Fully Nonlinear Equations in Conformal Geometry

Geometric Flows and
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Lecture III:
The σ_k-Yamabe problem: Further properties

References:

1. Local estimates for some fully nonlinear elliptic equations, by Sophie Chen, *Int. Math. Res. Not.* 2005, no. 55, 3403–3425.

2. Estimates and existence results for some fully nonlinear elliptic equations on Riemannian manifolds, by J. Viaclovsky, *Comm. Anal. Geom.* **10** (2002), no. 4, 815–846.

Back to the σ_k-Yamabe equation

• Recall we are interested in

$$\sigma_k\Big[A(g) + \nabla^2 u + du \otimes du - \frac{1}{2}|du|^2 g\Big] = \lambda e^{-2ku}. \quad (*)$$

• By the preceding material, we can say that this equation is elliptic provided the conformal Schouten tensor

$$A(\hat{g}) = A(g) + \nabla^2 u + du \otimes du - \frac{1}{2}|du|^2 g \in \Gamma_k^+.$$

• Think of this as a generalization of k-convexity.

• Note that $\lambda > 0$ is a necessary condition! (More on this point later).

• In practice, the way one solves equation $(*)$ is to *begin* with a manifold whose Schouten tensor satisfies $A \in \Gamma_k^+$ at each point, then *deform* it (through conformal metrics satisfying the ellipticity condition) to a solution of $(*)$.

Definition 1. *We say that (M, g) is admissible (or k-admissible) if the Schouten tensor $A \in \Gamma_k^+$ at each point.*

Examples.

1. The round sphere is admissible (for any $k \geq 1$), since $A = \text{diag}\{\frac{1}{2}, \ldots, \frac{1}{2}\}$.

2. The product manifold $\mathbb{S}^2 \times \mathbb{S}^1$ is <u>not</u> admissible for $k \geq 2$. In this case, the Schouten tensor is $A = \text{diag}\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\}$. Thus

$$\sigma_1(A) = 1/2,\ \sigma_2(A) = -1/4,\ \sigma_3(A) = -1/8.$$

In fact, we will see that the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ *never* admits an admissible metric for $k \geq 2$! In particular, admissibility is a geometric/topological condition.

What is the Geometric Meaning of Admissibility?

• Here is a nice result of Guan-Viaclovsky-Wang (generalizing an earlier result of Chang-Gursky-Yang):

Theorem 1. *Suppose (M, g) is k-admissible. Then the Ricci curvature satisfies*

$$Ric \geq \frac{k - n/2}{n(k-1)} R \cdot g.$$

In particular: if $k > n/2$, then the Ricci curvature of g is positive.

• This albegraic lemma has some important consequences. It will play an important role in the existence results we discuss later; for now we just want to point out a topological consequence:

Corollary 1. *The product manifold $M^3 = \mathbb{S}^2 \times \mathbb{S}^1$ does not admit an admissible metric if $k \geq 2$.*

Proof. Since

$$k \geq 2 > 3/2 = \frac{\text{dim } M^3}{2},$$

it follows that an admissible metric would have positive Ricci curvature. Then the Bochner theorem would imply that the first Betti number of M^3 vanishes, a contradiction. □

* * *

• Now would be a good time to give a more precise formulation of our problem:

The σ_k-Yamabe problem: Assume (M, g) is k-admissible. Find a k-admissible conformal metric $\widehat{g} = e^{-2u} g$ such that the σ_k-curvature satisfies

$$\sigma_k(A(\widehat{g})) = 1.$$

• Notice we have normalized the constant to be 1.

One problem, two equations: The negative cone

• When stating the definition of ellipticity for the equation

$$F[u] = F(x, u, \nabla u, \nabla^2 u) = 0,$$

the important point is that the linearized operator

$$F_{ij}(x, z, p, r) \equiv \frac{\partial F}{\partial r_{ij}}(x, z, p, r)$$

is *definite*—i.e., all of its eigenvalues must be positive or negative. For example, for the Monge-Ampere equation, we could just as well require u to be concave (i.e., $\nabla^2 u < 0$) instead of convex–it is simply a matter of convention.

• Similarly, for the Hessian equations we could require that $-\nabla^2 u$ be k-convex (at the price of keeping careful track of all the signs!).

• This is not the case with the σ_k-curvature equations. More precisely: the equation is also elliptic provided

$$-A(\hat{g}) = -\{A(g) + \nabla^2 u + du \otimes du - \frac{1}{2}|du|^2 g\} \in \Gamma_k^+.$$

However, this change in sign results in a new equation! This due to the fact that the non-Hessian terms are not linear in the solution u. To see this, let $w = -u$; then the condition above is equivalent to

$$-A(\hat{g}) = -A(g) + \nabla^2 w - dw \otimes dw + \frac{1}{2}|dw|^2 g \in \Gamma_k^+.$$

Definition 2. *We say that* (M, g) *is* negative k*-admissible if* $-A(g) \in \Gamma_k^+$ *at each point.*

Example. A hyperbolic manifold is negative admissible since $A = \mathrm{diag}\{-\frac{1}{2}, \dots, -\frac{1}{2}\}$.

• This definition leads to...

The σ_k-Yamabe problem (negative cone): Assume (M, g) is negative k-admissible. Find a k-admissible conformal metric $\widehat{g} = e^{2w} g$ such that the σ_k-curvature satisfies

$$\sigma_k(A(\widehat{g})) = 1.$$

• This is equivalent to solving the PDE

$$\sigma_k\Big[- A(g) + \nabla^2 w - dw \otimes dw + \frac{1}{2}|dw|^2 g\Big] = e^{2kw}. \qquad (-*)$$

• To highest order, this equation looks the same as $(*)$: we apply σ_k to the Hessian of the solution plus first and zeroth order terms. However, for reasons that are actually quite deep, the existence and regularity theory for this equation is very different. Indeed, at this time there are no general existence results for the negative cone equation.

Variational aspects

• A feature of the Yamabe problem that plays a crucial role in its solution is the fact that the equation is variational; i.e., the Yamabe equation is the Euler-Lagrange equation of a certain functional.

• The situation for the $SKYP$ is more complicated; indeed, the variational theory is almost a subject in its own right, and has fascinating connections with mathematical physics and the theory of conformally compact Einstien manifolds. We cannot go into very much detail here, other than to state the basic results in the thesis of Viaclovsky.

• To this end, define the functionals

$$\mathcal{F}_k[g] = \int \sigma_k(A(g))\ dV(g).$$

We will view $\mathcal{F}_k$ as a functional on the space $[g]_1 = \{\hat{g} = e^{-2u}g \mid Vol(\hat{g}) = 1\}$, i.e., the metrics of unit volume in a fixed conformal class.

Theorem 2. *(Viaclovsky)* (i) *If* $k = 1$ *or* 2, *and* $k \neq n/2$, *then a metric* g *has constant* σ_k-*curvature if and only if it is a critical point of* $\mathcal{F}_k$ *restricted to* $[g]_1$.

(ii) *If* (M, g) *is locally conformally flat, then the same statement holds for all* $k \neq n/2$.

Sketch of the proof. Assuming a certain Lemma of R. Reilly (see the references from Lecture II), we can give an idea of the proof.

Let $g_t = e^{-2t\phi} g$. Then the volume form $dV(g_t) = e^{-nt\phi} dV(g)$, hence

$$\frac{d}{dt} dV(g_t)|_{t=0} = -n\phi \; dV(g).$$

Also, by the conformal transformation law for the $(1,1)$-Schouten tensor,

$$\begin{aligned} \frac{d}{dt} A(g_t)_i^j|_{t=0} &= \frac{d}{dt}\{g_t^{jk} A(g_t)_{ik}\}|_{t=0} \\ &= \nabla_i \nabla^j \phi + 2\phi A(g)_i^j. \end{aligned}$$

Therefore, by the definition of the Newton transform and the homogeneity of the symmetric functions,

$$\begin{aligned}\frac{d}{dt}\sigma_k(A(g_t))\Big|_{t=0} &= T_{k-1}(A(g))^i_j\{\nabla_i\nabla^j\phi + 2\phi A(g)^j_i\} \\ &= T_{k-1}(A(g))^i_j\nabla_i\nabla^j\phi + k\phi\sigma_k(A(g)).\end{aligned}$$

It follows that the first variation of $\mathcal{F}_k$ is given by

$$\begin{aligned}\mathcal{F}_k' = \int \Big[&T_{k-1}(A(g))^i_j\nabla_i\nabla^j\phi \\ &+ (2k-n)\phi\sigma_k(A(g))\Big]\, dV(g).\end{aligned}$$

Note that we can integrate by parts in the first term and rewrite this as

$$\begin{aligned}\mathcal{F}_k' = \int \Big[&\{-\nabla^j T_{k-1}(A(g))^i_j\}\nabla_i\phi \\ &+ (2k-n)\phi\sigma_k(A(g))\Big]\, dV(g).\end{aligned}$$

The proof will follow from the following result of Reilly:

Proposition 1. *(Reilly)* (i) *If* $k = 1$ *or* 2*, then* T_{k-1} *is divergence-free:*

$$\nabla^j T_{k-1}(A(g))_j^i = 0. \qquad (1)$$

(i) *If* $k \geq 3$ *and* (M, g) *is locally conformally flat, then (1) also holds.*

Applying this Proposition, we see that under the assumptions of the Theorem,

$$\mathcal{F}_k' = (2k - n) \int \phi \sigma_k(A(g)) \, dV(g).$$

It follows that if $k \neq n/2$, then a metric is critical for $\mathcal{F}_k\Big|_{[g]_1}$ if and only if its σ_k-curvature is constant.

QED

• The proof of the Theorem implies the following remarkable corollary:

Corollary 2. (i) *In dimension four, the integral*

$$\int \sigma_2(A)\ dV$$

is conformally invariant.

(ii) *If the dimension* $n \geq 6$ *is even and* (M, g) *is locally conformally flat, then the integral*

$$\int \sigma_{n/2}(A)\ dV$$

is conformally invariant.

Remark. In both cases the integrals are related to the Chern-Gauss-Bonnet formula. When $n = 4$,

$$2\pi^2\chi(M^4) = \int \frac{1}{4}|W|^2\ dV + \int \sigma_2(A)\ dV,$$

while if $n \geq 6$ is even and (M^n, g) is locally conformally flat,

$$\chi(M^n) = c_n \int \sigma_{n/2}(A)\ dV.$$

This fact was first proved by Branson-Gilkey.

A priori estimates, part I:

<u>C^0-estimates</u>

• The rest of our lectures will be devoted to proving various *a priori* estimates for solutions the $SKYP$. As a consequence, we will be able to sketch the proof of an existence result for $k > n/2$.

• To develop a "feeling" for how these estimates work, we will begin by studying C^0-bounds for solutions. This will amount to an elementary application of the maximum principle, but along the way we will see how the algerbaic properties of the symmetric functions play an important role in the theory.

• Our first result is:

Theorem 3. *Let $w \in C^2$ be a solution of $(-*)$. The there is a constant $C_0 > 0$ such that*

$$-C_0 \leq w \leq C_0.$$

Proof. Denote

$$F[W] = \sigma_k^{1/k}[W].$$

Then equation $(-*)$ can be written

$$F[(-A) + \nabla^2 w - dw \otimes dw + \frac{1}{2}|dw|^2 g] = e^{2w}.$$

Let $p \in M$ be a point at which w attains its maxiumum value. Then

$$dw(p) = 0, \quad \nabla^2 w(p) \leq 0.$$

In particular, at p we have

$$F[(-A)(p) + \nabla^2 w(p)] = e^{2w(p)}.$$

Let

$$W_t = (-A)(p) + t\nabla^2 w(p),$$

and $f(t) = F[W_t]$. Note that

$$W_0, W_1 \in \Gamma_k^+ \ \Rightarrow \ W_t \in \Gamma_k^+,$$

since the Γ_k^+ are convex.

Now,

$$
\begin{aligned}
f(1) - f(0) &= e^{2w(p)} - \sigma_k((-A)(p))^{1/k} \\
&= \int_0^1 \frac{d}{dt} F[W_t]\ dt \\
&= \int_0^1 F^{ij}[W_t] \frac{d}{dt}(W_t)_{ij}\ dt \quad \text{(chain rule)} \\
&= \int_0^1 F^{ij}[W_t] \nabla_i \nabla_j w(p)\ dt
\end{aligned}
$$

Recall that $\partial\sigma_k/\partial r_{ij} = T^{ij}_{k-1}$, the Newton transform, hence

$$F^{ij}[W_t] = \frac{1}{k}\sigma_k(W_t)^{1/k-1} T^{ij}_{k-1}[W_t] > 0,$$

since $W_t \in \Gamma_k^+$. On the other hand, $\nabla_i \nabla_j w(p) \leq 0$, hence

$$e^{2w(p)} - \sigma_k((-A)(p))^{1/k} \leq 0.$$

Therefore, we have an upper bound for w depending only on the background metric g.

The proof of the lower bound is immediate: just apply the same argument at a minimum point. $\square$

Remarks.

1. The preceding estimate does not hold for solutions of $(*)$: recall the example of the sphere! Here we see the first of many differences between the positive and negative cone equations.

2. Using the C^0-bound, Viaclovsky also proved *a priori* C^1-bounds:

Proposition 2. *Let* $w \in C^2$ *be a solution of* $(-*)$. *The there is a constant* $C_1 = C(C_0) > 0$ *such that* $|\nabla w| \leq C_1$.

3. However, this is the end of the story (or the beginning, depending on your point of view), in the sense that there are counterexamples to (local) C^2-estimates:

Proposition 3. *(Sheng-Trudinger-Wang) There is a sequence $\{w_k\}$ of (negative admissible) solutions of the negative cone equation, defined in the unit ball $B(1) \subset \mathbb{R}^n$, with the following properties:*

$$\max_{B(1)} |w_k| + |\nabla w_k| \leq C,$$

and

$$|\nabla^2 w_k(0)| \to +\infty.$$

• It turns out that the σ_k-Yamabe equations are related to equations that arise in the theory of optimal transportation, and this loss of regularity is something well understood in the theory. Ultimately, it is a consequence of the nonlinear structure of the gradient terms–they are not lower order, after all!

• For the remaining lectures, therefore, we will restrict our attention to the positive cone equation $(*)$, beginning with the fundamental local estimates of Guan-Wang and S. Chen.