Fully Nonlinear Equations in Conformal Geometry

Geometric Flows and
Geometric Operators
Centro di Ricerca Matematica
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May, 2009

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Lecture I:
Background from Conformal Geometry

References:

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2. S.-Y. Alice Chang, "The Moser-Trudinger inequality and Applications to some problems in conformal geometry", in *Nonlinear Partial Differential Equations in Differential Geometry*, American Math. Soc., IAS/Park City Mathematics Series, 1996.

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Conformal changes of metric

Definition 1. *Let* (M, g) *be a Riemannian manifold. A metric* h *is* pointwise conformal *to* g *(or just* conformal*) if there is a function* f *such that*

$$h = e^f g.$$

• The function e^f is referred to as the *conformal factor*. We used the exponential function to emphasize the fact that we need to multiply by a positive function (since h must be positive definite). However, in some cases it will be more convenient to write the conformal factor differently.

• We can introduce an equivalence relation on the set of metrics: $h \sim g$ iff h is pointwise conformal to g.

The Uniformization Theorem

Let (M, g) be a closed (no boundary), compact, two-dimensional Riemannian manifold. Let K denotes its Gauss curvature.

Theorem 1. *(The Uniformization Theorem) There is a conformal metric $h = e^{2w}g$ with constant Gauss curvature.*

• Let $K_h = const.$ denote the Gauss curvature of the metric h. The sign of K_h is determined by the Gauss-Bonnet formula:

$$\begin{aligned} 2\pi\chi(M) &= \int K_h \, dA_h \\ &= K_h \cdot Area(h). \end{aligned}$$

• Why is the Uniformization Theorem important?

• Recall the classical result of Hopf:

Theorem 2. *(Hopf) Any complete, simply connected manifold of constant sectional curvature is isometric to Euclidean space $\mathbb{R}^n$, hyperbolic space $\mathbb{H}^n$, or the sphere $\mathbb{S}^n$ (up to scaling).*

• In particular, since $h = e^{2w}g$ has constant curvature the universal cover $\tilde{M}$ is isometric to either $\mathbb{S}^2, \mathbb{R}^2$, or $\mathbb{H}^2$, each case being determined by the sign of the Euler characteristic. Thus, the Uniformization Theorem provides strong geometric/topological information about the surface (M, g).

<u>Question</u> How does one prove this?

• Let (M, g) be a surface, and $h = e^{2w}g$ a conformal metric. Then the Gauss curvature K_h of h and the Gauss curvature K_g of g are related by

$$K_h = e^{-2w}\Big[- \Delta_g w + K_g \Big],$$

where Δ_g is the Laplace-Beltrami operator w.r.t g. Therefore, if we can solve the PDE

$$-\Delta w + K = \lambda e^{2w}$$

for a constant λ, then we have proved the Uniformization Theorem.

• There are now many proofs of this result, using a remarkable array of techniques. (Complex analysis, spectral theory, Ricci flow, etc.)

Question: What about higher dimensions?

The curvature tensor

Let (M, g) be a Riemannian manifold, and let $Riem$ denote the curvature tensor. Recall that the curvature tensor is a map $Riem : \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$ defined by

$$Riem(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where $\mathfrak{X}(M)$ denotes the space of smooth vector fields, and $[\cdot,\cdot]$ is the Lie bracket.

• We will view $Riem$ as a $(0,4)$-tensor; in a local coordinate system the components will be denoted

$$R_{ijkl} = g_{km} R^m_{ijl},$$

where

$$R^m_{ijl} = Riem(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^l} \otimes \frac{\partial}{\partial x^m}$$

• We will not need these complicated formulas very much, but here is a very useful little Lemma that gives us some "feel" for how the curvature comes up in our calculations:

Lemma 1. *Let ω be a 1-form; then for vector fields X, Y, Z we have*

$$\nabla_X \nabla_Y \omega(Z) - \nabla_Y \nabla_X \omega(Z) = \omega(R(X,Y)Z).$$

Proof. A short exercise using the definition of R and the properties of ∇. □

Corollary 1. *If $\omega = df$, then in local coordinates*

$$\nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f = \sum_m R^m_{kij} \nabla_m f.$$

<u>In particular</u>: Third derivatives do not commute unless $Riem = 0$, i.e., the manifold is flat.

Ricci and scalar curvature

• The Ricci curvature is the contraction of $Riem$. In local coordinates

$$R_{ij} = g^{kl} R_{ikjl}.$$

By definition, Ric is a symmetric $(0,2)$-tensor.

• For the round metric on $\mathbb{S}^n$, $R_{ij} = (n-1)g_{ij}$.

• The scalar curvature is the contraction of Ric. In local coordinates

$$R = g^{ij} R_{ij}.$$

• On the sphere, $R = n(n-1)$.

• In two dimensions, the curvature tensor $Riem$ is determined by the Gauss curvature K. In fact,

$$Ric = Kg, \quad R = 2K.$$

Curvature and conformal changes of metric

• Let $\widehat{g} = e^{-2u} g$ be conformal metrics, and let $Ric(\widehat{g})$ and $R(\widehat{g})$ denote the Ricci and scalar curvatures of $\widehat{g}$. Then

$$
\begin{aligned}
Ric(\widehat{g}) =& Ric(g) + (n-2)\nabla^2 u + \Delta u g \\
&+ (n-2) du \otimes du - (n-2)|\nabla u|^2 g,
\end{aligned}
$$

$$
\begin{aligned}
R(\widehat{g}) =& e^{2u}\Big\{ R(g) + 2(n-1)\Delta u \\
&- (n-1)(n-2)|\nabla u|^2 \Big\}.
\end{aligned}
$$

• These formulas are quite complicated; the formula for $Riem(\widehat{g})$ is even worse! However, in conformal geometry there are two other tensors that play a central role, and the corresponding transformation formulas for these tensors are *much* easier to understand.

Definition 2. *The* <u>*Schouten*</u> <u>*tensor*</u> *of* (M, g) *is*

$$A = \frac{1}{(n-2)}\Big(Ric - \frac{1}{2(n-1)}R \cdot g\Big).$$

• For the round metric on $\mathbb{S}^n$, $A_{ij} = \frac{1}{2}g_{ij}$, while for hyperbolic space $A_{ij} = -\frac{1}{2}g_{ij}$.

Definition 3. *The* <u>*Weyl*</u> <u>*curvature*</u> <u>*tensor*</u> *of* (M, g) *is*

$$W_{ijkl} = R_{ijkl} - g_{ik}A_{jl} + g_{il}A_{jk} + g_{jk}A_{il} - g_{jl}A_{ik}.$$

• All contractions of the Weyl tensor are zero: $g^{ij}W_{ikjl} = 0$, etc. Indeed, this is just another way to state the definiton of the Weyl tensor.

• As before, let $\widehat{g} = e^{-2u}g$ be conformal metrics, and let $A(\widehat{g})$ denote the Schouten tensor of $\widehat{g}$. Then

$$A(\widehat{g}) = A(g) + \nabla^2 u + du \otimes du - \frac{1}{2}|du|^2 g. \quad (1)$$

• This formula follows from the formulas for the Ricci and scalar curvatures.

• What about the Weyl tensor?

• Key fact: Let $\widehat{g} = e^{-2u}g$ be a conformal metric. Then

$$W(\widehat{g}) = e^{-2u}W(g). \quad (2)$$

Thus, up to a scale factor, the Weyl tensor is *conformally invariant*. This formula follows from the formula (1) above and the formula for the transformation of $Riem$.

What do these formulas tell us?

- Rewrite the definition of the Weyl tensor:

$$R_{ijkl} = W_{ijkl} + g_{ik}A_{jl} - g_{il}A_{jk} - g_{jk}A_{il} + g_{jl}A_{ik} \tag{3}$$

- Let $\widehat{g} = e^{-2u}g$ be a conformal metric. Then (3) tells us that $Riem(\widehat{g})$ is basically determined by $A(\widehat{g})$.

In other words, if we understand how the Schouten tensor transforms under a conformal change of metric, then we understand how the entire curvature tensor does: The Schouten tensor is the "piece" of the curvature that is not invariant under conformal changes of the metric.

Further properties Weyl & Schouten

• The following is a classical result from the theory of Riemann surfaces:

Theorem 3. *(Existence of isothermal coordinates) Let* (M^2, g) *be a two-dimensional Riemannian manifold, and* $p \in M^2$*. Then there is a local coordinate system* $\{x^1, x^2\}$ *near* p*, such that the metric* g *in this system is given by*

$$g = e^{-2\phi}[(dx^1)^2 + (dx^2)^2].$$

In particular, every Riemann surface is locally *conformal to the flat Euclidean metric.*

• We can skecth a proof of this result using the material we've covered so far. This will also help us to understand why the Uniformization Theorem cannot hold (in general) for higher dimensional manifolds.

Sketch of the proof Choose a small geodesic ball B around $p \in M^2$. On this ball, we can solve the PDE

$$\begin{aligned} -\Delta\phi + K_g = 0 \qquad & \text{in } B, \\ \phi = 0 \qquad & \text{on } \partial B. \end{aligned}$$

- Homework problem: Solve the above PDE!

- Let $h = e^{2\phi}g$. Using the formulas above, we find that the Gauss curvature of h is

$$K_h = e^{-2\phi}\Big[-\Delta\phi + K_g \Big] = 0.$$

Therefore, h is flat. This means that p has a small neighborhood $U \subset B$ which is isometric to a small neighborhood $V \subset \mathbb{R}^2$. In particular, on U we have Euclidean coordinates $\{x^1, x^2\}$ such that

$$h = (dx^1)^2 + (dx^2)^2.$$

Since $h = e^{2\phi}g$, it follows that

$$g = e^{-2\phi}[(dx^1)^2 + (dx^2)^2].$$

QED.

• In general, isothermal coordinates do not exist in higher dimensions: there is an integrability condition that the metric g must satisfy. Here is one way to think about this condition:

• Suppose we want to find a (locally defined) conformal factor ϕ so that

$$g = e^{-2\phi} ds^2,$$

where ds^2 is the flat metric. For the flat metric, the Weyl tensor (indeed, all the components of the curvature tensor) must vanish. Therefore,

$$W(g) = e^{-2\phi} W(ds^2) = 0.$$

So the vanishing of the Weyl tensor is a necessary condition for the existence of isothermal coordinates

• In fact, if $n \geq 4$, then the vanishing of the Weyl tensor is also sufficient. When $n = 3$, there is a condition on the Schouten tensor. Such manifolds are called *locally conformally flat*, for obvious reasons.

To summarize: In dimensions $n \geq 4$ we cannot conformally change to a metric of constant (sectional) curvature, *even locally*, unless the Weyl tensor vanishes.

• Another way to think about this is the following: The Ricci tensor (which is "part" of the curvature tensor) has $n(n-1)/2$ components, but in making a conformal change of metric we are only allowed to choose one function. Therefore, if $n \geq 3$, the problem is highly *overdetermined*.

• A more natural problem is to find a conformal metric for which a *scalar* quantity (i.e., a function) is constant.

Question What are the natural candidates?

Toward an answer

• Recall the behavior of the curvature tensor under a conformal change of metric is determined by the Schouten tensor. So, another way to ask the question above is, what are the natural (scalar) quantities we can construction from a $(0,2)$-tensor (i.e., bilinear form)?

• The algebraically natural response is to look at the elementary symmetric functions of the eigenavlues of the form.

• Since the Schouten tensor is symmetric, it has n real eigenvalues at each point:

$$A = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix}$$

By the definition of the Schouten tensor,

$$\text{trace}(A) = \alpha_1 + \cdots + \alpha_n = \frac{1}{2(n-1)} R.$$

• This suggests the following generalization of the Uniformization Theorem:

The Yamabe problem: Given a closed Riemannian manifold of dimension $n \geq 3$, find a conformal metric $\widehat{g} = e^{-2u}g$ such that the scalar curvature $R(\widehat{g})$ is constant.

• Like the Uniformization Theorem, solving the Yamabe problem is equivalent to solving a nonlinear PDE. From the formulas above, the PDE in this case is

$$2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2 + R(g) = \lambda e^{-2u}$$

for some constant λ.

• This formula can be simplified if we write $\widehat{g} = v^{4/(n-2)}g$, where $v > 0$. Then v should satisfy

$$-\frac{(n-2)}{4(n-1)}\Delta v + Rv = \lambda v^{\frac{n+2}{n-2}}.$$

• In the next lecture, we will discuss some PDE aspects of this problem, and indicate the main technical difficulty ("bubbling").

• Next, we will formulate another generalization of the Uniformization Theorem by considering other symmetric functions of the eigenvalues of the Schouten tensor. This problem, known as the σ_k-Yamabe problem, will be the main focus of the rest of the lectures.

• Along the way, we will learn some background material in the theory of fully nonlinear PDEs, and go through the details of the some of the main regularity estimates.