Fully Nonlinear Equations in Conformal Geometry

Geometric Flows and
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Lecture II: The σ_k-Yamabe problem

References:

1. Conformal geometry, contact geometry, and the calculus of variations, by J. Viaclovsky, *Duke Math. J.* **101** (2000), 283–316.

2. On the Hessian of a function and the curvatures of its graph, by R.C. Reilly, *Michigan Math. J.* **20** (1973), 373–383.

3. An inequality for hyperbolic polynomials, by L. Gårding, *J. Math. Mech.* **8** (1959), 957–965.

• At the end of Lecture I, we formulated the

The Yamabe problem: Given a closed Riemannian manifold of dimension $n \geq 3$, find a conformal metric $\widehat{g} = e^{-2u}g$ such that the scalar curvature $R(\widehat{g})$ is constant.

• Recall that (up to a constant) the scalar curvature is the trace of the Schouten tensor.

• In this lecture, we want to consider the problem of prescribing other symmetric functions of the eigenvalues of the Schouten tensor; this is known as the σ_k-Yamabe problem.

• However, we first want to point out the main technical difficulty in studying the Yamabe problem, since the same issue arises in the fully nonlinear version.

Definition 1. *Let* (M, g) *and* (N, h) *be two Riemannian manifolds. A diffeomorphism* $\varphi : M \to N$ *is called* conformal *if*

$$\varphi^* h = e^f g.$$

We say that (M, g) and (N, h) are *conformally equivalent*.

• Note h and g are pointwise conformal if and only if the identity map is conformal.

Example 1 Let $\delta_\lambda(x) = \lambda^{-1}x$ be the dilation map on $\mathbb{R}^n$, where $\lambda > 0$. Then δ_λ is easily seen to be conformal; in fact,

$$\delta_\lambda^* ds^2 = \lambda^{-2} ds^2,$$

where ds^2 is the Euclidean metric.

Example 2 Let $P = (0, \ldots, 0, 1)$ be the north pole of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Let $\sigma : \mathbb{S}^n \setminus \{P\} \to \mathbb{R}^n$ denote stereographic projection, defined by

$$\sigma(\zeta^1, \ldots, \zeta^n, \xi) = \Big(\frac{\zeta^1}{1-\xi}, \ldots, \frac{\zeta^n}{1-\xi}\Big).$$

Then $\sigma : (\mathbb{S}^n \setminus \{P\}, g_0) \to (\mathbb{R}^n, ds^2)$ is conformal, where g_0 is the standard metric on $\mathbb{S}^n$.

Since the composition of conformal maps is again conformal, we can use σ to construct conformal maps of $\mathbb{S}^n$ to itself: for $\lambda > 0$, let

$$\varphi_\lambda = \sigma^{-1} \circ \delta_\lambda \circ \sigma : \mathbb{S}^n \to \mathbb{S}^n.$$

Then

$$\varphi_\lambda^* g_0 = \Psi_\lambda^2 g_0,$$

where

$$\Psi_\lambda(\zeta, \xi) = \frac{2\lambda}{(1+\xi) + \lambda^2(1-\xi)}.$$

Note

$$(\zeta, \xi) = (\mathbf{0}, \mathbf{1}) \Rightarrow \mathbf{\Psi}_\lambda \to \infty \quad \text{as } \lambda \to \infty,$$
$$(\zeta, \xi) \neq (\mathbf{0}, \mathbf{1}) \Rightarrow \mathbf{\Psi}_\lambda \to \mathbf{0} \quad \text{as } \lambda \to \infty.$$

• The set of conformal maps of a given Riemannian manifold is a Lie group; the construction above shows that the conformal group of the sphere is *non-compact*. This fact distinguishes the sphere:

Theorem 1. *(Lelong-Ferrand) A compact Riemmanian manifold with non-compact conformal group is conformally equivalent to the sphere with its standard metric.*

This fact is the source of the analytic difficulty we mentioned above.

• More concretely, let $h = \varphi^* g_0 = \Psi_\lambda^2 g_0$. Then h has the same scalar curvature as the standard metric (since it is just the pull-back of the round metric by a diffeomorphism). Therefore, writing

$$h = v_\lambda^{4/(n-2)} g_0,$$

where

$$v_\lambda = \Psi_\lambda^{\frac{(n-2)}{2}},$$

we have a family $\{v_\lambda\}$ of solutions to

$$-\frac{(n-2)}{4(n-1)}\Delta v_\lambda + n(n-1)v_\lambda = n(n-1)v_\lambda^{\frac{n+2}{n-2}}.$$

(This is the Yamabe equation.) As we observed above, if P is the North pole, then

$$v_\lambda(P) \to \infty \quad \text{as } \lambda \to \infty,$$

whereas if $x \neq P$, then

$$v_\lambda(x) \to 0 \quad \text{as } \lambda \to \infty.$$

• Good news: There are many solutions of the Yamabe equation!

• Bad news: It will be impossible to prove *a priori* estimates (often an essential part of existence theory).

• Of course, the non-compactness of the set of solutions arises precisely because of the influence of the conformal group. Thus, on manifolds other than the sphere, one would expect that the set of solutions is compact. Put another way, ideally we would like to show that non-compactness implies the underlying manifold is $(\mathbb{S}^n, g_0)$.

• We will now formulate the σ_k-Yamabe problem, a fully nonlinear version of the Yamabe problem. Although some new technical issues will arise due to the nonlinear structure of the equation, the problem of the sphere will persist.

The σ_k-Yamabe problem

• To consider the problem of prescribing other symmetric functions of the eigenvalues of the Schouten tensor, we will need some new definitions and notation.

Definition 2. *For* $1 \leq k \leq n$, *define* $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ *by*

$$\sigma_k[x_1, x_2, \ldots, x_n] = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k},$$

the kth elementary symmetric polynomial.

Definition 3. *The* σ_k*-curvature (or* k*-scalar curvature) of* (M, g) *is the function*

$$\sigma_k(A) = \sigma_k[\lambda_1, \ldots, \lambda_n],$$

where $\lambda_1, \ldots, \lambda_n$ *are the eigenvalues of the Schouten tensor.*

Example. The σ_1-curvature is (up to a multiple) the scalar curvature:

$$\sigma_1(A) = \frac{R}{2(n-1)}.$$

$* \quad * \quad *$

The σ_k-Yamabe problem: Given a closed Riemannian manifold of dimension $n \geq 3$, find a conformal metric $\widehat{g} = e^{-2u}g$ such that the σ_k-curvature $\sigma_k(A(\widehat{g}))$ is constant.

$* \quad * \quad *$

• Recall if $\widehat{g} = e^{-2u}g$ is a conformal metric then

$$A(\widehat{g}) = A(g) + \nabla^2 u + du \otimes du - \frac{1}{2}|du|^2 g. \quad (1)$$

• Therefore, to solve the σ_k-Yamabe problem $(SKYP)$ we need to solve the following PDE:

$$\sigma_k\Big[A + \nabla^2 u + du \otimes du - \frac{1}{2}|du|^2 g\Big] = \lambda e^{-2ku} \quad (*)$$

where $A = A(g)$.

• The exponential weight is due to the fact that we are computing the eigenvalues of the the linear map (= $(1,1)$-tensor) induced by $A(\widehat{g})$; i.e., the eigenvalues of

$$\begin{aligned} A(\widehat{g})_i^j &= \widehat{g}^{jk} A(\widehat{g})_{ik} \\ &= e^{2u} g^{jk} \{A_{ik} + \nabla_i \nabla_k u + \nabla_i u \nabla_k u - \frac{1}{2}|du|^2 g_{ik}\}. \end{aligned}$$

• When $k = 1$, this reduces to the Yamabe equation, which is semilinear (that is, linear in the second derivatives, but nonlinear in the lower order terms). However, when $k \geq 2$, $(*)$ becomes *fully nonlinear*, meaning the equation is also nonlinear in the terms involving the second derivatives. For example, when $k = 2$ the leading terms are

$$-\frac{1}{2}|\nabla^2 u|^2 + \frac{1}{2}(\Delta u)^2 + \cdots = \lambda e^{-4u}.$$

• As a consequence of this nonlinear structure, equation $(*)$ is not, in general, elliptic. We will spend the rest of today's lecture trying to understand the definition of ellipticity for general second order equations, then applying these ideas to equation $(*)$.

Fully Nonlinear Equations

A general second order PDE can be written

$$F[u] = F(x, u, \nabla u, \nabla^2 u) = 0, \tag{2}$$

where $F = F(x, z, p, r)$ is defined on a subset of $\mathcal{U} = \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n \times n}$, where $\mathbb{R}^{n \times n}$ is the space of real symmetric matrices.

Definition 4. *Equation (2) is elliptic in $\mathcal{O} \subset \mathcal{U}$ if the matrix*

$$F^{ij}(x, z, p, r) \equiv \frac{\partial F}{\partial r_{ij}}(x, z, p, r) > 0$$

for all $(x, z, p, r) \in \mathcal{O}$.

Example

A linear, second order PDE can be written

$F[u] = a^{ij}(x)\partial_i\partial_j u + b^k(x)\partial_k u + c(x)u - g(x) = 0,$

where

$F(x, z, p, r) = a^{ij}(x)r_{ij} + b^i(x)p_i + c(x)z - g(x).$

Therefore,

$$\frac{\partial F}{\partial r_{ij}}(x, z, p, r) = a^{ij}(x).$$

So ellipticity in this sense coincides with the usual notion.

• If the eigenvalues of $\frac{\partial F}{\partial r_{ij}}$ are uniformly bounded above and below by positive constants, then F is *uniformly elliptic* in $\mathcal{O}$.

• In general, ellipticity may *depend on the solution*.

Example *The Monge-Ampere Equation*. Let $\Omega \subset \mathbb{R}^n$, and $u \in C^2(\Omega)$. Consider

$$F[u] = \det(\nabla^2 u) - f(x) = 0. \qquad (3)$$

In this case,

$$\frac{\partial F}{\partial r_{ij}} = F^{ij} = ((\nabla^2 u)^{ij}),$$

where $(\nabla^2 u)^{ij}$ denotes the inverse of the Hessian matrix. Therefore, (3) will be elliptic provided the Hessian matrix is positive definite; i.e., if u is convex.

Note that *uniform* ellipticity depends on *a priori* bounds on the second derivatives of u!

• It will be illuminating to discuss certain generalizations of the Monge-Ampere equation, known as the *Hessian equations*.

Hessian Equations. Let $\Omega \subset \mathbb{R}^n$, and $u \in C^2(\Omega)$. Consider

$$F[u] = \sigma_k(\nabla^2 u) - f(x) = 0.$$

Thus, F is given by

$$F(x, z, p, r) = \sigma_k(r_{ij}) - f(x).$$

To check the ellipticity properties of this equation we need to understand the matrix

$$\begin{aligned} F^{ij} &= \frac{\partial F}{\partial r_{ij}} \\ &= \frac{\partial}{\partial r_{ij}} \sigma_k(r). \end{aligned}$$

in particular, when is $F^{ij} > 0$? This will require (unfortunately) some more notation and definitions.

Definition 5. *Let $A : V \to V$ be symmetric linear transformation. For $0 \leq q \leq n$, define*

$$T_q(A) = \sigma_q(A) \cdot I - \sigma_{q-1}(A) \cdot A + \cdots + (-1)^q A^q,$$

the qth Newton transformation *associated with* A*.*

<u>Key identity</u>: if $F[A] = \sigma_k(A)$, then

$$F^{ij} = \frac{\partial F}{\partial A_{ij}} = T^{ij}_{k-1}(A)$$

(see reference # 2). That is, the Newton transforms are what we get when we differentiate the symmetric functions.

• In particular: the Hessian equation is elliptic when $T_{k-1}(\nabla^2 u)^{ij} > 0$. But when does this hold?

Some Linear Algebra

Definition 6. *For* $1 \le k \le n$, *let*

$$\Gamma_k^+ = \{\mathbf{x} \in \mathbb{R}^n | \ \sigma_j(\mathbf{x}) > 0 \text{ for } 1 \le j \le k\}.$$

Facts

1. Γ_k^+ convex, open cone with vertex $= \{0\}$.

2. $\Gamma_n^+ \subset \Gamma_{n-1}^+ \subset \cdots \subset \Gamma_1^+$

3. $\Gamma_n^+ = \{\mathbf{x} \in \mathbb{R}^n \mid x^1 > 0, \ldots, x^n > 0\}$, the positive quadrant.

Definition 7. *Let $A : V \to V$ be symmetric linear transformation. We say $A \in \Gamma_k^+$ if the eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of A are in Γ_k^+.*

Remarks.

1. $A \in \Gamma_1^+$ means trace$(A) > 0$.

2. $A \in \Gamma_n^+$ means $\lambda_i > 0$ for all i; i.e., A is positive definite.

3. In general, if $A \in \Gamma_k^+$, then the larger k is, the "more positive" A is.

Homework problem. Let $A \in \mathbb{R}^{3\times 3}$ be symmetric. Show that $A \in \Gamma_2^+$ implies that at most one eigenvalue of A is negative. What else can you say?

Key Fact: If $A \in \Gamma_k^+$, then the k^{th}-Newton transform $T_{k-1}(A)$ is positive definite.

Definition 8. *If $u \in C^2(\Omega)$ and $\nabla^2 u(x) \in \Gamma_k^+$ for each $x \in \Omega$, we say that u is k-*convex*.*

• For example, an n-convex function is convex in the usual sense; i.e., $\nabla^2 u > 0$.

• To summarize our discussions of the Hessian equations:

The Hessian equations are elliptic when we restrict to solutions u which are k-convex.

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• We will begin the next lecture by introducing the appropriate notion of ellipticity for the σ_k-Yamabe equation. This ellipticity condition (termed 'admissibility') is a generalization of k-convexity.

• We will also use much of the algebraic material we covered today in order to derive some basic estimates for solutions of $(*)$.

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Further homework problems

1. Show that (in any dimension),

$$\sigma_2(A) = -\frac{1}{2}|A|^2 + \frac{1}{2}(\sigma_1(A))^2, \qquad (4)$$

where $|A|^2 = \sum_{i,j} A_{ij}^2$.

2. Use the definition of the Newton transform to write the formula for $T_1(A)$.

3. Use (4) to calculate $\partial\sigma_2(A)/\partial A_{ij}$. Compare this with your answer in the preceding problem.

4. Prove Newton's inequality: If $A \in \sigma_n(A)$, then

$$\sigma_n(A)^{1/n} \leq \frac{1}{n}\sigma_1(A).$$