surface, and to the energy spectrum, which is measurable.

1. Eigenvalues of the Laplace operator   Ref Chavel: eigenv. in Riem geom ①

We consider a compact closed manifold with metric g. In a local system of coordinates the Laplace-Beltrami operator of (M,g) is defined as

$\Delta_g = \frac{1}{\sqrt{\det g}} \partial_i \left(g^{ij} \sqrt{\det g}\, \partial_j u \right)$. This is self adjoint and

$$-\int_M (\Delta_g u) v \, dV_g = \int g(\nabla u, \nabla v)\, dV_g = \int_M g^{ij} \partial_i u \, \partial_j v \sqrt{\det g}\, dx$$

It is well known that $L^2(M)$ admits an o.n. basis of eigenfunctions $(\phi_i)_i$ which satisfy $-\Delta \phi_i = \lambda_i \phi_i$, $\lambda_i \to +\infty$ as $i \to +\infty$.

<u>Rayleigh and min-max methods</u>

What is an efficient method to estimate the λ_j's?

<u>Thm</u> Let $(\lambda_i)_i$ be the eigenvalues of $-\Delta$ counted with multiplicity.

Then $\forall K$ $\lambda_K = \inf \left\{ \left\{ \sup_{u \in V_K} \frac{\int |\nabla u|^2}{\int u^2} \mid u \neq 0 \right\} \mid V_K \text{ } K \text{ dim subspace of } H^1(M) \right\}$

Moreover $\lambda_K = \sup_{\tilde{V}_{K-1}} \left\{ \inf_{u \perp_{L^2} \tilde{V}_{K-1}} \left\{ \frac{\int |\nabla u|^2}{\int u^2} \mid u \neq 0 \right\} \mid \tilde{V}_{K-1} \text{ } (K-1)\text{-dim subspace of } L^2(M) \right\}$

<u>Rem</u> ① $\lambda_1 = \inf \left\{ \frac{\int |\nabla u|^2}{\int u^2} \mid u \neq 0 \quad u \in H^1(M) \right\}$

② The same conclusion holds if Ω is a domain in M or in $\mathbb{R}^n$, with Dirichlet boundary conditions $(H_0^1(\Omega))$, or Neumann

<u>Cor 1</u> (Domain monotonicity of eigenvalues) Suppose $\Omega \subseteq M$ $(\mathbb{R}^n)$ is a smooth domain and $\Omega_1, \ldots, \Omega_m$ be pairwise disjoint piecewise

smooth domains whose bdries intersect $\partial\Omega$ transv.. Given an eigenvalue problem on Ω (Dirichlet or Neumann) with eigenvalue λ_k, for all $r = 1, \dots, m$ consider the eigenv. problem on Ω_r obtained imposing Dirichlet data in $\partial\Omega_r \cap \Omega$ and the same data on $\partial\Omega \cap \bar{\Omega}_r$. Arrange all the eigenv. ν_i (with mult.) in increasing order. Then $\forall K$ $\lambda_k \leq \nu_k$. (2)

Pf Let $\phi_1, \dots, \phi_{k-1}$ be eigenf. corresp. to $\lambda_1, \dots, \lambda_{k-1}$. For $j = 1, \dots, k$, let ψ_j be an eigenf. corresp. to ν_j and extended to 0 in Ω.

Then $\psi_j \in H^1_{(0)}(\Omega)$, and $\psi_1, \dots, \psi_k$ can be chosen to be o.n. in $L^2(\Omega)$.

We can find $\alpha_1, \dots, \alpha_k$ not all zero s.t. $\sum_{j=1}^{k} \alpha_j (\psi_j, \phi_\ell) = 0$ $\ell = 1, \dots, k-1$.

Hence ($f = \sum \alpha_j \psi_j$) $\lambda_k \|f\|^2_{L^2} \leq \int_\Omega |\nabla f|^2 = \sum_{j=1}^{k} \nu_j \alpha_j^2 \leq \nu_k \|f\|^2_{L^2(\Omega)}$. □

Particular case Let $\tilde{\Omega} \subseteq \Omega$ and let $\lambda_k^D(\tilde{\Omega})$ be the Dirichlet eigenv. in $\tilde{\Omega}$. Then, if $\lambda_k(\Omega)$ is any eigenv (D. or N.) then $\lambda_k^D(\tilde{\Omega}) \geq \lambda_k(\Omega)$.

Cor 2 Suppose we are as in Cor 1, and assume $M = \bar{\Omega}_1 \cup \dots \cup \bar{\Omega}_m$. Assume Neumann on $M \cap \partial\Omega_r$ and original data on $\partial\Omega_\ell \cap \partial M$. Then $\nu_k \leq \lambda_k$.

Pf Let ψ_ℓ be as before. If $f \in H^1(\Omega)$, $f \in H^1(\Omega_r)$ $\forall r$. If $f \perp_{L^2} \psi_1, \dots, \psi_{k-1}$, then
$\int_\Omega |\nabla f|^2 = \sum_{i=1}^{m} \int_{\Omega_r} |\nabla f|^2 \geq \sum_{r=1}^{m} \nu_k \int_{\Omega_r} f^2 = \nu_k \|f\|^2_{L^2(\Omega)}$. But $\exists$ a non zero
$f = \sum_{j=1}^{k} \alpha_j \phi_j$ $\perp_{L^2(\Omega)} \psi_1, \dots, \psi_{k-1}$ $\Rightarrow$ $\int |\nabla f|^2 \leq \lambda_k \|f\|^2_{L^2(M)}$ □

Weyl's asymptotic formula

Consider a rectangle in $\mathbb{R}^n$ $\Omega = (0,L_1)\times\cdots\times(0,L_n) \subseteq \mathbb{R}^n$. Then the Dirichlet eigenfunctions are of the form $\sin\frac{k_1\pi}{L_1}\times\cdots\times\sin\frac{k_n\pi}{L_n}$, with eigenv. $\pi^2\left(\frac{k_1^2}{L_1^2}+\cdots+\frac{k_n^2}{L_n^2}\right)$. Let $N(\lambda)$ be the number of eigenvalues $\leq\lambda$ counted with multiplicity. Then $N(\lambda)\simeq\omega_n\lambda^{n/2}\,Vol(\Omega)\,(2\pi)^{-n}$ $\boxed{Ex1}$

The same holds for Neumann b.c.

Prop For a bounded domain of $\mathbb{R}^n$ one has $N_\Omega^D(\lambda)\sim\lambda^{n/2}\omega_n Vol(\Omega)(2\pi)^{-n}$

Pf Lower bound Let $G_1,\dots,G_\ell$ be disjoint open rectangles and let $N_j(\lambda)=N_{G_j}^D(\lambda)$. Then by Cor 1 we have $N_\Omega^D(\lambda)\geq\sum_{j=1}^{\ell}N_j(\lambda)$.

So we have $\liminf N(\lambda)/\lambda^{n/2}\geq\sum_{j=1}^{\ell}\liminf N_j(\lambda)/\lambda^{n/2}=\frac{\omega_n}{(2\pi)^n}\sum_{j=1}^{\ell}Vol(G_j)$.

Upper bound Let $G_1,\dots,G_\ell$ be disjoint open rect. with $\Omega\subseteq int(\bar G_1\cup\cdots\cup\bar G_\ell)$.

Let $M(\lambda)$ be the summatory function for the Neuman eigenv. of $int(\bar G_1\cup\cdots\cup\bar G_\ell)$, and $M_j(\lambda)=M_{G_j}^N(\lambda)$. Then $N_\Omega^D(\lambda)\overset{Cor\,1}{\leq}M(\lambda)\overset{Cor\,2}{\leq}\sum_{j=1}^{\ell}M_j(\lambda)$

which implies $\limsup\frac{N(\lambda)}{\lambda^{n/2}}\leq\frac{\omega_n}{(2\pi)^n}\sum_{j=1}^{\ell}Vol(G_j)$. $\square$

Rem The same can be proved for Neumann boundary conditions

Proof for a manifold (sketch) (with a triangulation) (or divide it into piecewise lip. (decomposition)), Let $(\Omega_i)_i$ be a triangulation (4)

of M, s.t. all triangles (Ω_i) are contained in some geodesic ball. Then in each

Ω_i $g = dx^2 + o(1)\,dx^2$, which implies $\lambda_k^{D(N)}(\Omega_i) = \lambda_k^{D(N)}(\tilde{\Omega}_i)(1+o(1))$ $\forall k$,

where $\tilde{\Omega}_i$ is the pre image of Ω_i through an exponential map.

Then, as for the above arguments we have that $\lambda_k(M) \leq \nu_k^D(\bigcup_i \Omega_i)$,

and that $\lambda_k(M) \geq \nu_k^N(\bigcup_i \Omega_i)$. So we get the asymptotics. $\square$

2. Heat Kernel for compact manifolds

We consider the heat operator $Hu := \Delta u - \frac{\partial u}{\partial t}$ on $M \times (0,\infty)$, M cpt. man.

Def A fundamental solution of the heat equation (or heat Kernel) is a continuous function $p = p(x,y,t)$ def. on $M \times M \times (0,+\infty)$ which is C^2 w.r.t. x, C^1 w.r.t. t and which satisfies $H_x p = 0$, $\lim_{t\to 0} p(\cdot,y,t) = \delta_y$

(or $\lim_{t\to 0} \int_M p(x,y,t) f(x)\, dV(x) = f(y)$ $\forall f$ continuous on M)

Sturm-Liouville decomposition Consider a complete o.n. basis in $L^2(M)$ consisting of eigenfunctions of $-\Delta$, $\phi_0, \phi_1, \cdots$. Then one has

$p(x,y,t) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \phi_j(x)\phi_j(y)$ $\boxed{\text{Ex}}$ (the convergence is uniform in every norm)

. From the properties of p we have that if $Hu = 0$, $u|_{t=0} = u_0$, then

$u(x,t) = \int_M p(x,y,t)\, u_0(y)\, dy$. Let us verify it for the above function. (5)

If $u_0(x) = \sum_{k=0}^{\infty} c_k \phi_k(x)$, then $u(x,t) = \sum_{k=0}^{\infty} c_k e^{-\lambda_k t} \phi_k(x)$.

Also $\int_M p(x,y,t)\, u_0(y)\, dy = \sum_{k,j=0}^{\infty} e^{-\lambda_j t} \phi_j(x) \int \phi_j(y)\, c_k \phi_k(y)\, dy = \sum_{j=0}^{\infty} e^{-\lambda_j t} c_j \phi_j(x)$ Δ

- In particular we get $\int_M p(x,x,t)\, dV(x) = \sum_{j=0}^{\infty} e^{-\lambda_j t}$
- As $t \to \infty$ $u(x,t) \xrightarrow[t\to\infty]{} \fint_M u_0$ unif in every norm.
- $\forall\, x,y,t$ one has $p(x,y,t) = p(y,x,t)$

<u>The Minakshisundaram-Pleijel expansions</u>

In $\mathbb{R}^n$ the heat kernel is given by $e(x,y,t) = (4\pi t)^{-n/2} e^{-|x-y|^2/4t}$

The idea is that for t small the heat kernel on a cpt. manifold has the same asymptotics, but with (main) corrections which depend on the local geometry.

Let us set $E(x,y,t) = (4\pi t)^{-n/2} e^{-d(x,y)^2/4t}$. Fixing x. We look for approx sol. like

$E_k(x,y,t) = E(x,y,t)\,(U_0 + U_1 t + \cdots U_k t^k)$.

The laplacian in a coordinate system around x is given by $\frac{1}{\sqrt{\det g}}\,\partial_i(\sqrt{\det g}\, g^{ij}\partial_j u)$,

and in geodesic coordinates by $\frac{1}{\sqrt{\det g}}\,\partial_r(\sqrt{\det g}\,\partial_r u) + \Delta_{S(x,r)}(u|_{S(x,r)})$.

- In Euclidean space $\sqrt{\det g} = r^{n-1}$.

Set $U_0(x,x) = 1$, and define recursively U_j by

$$2\frac{dU_j}{dr} + \frac{r}{2}\,\frac{d\log\sqrt{g}}{dr}\,U_j + j\,U_j = \Delta U_{j-1} \qquad (U_{-1} \equiv 0)$$

Claim $H_y E_K(x,y,t) = \Delta U_K t^{\cdot} E(x,y,t)$ $\quad \frac{1}{z^{n-1}}$

if $\Phi = \Phi(z)$

Pf $(K=1)$. We have $\Delta(\phi U) = U\left(\phi_{zz} + \frac{n-1}{z}\phi_z + \frac{\partial \varphi}{\partial z}\,\phi_z\right) + 2U_z\phi_z + \phi\,\Delta U$

Moreover E satisfies $E_z = -\frac{1}{2}\frac{z}{t}E$; $E_{zz} + \frac{n-1}{z}E_z - E_t = 0$

$\Rightarrow H_y E_1(x,y,t) = \frac{E}{t}\left(-zU_{0z} - \frac{1}{2}z\frac{\partial \log\sqrt{\det g}}{\partial z}U_0\right)$

$+ E\left(-zU_{1z} - \frac{1}{2}z\frac{\partial \log\sqrt{\det g}}{\partial z} - U_1\right) + E\,\Delta U_0 + Et\,\Delta U_1$

$U_0 = \varphi^{-1/2}$

$\varphi_2 \frac{\sqrt{\det g}}{z^{n-1}}$

Fact For $K > \frac{n}{2}$ E_K is a parametrix for the Heat operator

$\left(\cancel{H}\ P \in C^\infty(M\times M\times(0,\infty)) \cap C(M\times M\times[0,\infty))\right.$, $P(\cdot,y,t) \sim \delta_y$, $P(x,\cdot,t) \doteq \delta_x$ as $t \to 0$ $)$

- $H_y E_K = O\left(e^{-\frac{x^2}{4t}}\right) t^{K-\frac{n}{2}}$ $\rightsquigarrow$ apply Schauder theory to find a rigorous estimate for the heat kernel

$\boxed{\text{Ex}}$ Prove that $U_1(x,x) = \frac{1}{6}R_g(x)$ (scalar curvature)

Hint : use $g_{ij} = \delta_{ij} + \frac{1}{3}R_{kij\ell}\,x^k x^\ell + o(|x|^2)$ in geod. coord.

$(R_g = 2K_g$ for $n=2)$

The determinant of the Laplacian Ref OPS, JFA '88

· Formally, this is defined as $\prod_{j=1}^{\infty} \lambda_j$ (λ_j eigenv. of $-\Delta$)

We need a regularization: consider $Z(s) = \sum_{j=1}^{\infty} \lambda_j^{-s}$, so $\det' \Delta = e^{-Z'(0)}$

In fact $Z'(s) = \frac{d}{ds} \sum_{j=1}^{\infty} e^{-s \log \lambda_j} = \sum_{j=1}^{\infty} (-s) \log \lambda_j \, e^{-s \log \lambda_j}$.

By the Weyl's asymptotic formula the series converges absolutely for $Re(s) > 1$.

We can write that $Z(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-\lambda_j t} t^s \frac{dt}{t}$ $\quad \left(\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt \right)$

This can be also written as $Z(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} (Tr(e^{\Delta t}) - 1) t^s \frac{dt}{t}$, so

$$Z(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} Tr\left(e^{\Delta t} - \frac{1}{A(\Sigma)}\right) t^s \frac{dt}{t}.$$

· Recall that we have the expansion $\sum_{j=0}^{\infty} e^{-\lambda_j t} \varphi_j^2(x) = \frac{1}{4\pi t} + \frac{K(x)}{12\pi} + O(t)$,

which implies $Tr(e^{\Delta t}) = \frac{A}{4\pi t} + \frac{\chi(\Sigma)}{6} + O(t)$.

It follows that $Z(s) = \frac{1}{\Gamma(s)} \left\{ \frac{A}{4\pi(s-1)} + \left(\frac{\chi(\Sigma)}{6} - 1 \right) \frac{1}{s} + \text{holom. in } s \right\}$ near zero.

Therefore Z is regular at $s = 0$, and $e^{-Z'(0)}$ is well defined. $\left(Z(0) = \frac{\chi(\Sigma)}{6} - 1 \right)$.

Conformal changes of metrics

Let g be a fixed metric on Σ, and consider the conformal metric $\tilde{g} = e^{2w} g$

Then one has $dV_{\tilde{g}} = e^{2w} dV_g$, $\Delta_{\tilde{g}} = e^{-2w} \Delta_g$ $\quad K_{\tilde{g}} = e^{-2w}(-\Delta_g w + K_g)$

If we differentiate $Z(s)$ with respect to w we find

$$\delta Z(s) = \frac{1}{\Gamma(s)} \int_0^\infty \delta \, \mathrm{Tr}\left(e^{\Delta t} - \frac{1}{A}\right) t^s \frac{dt}{t} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\left(\delta\Delta \, e^{\Delta t}\right) t^s dt \qquad (8)$$

But we have $\delta\Delta = -2e^{-2w}\delta w \, \Delta_g = -2\delta w \, \Delta$, so we get

$$\delta Z(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\left(-2\delta w \, \Delta e^{\Delta t}\right) t^s dt = -\frac{1}{\Gamma(s)} \int_0^\infty \frac{d}{dt} \mathrm{Tr}\left(2\delta w\left(e^{\Delta t} - \frac{1}{A}\right)\right) t^s dt \overset{\text{int. by parts}}{=} \frac{2s}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\left(\delta_w\left(e^{\Delta t} - \frac{1}{A}\right)\right) t^s \frac{dt}{t}$$

- The function $s/\Gamma(s)$ has a second order zero at $s=0$, so we can diff. w.r.t. s

and calculate at $s=0$. In this way we get

$$\delta Z'_{|s=0} = 2\int_\Sigma (\delta w)\left(\frac{K_g(x)}{12\pi} - \frac{1}{A}\right) e^{2w} dV_g = \frac{1}{6\pi} \int_\Sigma \delta w\left(-\Delta_g w + K_g\right) dV_g - \delta \log A, \quad \text{so}$$

$$\delta\left(-\log \det{}' \Delta_g\right) = \frac{1}{6\pi}\left\{\int_\Sigma \underbrace{\delta w \left(-\Delta_g w\right) dV_g}_{\delta\left(\frac{1}{2}|\nabla_g w|^2\right)} + \int_\Sigma K_g \delta w \, dV_g\right\} - \delta \log A$$

Integrating we find that (<u>Polyakov, Ray-Singer formula</u>)

$$\log\left(\frac{\det{}' \Delta_{\tilde{g}}}{\det{}' \Delta_g}\right) = \det{}' \Delta_{\tilde{g}} - \det{}' \Delta_g = -\frac{1}{6\pi}\left\{\frac{1}{2}\int_\Sigma |\nabla_g \varphi|^2 dV_g + \int_\Sigma K_g \varphi \, dV_g\right\} + \log \frac{A_{\tilde{g}}}{A_g}$$

<u>Extremal metrics</u> (Osgood - Phillips - Sarnak)

Let us introduce the Functional $F(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 dV_g + \int_\Sigma K_g u \, dV_g - \pi \chi(\Sigma) \log \int_\Sigma e^{2u} dV_g$

- This functional is invariant by translation, and also $\min F \iff \max \det{}' \Delta$
- We can normalize conveniently by $\int_\Sigma e^{2u} = \text{const}$ (Vol. constr.) or $\int_\Sigma u \, dV_g = 0$ (linear constr.)

<u>Theorem</u> ([OPS]) Within all metrics in a given conformal class and with the same volume, the constant curvature one has maximum determinant.

<u>Easy case: $\chi(\Sigma) \leq 0$</u> In this case the log term has the good sign

In this case it is convenient to normalize g so that the unit volume is 1. (9) and $\int_\Sigma u\, dV_g = 0$

Then we can use Jensen's inequality $\int_\Sigma e^{2u} dV_g \geq \exp\left(2\int_\Sigma u\, dV_g\right) = 0$

We also have, by Holder's ineq. $\int e^{2\alpha u + (1-\alpha)2v} \leq \left(\int e^{2u}\right)^{\alpha}\left(\int e^{2v}\right)^{1-\alpha}$, $\alpha \in [0,1]$

$\Rightarrow$ For $\chi(\Sigma) \leq 0$ $F(\cdot)$ is strictly convex. $\Rightarrow$ minimum is unique

Existence of a minimizer Let u_m be s.t. $F(u_m) \to \inf F$

By the Poincaré inequality we have that $(u_m)_m$ is uniformly bounded in $H^1(\Sigma)$

Hence $\exists\, u_0 \in H^1(\Sigma)$ s.t. $u_m \rightharpoonup u_0$, $\int_\Sigma |\nabla u_0|^2 dV_g \leq \liminf \int_\Sigma |\nabla u_m|^2 dV_g$

$u_m \to u_0$ strongly in L^2 and pointwise a.e., plus $\int e^{2u_0} \leq \liminf \int e^{2u_m}$

by Fatou's lemma. Hence $F(u_0) = \inf F$, so u_0 is a minimizer.

Looking at the Euler equation at u_0 we find that u_0 solves

$-\Delta_g u_0 + K_g = \dfrac{2\pi \chi(M)}{\int_\Sigma e^{2u_0} dV_g} e^{2u_0}$. By elliptic reg. the solution

is also classical. By the trasf. law of Gauss curvature

we get that the extremal metric has constant curvature.

Difficult case: Σ simply connected

- We assume here the uniformization theorem, namely that every metric

on S^2 is conformal to the standard one g_0. We need to show that

if the determinant is maximal then the metric is isometric to g_0

Remark The functional is invariant under the action of the Möbius group (10)

$\tau : S^2 \to S^2$ is defined via composition of the stereographic projection and a dilation in $\mathbb{R}^2$. $u(x) \to u_\sigma(x) = u(\sigma(x)) + \frac{1}{2}\log(|d\sigma(x)|)$.

Then $F(u_\sigma) = F(u)$ $\forall u, \sigma$. $\Rightarrow$ F is no more convex. $\square$

• Since $F(0) = 0$, the inequality is equivalent to

$$\int_{S^2} |\nabla_{g_0} u|^2 dV_{g_0} + 2\int_{S^2} u \, dV_{g_0} - 4\pi \log \fint_{S^2} e^{2u} dV_{g_0} \geq 0$$ Onofri inequality,

with equality iff u is of the type 0_σ.

Pf ([OPS]). It is sufficient to assume that $\int_{S^2} u \, dV_{g_0} = 0$, and to prove that, for all such u and $\forall \varepsilon > 0$ one has $G_\varepsilon(u) \geq 0$, where

$$G_\varepsilon(u) = (1+\varepsilon)\int_{S^2} |\nabla_{g_0} u|^2 dV_{g_0} - 4\pi \log \fint_{S^2} e^{2u} dV_{g_0} .$$

One has indeed the weaker Moser's inequality $G(u) := G_0(u) \geq -C$ for some $C > 0$.

• From Moser's inequality it follows that $G_\varepsilon(u)$ attains a minimum u_ε.

By symmetrization we can assume that u_ε is axially symmetric in the N-S direction and angular decreasing. Moreover it satisfies

(E_ε)

$$-(1+\varepsilon)\Delta_{g_0} u_\varepsilon + 1 = \frac{4\pi e^{2u_\varepsilon}}{\int_{S^2} e^{2u_\varepsilon} dV_{g_0}} .$$ We will show that $u_\varepsilon = 0$.

• We can set $v_\varepsilon = a + u_\varepsilon$ so that $(a \in \mathbb{R})$

v_ε satisfies the ODE

$(ODE_\varepsilon) \quad -(1+\varepsilon)\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dv_\varepsilon}{d\theta}\right) = e^{2v_\varepsilon} - 1 \quad , \qquad 0 \le \theta \le \pi.$ (11)

__Lemma__ Suppose v is a solution of $\; -\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dv}{d\theta}\right) = C(e^{2v}-1)$

If $C \neq 1$, then $v \equiv$ const.

__Cor__ If u_ε is a solution of (E_ε) with zero mean value, then $u_\varepsilon = 0$.

__Pf of Lemma__ We have $-\frac{1}{\sin\theta}(\sin\theta\, v')' = C(e^{2v}-1)$.

Differentiating we obtain $-\left[\frac{1}{\sin\theta}(\sin\theta\, v')'\right]' = 2Ce^{2v}v'$, so using

the equation $-\left[\frac{1}{\sin\theta}(\sin\theta\, v')'\right]' = 2v'\left(-\frac{1}{\sin\theta}(\sin\theta\, v')' + C\right)$

Set $\tilde{v} = \sin\theta\, v'$, so $-\left[\frac{1}{\sin\theta}\tilde{v}'\right]' = \frac{2\tilde{v}}{\sin\theta}\left(-\frac{1}{\sin\theta}\tilde{v}' + C\right)$, which implies

$\sin^2(\theta)\left(\frac{\tilde{v}'}{\sin\theta}\right)' = 2\tilde{v}\tilde{v}' - 2C\tilde{v}\sin\theta$. We have that $v(0) = v(\pi) = 0$,

so integr. by parts we get $-2\int_0^\pi \tilde{v}'\cos\theta = -2C\int_0^\pi \tilde{v}\sin\theta$

Since $\tilde{v}$ has a fixed sign, the integral is non zero $\Rightarrow C = 1$. $\square$

$\circ$ We only need to show when equality holds. Clearly, $u = 0$ and its

Mobius equivalents realize equality. Let us show there are no others.

A minimizer u satisfies $-\Delta_{g_0} u + 1 = 4\pi\frac{e^{2u}}{\int_{S^2} e^{2u}dV_{g_0}}$. Then the metric

$\tilde{g} = e^{2u}g_0$ has constant Gauss curvature, so it must arise from a Mobius map. $\square$

(Obata)

Some remarks about manifolds with boundary

When $\partial\Sigma \neq \emptyset$ one can consider the eigenvalues of the Laplace operator with Dirichlet boundary conditions. In this case the heat kernel satisfies

$$\mathrm{Tr}\,(e^{\Delta t}) = \frac{|\Sigma|}{8\pi t} - \frac{1}{8}\frac{1}{(\pi t)^{1/2}}|\partial\Sigma| + \frac{1}{6}\chi(\Sigma) + O(\sqrt{t})$$

In this case, if $\tilde{g} = e^{2w} g$ one has that the determinant becomes

$$\log \frac{\det \Delta_{\tilde{g}}}{\det \Delta_g} = -\frac{1}{6\pi}\left\{ \frac{1}{2}\int_\Sigma |\nabla_g w|^2 dV_g + \int_\Sigma K_g\, w\, dV_g + \int_{\partial\Sigma} k_g\, w\, dl_g \right\} - \frac{1}{4\pi}\int_{\partial\Sigma} \partial_{\nu_g} w\, dl_g ,$$

where k_g is the geodesic curvature of $\partial\Sigma$.

Thm ([OPS]) Suppose $\partial\Sigma \neq \emptyset$. Then, in a given conformal class of metrics with given area the determinant is maximized by constant curvature metrics such that $\partial\Sigma$ has zero geodesic curvature. If we instead consider conformal metrics with given boundary length, the determinant is maximal when Σ is flat and $\partial\Sigma$ has constant geodesic curvature.

The "difficult case" is when Σ has the topology of the unit disk. If D is the unit disk, one has to use the Lebedev-Milin inequality

$$\log \fint_{\partial D} e^u\, d\theta \le \frac{1}{4\pi}\int_D |\nabla u|^2 + \fint_{\partial D} u\, d\theta \quad \forall u \in H^1(D), \text{ with "=" iff Mobius}$$

Isospectral surfaces

We consider a family of Riemannian metrics on a given surface Σ with the property that they have the same spectrum. The question is related to a classical problem raised by M. Kac in 1966 asking whether it is possible to hear the shape of a drum. About planar domains there are conterexaples by Gordon-Wilson-Wolpert, and for compact surfaces
Tori Milnor Sup periodische Groups H, K Lie Groups
exaples by Vigneras, Sunada and Gordon-Wilson: in the latter case there exist even continuous families of metrics.

We fix Σ compact closed surface and a background (smooth) metric g. We use this fixed metric to express the convergence in C^k norm of metric tensors.

Def We say that two metrics are isometric if they are equivalent by pull back with a C^∞ diffeomorphism. Dealing with cpt, we look at suit. representatives.

- The situation is also different in the case of the sphere or for other surfaces, since for the sphere all metrics are conformally equivalent.

Thm ([OPS]) On a given surface Σ isospectral metrics are compact in every C^k topology (choosing suitable representatives)

- The proof uses the determinant to prove H^1 bounds and heat invariants to prove H^k estimates.

Case of the sphere: all metrics are conformally equivalent, and we have invariance under the action of the Möbius group.

Def A metric $\tilde{g} = e^{2u} g_0$ on S^2 is said to be balanced if $\int_{S^2} e^{2u} x_j \, dV_{g_0} = 0 \quad \forall j = 1,2,3.$

It is possible to prove using degree theory that every metric has a balanced representative, choosing suitably the Möbius action. Moreover one has

Prop ([Aubin]) If $\tilde{g} = e^{2u} g_0$ is a balanced metric, then $\forall \varepsilon > 0 \; \exists C_\varepsilon > 0$ s.t.

$$\log \fint_{S^2} e^{2u} dV_{g_0} \leq \frac{1+\varepsilon}{2} \frac{1}{4\pi} \int_{S^2} |\nabla u|^2 dV_{g_0} + 2 \fint_{S^2} u \, dV_{g_0} .$$

- The 0-th order heat invariant determines the area of the surface, so we can assume that all the areas are equal to 4π.

Also, $\det' \Delta_{\tilde{g}}$ is a spectral invariant, and by Polyakov's formula

$$\log \det{}' \Delta_{\tilde{g}} = -\frac{1}{12\pi} \left(\int_{S^2} |\nabla u|^2 dV_{g_0} + 2 \int_{S^2} u \, dV_{g_0} \right) + \log A_g(\Sigma) + C$$

Using this and the area bound we deduce that

$\int_{S^2} |\nabla u|^2 dV_{g_0} + 2 \int_{S^2} u \, dV_{g_0}$ is uniformly bounded, and moreover from Aubin's inequality we find that $\int_{S^2} |\nabla u|^2 dV_{g_0}$ is unif. bounded.

This also implies a uniform bound for $\left| \int_{S^2} u \, dV_{g_0} \right|$, so we get uniform bounds on $\|u\|_{H^1(S^2)}$.

Higher heat invariants

Using the M-P expansion it is possible to prove that $Tr(e^{\Delta t}) = \frac{1}{t}\sum_{j=0}^{e} a_j t^j + O(t^e)$,

where $a_j = \int_\Sigma U_j^{(x,x)} dV_g$, where the $U_j^{(x,x)}$'s are universal polynomials (indep. of Σ)

of degree $2j$ in K_g and Δ_g. We saw $a_0 = \frac{|\Sigma|}{4\pi}$, $a_1 = \frac{\chi(\Sigma)}{6}$.

McKean and Singer proved that $a_2 = \frac{\pi}{60}\int_\Sigma K_g^2 dV_g$, while Gilkey proved the formula in the general case.

- Using integration by parts one finds that the highest order term in the metric is given by $c_j \int_\Sigma K_g \Delta^{j-2} K_g dV_g = \pm c_j \int_\Sigma (\Delta^{\frac{j-2}{2}} K_g)^2 dV_g$

In [OPS] it was shown that $c_j \neq 0 \ \forall j$, using special metrics on the torus.

Estimates in higher norms

Recall that the Gauss curvature of a conformal metric $\tilde{g} = e^{2u} g_0$ is given by

$$K_{\tilde{g}} = e^{-2u}(K_{g_0} - \Delta_{g_0} u) = e^{-2u}(1 - \Delta_{g_0} u).$$

From the control on the second heat invariant we get that $\int_{S^2} e^{-2u}(1-\Delta_{g_0} u)^2 dV_{g_0} \leq C$

From the M-T inequality we have that $\int e^{-2u}(\Delta_{g_0} u)^2 dV_{g_0} \leq C$.

We then write $u(x) - \fint u = \int_{S^2} G(x,y)\Delta_{g_0} u(y) dV_g(y)$, where G is the Green's function of the Laplacian. $G(x,y) \simeq \frac{1}{2\pi}\log dist(x,y)$ as $dist(x,y) \to 0$.

We can write $\int G(x,y)\,\Delta_{g_0} u(y)\,dV(y) = \int G(x,y)\,\Delta_{g_0} u(y)\, e^{-u(y)}\, e^{u(y)}\, dV_{g_0}(y)$ (16)

and use Holder's inequality to get $|u(x) - \fint u|^2 \le \left(\int_{S^2} G^4(x,y)dy\right)^{1/2} \left(\int_{S^2} e^{4u(y)} dy\right)^{1/2} \int_{S^2} e^{-2u(y)} (\Delta_{g_0} u)^2 dy$

By the control on the average we get uniform C^0 bounds.

Half integer means gradient

$n \geq 2$

• We next prove by induction that $\int_{S^2} (\Delta_g^{\frac{n-2}{2}} K_g)^2 dV_g \le C_n$, $\int_{S^2} (\Delta_g^{\frac{r-2}{2}} K_g)^{\ell} dV_g \le C_n$ $\ell \le n$, $2 \le r < n$.

We have it for $n=2$ from the second heat invariant: assume it true for $n \geq 2$ and let us prove that $\int_{S^2} (\Delta^{\frac{n-1}{2}} K_g)^2 dV_g$ is bdd.

From the $(n+1)$-th heat invariant (laplacians) we get that $\underset{\#}{\overset{0}{R_{n+1}}} = C_{n+1} \int_{S^2} (\Delta^{\frac{n-1}{2}} K_g)^2 + \text{l.o.t.}$

The l.o.t. contain either derivatives of order less than $\frac{n-2}{2}$, or there is a term like $T_{n+1} := \int_{S^2} K_g (\Delta^{\frac{n-2}{2}} K_g)^2 dV_g$. Derivatives of order less than (at most n terms, or $\int K_g^{n+1}$ (Sol.)) $\frac{n-2}{2}$ can be treated using the induction hypothesis. For T_{n+1} instead we can use ~~Gagliardo-Nirenberg~~

$$|T_{n+1}| \le \|K_g\|_{L^2} \|(\Delta^{\frac{n-2}{2}} K_g)\|^2_{L^2} \le C \|\Delta^{\frac{n-2}{2}} K_g\|_{L^4} \le C \|\Delta^{\frac{n-2}{2}} K_g\|^{3/2}_{H^1} \|\Delta^{\frac{n-2}{2}} K_g\|^{1/2}_{L^2}$$

$$\le C \|\Delta^{\frac{n-2}{2}} K_g\|^{3/2}_{H^1} \le C.$$

, thus $\int_{S^2} (\Delta^{\frac{n-1}{2}} K_g)^2 dV_g \le C$.

The other estimate follows from the Sobolev ineq. $\|u\|_{L^p} \le C_p \|\nabla u\|_{L^2} + \left|\int_{S^2} u\, dV_g\right|$.

Since $\Delta_g = e^{-2u} \Delta_{g_0}$, from the C^0 bound on u we get $\|\Delta_{g_0} K_g\|_{L^2} \le C$,

which implies $\|\Delta^2_{g_0} u\|_{L^2} \le C \Rightarrow$ (By induction) $\|u\|_{H^k} \le C_K$

Higher genus

Suppose now Σ is a closed surface of genus $G \geq 1$. Let $\hat{g}_n$ be a sequence of isometry classes of isospectral metrics. We can assume that the area is fixed $= 1$.

In each conformal class $\exists !$ hyperbolic metric $\hat{\tau}_n$. By the extremality of the determinant we have $\det(\hat{\tau}_n) \geq \det(\hat{g}_n)$.

Let M_{G_1} be the moduli space of hyperbolic metrics with area 1. Wolpert '87 showed that det satisfies $\det(\hat{\tau}) \leq \frac{1}{\ell} \exp\left(-\frac{c_1}{\ell}\right)$, where ℓ is the length of the shortest geodesic and c_1 is a constant dep. only on the genus. Then we have $\ell_n \geq \frac{1}{C} > 0$.

A theorem of Mumford '71 implies that a subsequence of $\hat{\tau}_n$ converges to a $\hat{\tau}$ in M_G.

It is possible then to prove (Ahlfors) that the convergence is in C^∞.

Since $g_n = e^{2u_n} \tau_n$ for some u_n, it is sufficient to show that the u_n's converge in C^∞. Using Polyakov's formula we have that

$$-6\pi \log \det g_n = \frac{1}{2} \int_\Sigma |\nabla_{\tau_n} u_n|^2 \, dV_{\tau_n} + 2\pi(2-2G) \int_\Sigma u_n \, dV_{\tau_n} - 6\pi \log \det(\tau_n)$$

$(K_{\tau_n} = 2-2G)$. Moreover $\int_\Sigma dV_{\tau_n} = 1 = \int_\Sigma e^{2u_n} dV_{\tau_n}$

Jensen's inequality implies $\int_\Sigma u_n \, dV_{\tau_n} \leq 0$. Thus from the latter equations we get $\int_\Sigma u_n \, dV_{\tau_n} \leq 0$, and hence $\int |\nabla_{\tau_n} u_n|^2 dV_{\tau_n} \leq C$.

The M-T ineq. implies $1 = \int_\Sigma e^{2u_n} dV_{\tau_n} \leq C \exp\left(c_1 \int_\Sigma |\nabla_{\tau_n} u_n|^2 dV_{\tau_n} + c_2 \int u_n \, dV_{\tau_n}\right)$

This implies $\left|\int_\Sigma u_n \, dV_{\tau_n}\right| \leq C$ and hence $\left|\int_\Sigma u_n \, dV_{\tau_0}\right| \leq C$

The latter and the Dirichlet bound implies $\|u_n\|_{H^1} \leq C$.

We have reached the same initial step as before: now we conclude using the higher heat invariants. $\square$

Rem 1 A similar result holds for planar simply connected domains.
Using the Riemann mapping, one can consider flat metrics on the unit disk D of $\mathbb{R}^2$. Then, if the metrics are isospectral, one gets convergence of the Riemann mappings in $C^\infty(\bar{D})$.

Rem 2 There is an extension by Chang - Yang to 3D manifolds. Here one considers the conformal Laplacian $-8\Delta_g + R_g$ (with corresp. spectrum), and restricts to conformal metrics.

In this case $a_0 = Vol_g(M)$, $a_1 = \int R_g \, dV_g$ $\quad a_2 = A_1 \int R_g^2 \, dV_g + A_2 \int |Ric|^2 \, dV_g$

$a_k = A_{1,k} \int |D^{k-2} R_g|^2 \, dV_g + A_{2,k} \int |D^{k-2} Ric|^2 \, dV_g$

• One key step in the proof is a uniform lower bound on the first eigenvalue: this argument extends to surfaces in a given conformal class.
optimal Sobolev ineq. Moser iteration

Conformally covariant operators (M^n, g)

- It is a class of operators which transform nicely under a conformal change of metric. It is a somehow necessary condition if one wants to get Polyakov formulas.

<u>Def</u> An operator is called conformally covariant if $\exists\ a, b \in \mathbb{R}$ s.t. if $\tilde{g} = e^{2w} g$, then $A_{\tilde{g}} \varphi = e^{-bw} A_g (e^{aw} \varphi) \quad \forall \varphi \in C^\infty(M)$

<u>Examples</u>

1. $n=2$ Δ satisfies the above with $(a,b) = (0,2)$
2. $n \geq 3$ $L_g = -\frac{4(n-1)}{(n-2)} \Delta_g + R_g$ <u>conformal laplacian</u> : $(a,b) = \left(\frac{n-2}{2}, \frac{n+2}{2}\right)$
3. $n=4$ <u>Paneitz operator</u> $P_g \varphi = \Delta_g^2 \varphi + \operatorname{div}\left[\left(\frac{2}{3} R_g - 2\,\mathrm{Ric}\right) \nabla \varphi\right]$ $(a,b) = (0,4)$
4. $n \geq 2$ <u>Dirac operator</u> (on spinors) $D\!\!\!/$ $(a,b) = \left(\frac{n-1}{2}, \frac{n+1}{2}\right)$

<u>Heat Kernel and determinant of L_g ([Parker-Rosenberg])</u>

Let $H_L(x,y,t)$ be the heat kernel of $\frac{\partial}{\partial t} + L_g$. Then, again, $H_L(x,x,t) = (4\pi t)^{-n/2} \sum a_k(x) t^k$

<u>Thm</u> ([P-R]) Suppose n is even. Then $a_{\frac{n-2}{2}}$ is a pointwise conformal invariant.

Still for n even, $\int_M a_{n/2}\, dV_g$ is a global conformal invariant.

For n odd, the determinant is a global conformal invariant.

$\left(n=4 \quad a_0(x) = 1, \quad a_2(x) = \frac{1}{180}\left[|W_g|^2 - \Delta R_g + \frac{1}{4}|\mathrm{Ric}|^2 - \frac{1}{12} R_g^2 \right] \right) \rightsquigarrow$ Chern-Gauss-Bonnet

Log determinants in 4D

$$\log \det A_g = -Z_A'(0) \qquad Z_A(s) = \sum_{\lambda_j \neq 0} |\lambda_j|^{-s} \qquad A_\varepsilon = A_{e^{2\varepsilon w} g}$$

Branson and Oersted proved that $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} Z'_{A_\varepsilon}(0) = 2\ell \int w \left(a_2[A] - \sum_{\lambda_j = 0} |\varphi_j|^2 \right)$

a_2 conf inv. $\qquad (\ell = a + 2\ell,\ \varphi_j$ Kernel$)$

Using Weyl's invariant theory one can show that a_2 is a linear combination of $|W_g|^2$, $|Ric|^2$, R_g^2 and ΔR_g. More precisely, it can be shown that

$$a_2 = \beta_1 |W_g|^2 + \beta_2 Q_g + \beta_3 \Delta R_g, \text{ where } Q_g = \frac{1}{12}\left(R_g^2 - 3|Ric|^2 - \Delta_g R_g\right) \text{ Q curvature}$$

Using the transformation laws for R_g, $|W_g|^2$ and Q_g under conformal changes

$$\tilde g = e^{2w} g \implies -6(\Delta w + |\nabla w|^2) + R_g = R_{\tilde g} e^{2w} \qquad |W_{\tilde g}|^2 = e^{-4w} |W_g|^2$$

one then finds, by integration $\qquad P_g w + 2 Q_g = 2 Q_{\tilde g} e^{4w}$

$$\log \frac{\det A_{\tilde g}}{\det A_g} = \gamma_1 \, I[w] + \gamma_2 \, II[w] + \gamma_3 \, III[w], \qquad \gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}, \text{ with}$$

$$I[w] = 4 \int_M |W_g|^2 w \, dV_g - \left(\int_M |W_g|^2 dV_g \right) \log ⨍_M e^{4w} dV_g$$

$$II[w] = \int_M w P_g w \, dV_g + 4 \int_M Q_g w \, dV_g - \left(\int_M Q_g \, dV_g \right) \log ⨍_M e^{2w} dV_g$$

$$III[w] = 12 \int_M (\Delta w + |\nabla w|^2)^2 dV_g - 4 \int_M R_g |\nabla w|^2 dV_g - 4 \int_M (\Delta_g R_g) w \, dV_g$$

- If $A = L_g$, then $(\gamma_1, \gamma_2, \gamma_3) = (1, -4, -\tfrac{2}{3})$, (Branson) P_g $(\gamma_1, \gamma_2, \gamma_3) = (-\tfrac{1}{4}, -14, \tfrac{8}{3})$

and if A is the square of the Dirac operator then $(\gamma_1, \gamma_2, \gamma_3) = (7, -88, -\tfrac{14}{3})$.

(the formula also holds for powers of conformally covariant operators)

Thm ([BCY]) On S^4 the standard metric g_0 minimizes both $\log\det(L_g)$ and $-\log\det(\not{D}^2)$ among all conformal metrics with fixed volume.

The main tool here is a sharp inequality due to Beckner

Lemma Suppose $f\in C^\infty(S^n)$ has an expansion $f=\sum_{k=0}^{\infty} Y_k(\theta)$ in spherical harmonics.

Then $\log \fint_{S^n} e^{f-\bar f} \le \frac{1}{2n}\sum_{k=1}^{\infty} B(n,k)\fint_{S^n}|Y_k|^2$, $B(n,k)=\dfrac{\Gamma(n+k)}{\Gamma(n)\Gamma(k)}$.

"=" iff $e^{2f/n}g_0=\varphi^* g_0$ for some Möbius map φ

. For $n=4$ one has $B(4,k)=\frac{1}{6}k(k+1)(k+2)(k+3)$. Since also the eigenvalue of $-\Delta$ on Y_k is $k(k+3)$ we get $\log \fint e^{4(w-\bar w)}\le \frac13\fint_{S^4} w\,\Delta(\Delta+2)w=\frac13\left(\fint_{S^4}(\Delta w)^2+2\fint|\nabla w|^2\right)$

The right hand side is the Paneitz operator, so we get extremality for II.

At this point, one should prove the same for III, since $\gamma_1=0$, and $\beta_2,\beta_3>0$.

To see this, recall that $R_{g_0}=12$, and that $(\Delta+2)e^w=\frac16 R_g e^{3w}$,

from which we get $-\dfrac{\Delta e^w}{e^w}=-2+\dfrac{R_g}{6}e^{2w}$. Also, $-\dfrac{\Delta e^w}{e^w}=-\Delta w-|\nabla w|^2$,

so $\mathrm{III}(w)=\displaystyle\int_{S^4}\left(\frac{-\Delta(e^w)}{e^w}\right)^2-4\int_{S^4}|\nabla w|^2=\int_{S^4}\Big[4-\tfrac{2}{3}R_g e^{2w}+R_g^2 e^{4w}+4\big(-2+\tfrac{R_g}{6}e^{2w}\big)$

$$=-4+\left(\int_{S^4}R_g^2 e^{4w}\right)\frac{1}{36}$$

It is well known that the Yamabe functional $\Big(\overbrace{\int_{S^4}|\nabla(e^w)|^2+\frac16 R_{g_0}\int_{S^4}e^{2w}}^{=\int_{S^4}R_g e^{4w}}\Big)\Big/\Big(\int_{S^4}e^{4w}\Big)^{1/2}$

attains the infimum at the metrics of the form $e^{2w}g_0=\varphi^* g_0$, φ Möbius map.
$\binom{\shortparallel}{2}$

Now the last formulas imply (assuming $\int_{S^4} e^{4w} = 1$)

$$6\,\mathrm{II}[w] = -4 + \int_{S^4} \left(R_{\tilde g}/6\right)^2 e^{4w} \geq -4 + \left(\int_{S^4} \tfrac{1}{6} R_{\tilde g} e^{4w}\right)^2 = Y_{S^4}^{\,2} - 4 = 0 \quad \square$$

<u>Remark</u> By a result of Gursky, the standard metric is not only the extremal one but also the unique critical point (using Bochner's formulas)

<u>Remark</u> On a general 4-manifold, one has the following (non sharp) inequality

$$\log \int_M e^{4(u-\bar u)} dV_g \leq \frac{1}{8\pi^2} \int_M (\Delta u)^2 dV_g + C$$

This is proved adapting a result by Adams for flat domains in $\mathbb{R}^n$. Some application is given to derive a priori estimates in $H^2(M)$ in terms of the determinant.

<u>Existence of extremal metrics</u>

If Q_g denotes the Q-curvature of (M,g), one has a Gauss-Bonnet formula

$$\int (Q_g + \tfrac{1}{8}|W_g|^2) dV_g = 4\pi^2 \chi(M). \quad \text{Since } |W_g|^2 \text{ is a pt. conf inv, } \int_M Q_g dV_g \text{ is conf. inv.}$$

Define the quantities (conf. inv.) $K_p = \int_M Q_g dV_g$, $K_d = -\gamma_1 \int_M |W_g|^2 dV_g - \gamma_2 \int_M Q_g dV_g$

<u>Thm</u> ([C-Y]) Suppose that $\gamma_2, \gamma_3 < 0$ and that $K_d < (-\gamma_2) 8\pi^2$. Then the supremum of $\gamma_1 \mathrm{I} + \gamma_2 \mathrm{II} + \gamma_3 \mathrm{III}$ is attained. If $K_d \leq 0$, the extremal metric for the determinant of the conformal Laplacian is also unique.

Pf of existence. Set $U = \gamma_1 |W_g|^2 + \gamma_2 Q_g - \gamma_3 \Delta R$ $\quad \left(\int_M U dV_g = -k_d \right).$

Then $\gamma_1 I + \gamma_2 II + \gamma_3 III = k_d \log ⨍ e^{4(w-\bar{w})} + 4\int U(w-\bar{w}) + \gamma_2 \int w P_g w + 12\gamma_3 Y(w)$,

When $k_d \geq 0$ $\quad F[w] \leq C_1 k_d + \frac{k_d}{8\pi^2}\int (\Delta w)^2 dV_g + 4\int U(w-\bar{w}) + \gamma_2 \int w P_g w + 12\gamma_3 Y(w)$

When $k_d \leq 0$ $\quad F[w] \leq 4\int U(w-\bar{w}) + \gamma_2 \int w P_g w + 12\gamma_3 Y(w)$.

$$\boxed{Y[w] = \int \left(\frac{\Delta(e^w)}{e^w}\right)^2 - \frac{1}{3}\int R_g |\nabla w|^2}$$

Computing, we have

$$\int_M w P_g w = \int_M (\Delta w)^2 + \tfrac{2}{3}\int_M R_g |\nabla w|^2 dV_g - 2\int_M R^{ij} w_i w_j \, dV_g, \quad \text{and}$$

$$Y(w) = \int_M (\Delta w)^2 + \int_M |\nabla w|^4 + 2\int \Delta w |\nabla w|^2 - \tfrac{1}{3}\int R_g |\nabla w|^2 dV_g .$$

- The functional is upper semicontinuous, and for a maximizing sequence we

$\tilde{k}_d = \max\{k_d, 0\}$

$$-C \leq F[w_e] \leq C_1 \tilde{k}_d + \left(\frac{\tilde{k}_d}{8\pi^2} + \gamma_2 + 12\gamma_3\right)\int (\Delta w_e)^2 + 12\gamma_3 \int |\nabla w_e|^4 + 4\int U(w_e - \bar{w}_e)$$

$$+ \left(\tfrac{2}{3}\gamma_2 - 4\gamma_3\right)\int R_g |\nabla w_e|^2 - 2\gamma_2 \int R^{ij}(w_e)_i (w_e)_j + 24\gamma_3 \int (\Delta w_e)|\nabla w_e|^2 \quad (\tilde{k}_d = k_d \vee 0)$$

- If $k_d < -\gamma_2 8\pi^2$ $\left(\frac{\tilde{k}_d}{8\pi^2} + \gamma_2 < 0 \right)$, using $2\int \Delta w |\nabla w|^2 \leq a\int (\Delta w)^2 + \frac{1}{a}\int |\nabla w|^4$ we get

($a > 1$ s.t. this is pos.)

$$\left(-a - 12(1-a)\gamma_3\right)\int (\Delta w_e)^2 - 12\gamma_3\left(1 - \tfrac{1}{a}\right)\int |\nabla w_e|^4 \leq C_1 \tilde{k}_d + 4\int U(w_e - \bar{w}_e) + C_2 \int R |\nabla w_e|^2$$

$$+ C_3 \int R^{ij}(w_e)_i (w_e)_j + C.$$

- The r.h.s are l.o.t. $\Rightarrow$ a priori bounds. $\square$

Pf uniqueness. The functional is concave. $\square$

A related result is the following.

Thm ([C-Y]) Suppose $P_g \geq 0$ and that $k_P < 8\pi^2$. Then $\inf II$ is achieved by some $w \in H^2(M)$. For this w, $\hat{g} = e^{2w} g$ has constant Q curvature.

Rem (i) If $K_P \leq 0$ then the extremal metric is unique.

(ii) Applications were found for studying the σ_2 equation. ($\sim$, Ric ≥ 0)

For the functional II, extremal metrics were found under generic assumptions.

Thm ([D-M]) Suppose P_g has only trivial Kernel, and that $K_P \neq 8k\pi^2$, $k = 1, 2, \dots$

Then $\exists\ \hat{g} = e^{2w} g$ which is stationary for II and has constant Q curvature.

The proof relies on min-max theory (or Morse) and some a-priori estimates for solutions of $P_g u + 2Q_g = 2K_P e^{4w}$ ([D-R], [M]). Here the condition $K_P \neq 8\pi^2 K$ is used to prove compactness.

- Existence of saddle type extremal metrics for general functionals F is not known at the moment. Branson proved that for P_g $(\gamma_1, \gamma_2, \gamma_3) = (-\frac{1}{4}, -14, 8/3)$

Example On the flat torus $\mathbb{T}^4$ the determinant of the Paneitz operator is $9\int(\Delta u)^2 + 32\int \Delta u |\nabla u|^2 + 16\int |\nabla u|^4$

0 is a local minimum but there are functions for which the energy is negative. Montain pass?

- There exist radial critical points in $\mathbb{R}^4$, and we can describe the asymptotic behavior of general solutions with finite H^2 norm.