Fully Nonlinear Equations in Conformal Geometry

Geometric Flows and
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Lecture IV:
Local C^2-estimates

References:

1. Classical solutions of fully nonlinear, convex, second-order elliptic equations, by L. Evans, *Comm. Pure Appl. Math.* **35** (1982), 333–363.

2. Boundedly inhomogeneous elliptic and parabolic equations in a domain, by N. Krylov, *Izv. Akad. Nauk SSSR Ser. Mat.* **47** (1983), 75–108.

3. Local estimates for some fully nonlinear elliptic equations, by Sophie Chen, *Int. Math. Res. Not.* 2005, no. 55, 3403–3425.

• In this lecture we will prove *a priori* estimates for (admissible) solutions of the equation

$$\sigma_k^{1/k}\Big[A + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g\Big] = f(x)e^{-2u(x)} \tag{1}$$

where $f(x) > 0$. When $f \equiv const.$, then u will be a solution of $(*)$.

• In fact, the estimates we will outline can easily be adapted to more general equations of the form

$$F(\nabla^2 u + c_1 du \otimes du + c_2|\nabla u|^2 g + S) = f(x, u), \tag{2}$$

where F is a symmetric function satisfying certain "structural" conditions. Equations not unlike (2) arise, for example, in geometric optics.

Preliminary Remarks

• We begin by outlining the goal of our estimates: what we hope to prove, and what the obstructions might be.

What are we trying to estimate, anyway?

• Owing to a fundamental result to Evans, Krylov (plus the classical Schauder estimates), we only need to worry about estimating derivatives up to order two. That is,

$$|u| + |\nabla u| + |\nabla^2 u| \leq C_2$$
$$\Downarrow$$
$$|u| + |\nabla u| + \cdots + |\nabla^k u| \leq C(k, C_2).$$

• To understand why, let's first state the Theorem of Evans (ref. #1) and Krylov (ref. #2).

Theorem 1. *(Evans, Krylov) Suppose* $u \in C^4$ *is a solution of*

$$F[x, u, Du, D^2u] = 0 \qquad (3)$$

in $B(1)$, *where we assume*

(i) F *is uniformly elliptic with respect to* u; *i.e., there are constants* $0 < \lambda < \Lambda$ *such that*

$$\lambda|\xi|^2 \le F^{ij}(x, u, Du, D^2u)\xi_i\xi_j \le \Lambda|\xi|^2 \qquad (4)$$

for all $\xi \in \mathbf{R}^n$, *where* $F^{ij} = \partial F/\partial r_{ij}$; *and*

(ii) F *is concave:*

$$F^{ij,kl}\eta_{ij}\eta_{kl} \le 0$$

for all $\eta \in \mathbf{R}^{n\times n}$.

Then u *satisfies*

$$\|u\|_{C^{2,\alpha}(B(1/2))} \le C(\|u\|_{C^2(B(1))}). \qquad (5)$$

• Now, suppose u is an admissible solution of (1) which satisfies an estimate of the form

$$|u| + |\nabla u| + |\nabla^2 u| \le C_2 \qquad (6)$$

say, on $B(1)$. It is a well known property that $\sigma_k^{1/k}$ is concave when restricted to Γ_k^+. This implies that u is a solution of an equation like (3), with F satisfying (ii).

To apply the E-K result, we still need to verify the uniform ellipticity condition (4). This follows from properties of the Newton transforms and the estimate (6) above. More precisely, we need the following fact:

Proposition 1. *Let* $F = \sigma_k^{1/k} : \mathbf{R}^n \to \mathbf{R}$. *Suppose* $A \in \Gamma_k^+$ *satisfies* (a) $F(A) \ge c_0 > 0$, *and* (b) $|\lambda(A)| \le C_1$. *Then there is a positive constant* $\epsilon = \epsilon(c_0, C_1) > 0$ *such that*

$$\epsilon|\xi|^2 \le F^{ij}(A)\xi_i\xi_j \le \epsilon^{-1}|\xi|^2 \qquad (7)$$

for all $\xi \in \mathbf{R}^n$.

• Since the estimate (6) clearly implies properties (a) and (b), the uniform ellipticity of our equation follows from (7). Consequently, the E-K result implies

$$\|u\|_{C^{2,\alpha}} \leq C_2'$$

on the half ball.

• The next step in proving higher order estimates is to differentiate the equation. To this end, we write the equation as

$$F(W) = f(x)e^{-2u},$$

where $F = \sigma_k^{1/k}$ and

$$W_{ij} = \nabla_i \nabla_j u + \nabla_i u \nabla_j u - \frac{1}{2}|\nabla u|^2 g_{ij} + A_{ij}. \quad (8)$$

Then, differentiating in the x^k-direction and using the chain rule gives

$$\nabla_k F(W) = F^{ij} \nabla_k W_{ij} = \nabla_k (f e^{-2u}). \quad (9)$$

By (8),

$$\nabla_k W_{ij} = \nabla_k \nabla_i \nabla_j u + \cdots ,$$

where "$\cdots$" denotes terms which are (at most) second order in u. Commuting derivatives as in Lecture I, we get

$$\begin{aligned} \nabla_k W_{ij} &= \nabla_i \nabla_j \nabla_k u + R_{kijm} \nabla_m u + \cdots \\ &= \nabla_i \nabla_j (\nabla_k u) + \cdots \end{aligned} \tag{10}$$

Let $w = \nabla_k u$. Substituting (10) into (9) gives

$$F^{ij} \nabla_i \nabla_j w = \Big(2^{nd}\text{-order terms in } u \Big). \tag{11}$$

• Thus, w satisfies a linear, second order uniformly elliptic equation with Hölder continuous data. By the Schauder estimates we obtain a bound

$$\|w\|_{C^{2,\alpha}} \leq C,$$

which implies a $C^{3,\alpha}$-estimate on u. We can therefore appeal to (11) again, now using the fact that our data is in $C^{1,\alpha}$.

• This gives us a bound on the $C^{3,\alpha}$-bound on w–and hence a $C^{4,\alpha}$-bound on u. By mathematical induction, it follows that we can estimate the C^k-norm of u, for any $k \geq 1$.

<u>Obstructions</u>

• Let's return to the case of the scalar curvature; i.e., $k = 1$. Recall that $\mathbf{S}^n$ we have a 1-parameter family $\{v_\lambda\}$ of solutions to

$$-\frac{(n-2)}{4(n-1)}\Delta v_\lambda + n(n-1)v_\lambda = n(n-1)v_\lambda^{\frac{n+2}{n-2}}.$$

If we let

$$u_\lambda = -\frac{2}{n-2}\log v_\lambda,$$

then this gives a 1-parameter of solutions to (1) with $k = 1$.

Note that as $\lambda \to \infty$, the conformal factor grows like

$$\max e^{-2u_\lambda} \sim \lambda^2,$$

while the gradient and Hessian of u grow like

$$\max |\nabla u_\lambda|^2 \sim \lambda^2, \quad \max |\nabla^2 u_\lambda| \sim \lambda^2.$$

In particular, this example shows that we can never hope to obtain global estimates on u and its derivatives without some kind of geometric assumption. However, we may be able to prove some kind of "bootstrap" estimate, along the lines of

$$\max \Big(2^{nd} \text{ derivatives of } u \Big) \leq C \max e^{-2u}.$$

Local Estimates

- Here is the estimate we are going to prove:

Theorem 2. *(Guan-Wang, Chen, X.J. Wang) Assume* $u \in C^4$ *is an admissible solution of (1) on* $B(1)$ *with* $k \geq 2$. *Then*

$$\max_{B(1/2)} \Big[|\nabla u|^2 + |\nabla^2 u| \Big] \leq C(1 + \max_{B(1)} e^{-2u}). \tag{12}$$

- We have fixed the radius of the ball to be 1, but a simple scaling argument can be used to give an estimate on a ball of any radius $r > 0$. The precise dependence on the radius is

$$\max_{B(r/2)} \Big[|\nabla^2 u| + |\nabla u|^2 \Big] \leq C\Big(r^{-2} + \max_{B(r)} e^{-2u} \Big).$$

• This estimate implies an "ϵ-regularity" result:

Proposition 2. *There is an* $\epsilon_0 > 0$ *such that any admissible solution of (1) on* $B(r)$ *with*

$$\int_{B(r)} e^{-nu}\, dvol \leq \epsilon_0 \tag{13}$$

satisfies

$$\max_{B(r/2)} e^{-2u} \leq C r^{-2}. \tag{14}$$

Consequently,

$$\max_{B(r/4)} \left[|\nabla^2 u| + |\nabla u|^2 \right] \leq C' r^{-2}.$$

• This result will be important in Lecture V.

The Proof

• The proof we present is due to Sophie Chen, and is more transparent than the proof of Guan-Wang. It is also more flexible, and can be adapted to more general equations of the form

$$F(W) = f(x, u),$$

where F is a symmetric function of the eigenvalues of W satisfying certain structural conditions.

• Proving the estimate on a ball requires the use of a "cut-off" function, introducing many terms into the relevant formulas. To keep things as simple as possible, we will prove a global estimate of the form

$$\max_M \left[|\nabla u|^2 + |\nabla^2 u| \right] \leq C(1 + \max_M e^{-2u}). \quad (15)$$

By examining the proof, it will easy to see how things should be modified to give the local version.

• We begin with two simple, but important, observations:

• First, since $W \in \Gamma_k^+$ at each point, it follows that $\sigma_1(W) > 0$. This implies

$$\sigma_1(A) + \Delta u - \frac{n-2}{2}|\nabla u|^2 > 0,$$

hence

$$|\nabla u|^2 \leq C[\Delta u + 1].$$

This implies two facts:

(1) Δu is bounded below, and

(2) An upper bound on Δu implies an upper bound on $|\nabla u|^2$.

• Next, since $k \geq 2$, we also have $\sigma_2(W) > 0$. From your homework you will recall that

$$0 < \sigma_2(W) = -\frac{1}{2}|W|^2 + \frac{1}{2}\sigma_1(W)^2.$$

This implies (after some messy algebra)

$$|\nabla^2 u|^2 \leq C[(\Delta u)^2 + |\nabla u|^4 + 1] \\ \leq C[(\Delta u)^2 + 1].$$

Therefore, if we obtain an upper bound for Δu, then we have upper bounds for $|\nabla^2 u|$ (and $|\nabla u|$). So that will be our goal.

- To begin the proof, let

$$H = \Delta u + |\nabla u|^2.$$

We will show that

$$H \leq C(1 + \max_M e^{-2u}).$$

- At a maximum point of H we have

$$0 = \nabla_j H = \nabla_j(\Delta u) + \nabla_j |\nabla u|^2,$$

and

$$0 \geq \nabla_i \nabla_j H = \nabla_i \nabla_j (\Delta u) + \nabla_i \nabla_j |\nabla u|^2.$$

We need to commute some derivatives, and this will introduce curvature terms (see Corollary 1 from Lecture I). For example,

$$\begin{aligned}\nabla_j(\Delta u) &= g^{k\ell}\nabla_j\nabla_k\nabla_\ell u \\ &= g^{k\ell}\Big[\nabla_k\nabla_j\nabla_\ell u + R^m_{jk\ell}\nabla_m u\Big] \\ &= \Delta\nabla_j u + O(|\nabla u|).\end{aligned}$$

Continuing in this way, we find

$$\begin{aligned}\nabla_i\nabla_j(\Delta u) &= \Delta[\nabla_i\nabla_j u]. + O(|\nabla^2 u| + |\nabla u|) \\ &= \Delta[\nabla_i\nabla_j u]. + O(H)\end{aligned}$$

We then replace the Hessian term with W_{ij} to get

$$\begin{aligned}\nabla_i\nabla_j(\Delta u) &= \Delta\{W_{ij} + \frac{1}{2}|\nabla u|^2 g_{ij} - \nabla_i u\nabla_j u\} \\ &\qquad + O(H+1) \\ &= \Delta W_{ij} + \frac{1}{2}\Delta|\nabla u|^2 g_{ij} - \Delta(\nabla_i u\nabla_j u) \\ &\qquad + O(H+1).\end{aligned}$$

Next, we compute

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla(\Delta u), \nabla u\rangle + O(H),$$

$$-\Delta(\nabla_i u \nabla_j u) = -2\nabla_k \nabla_i u \nabla_k \nabla_j u - \nabla_i(\Delta u)\nabla_j u - \nabla_j(\Delta u)\nabla_i u + O(H).$$

Therefore,

$$\nabla_i\nabla_j(\Delta u) = \Delta W_{ij} + |\nabla^2 u|^2 g_{ij} + \langle \nabla(\Delta u), \nabla u\rangle g_{ij} - \nabla_i(\Delta u)\nabla_j u - \nabla_j(\Delta u)\nabla_i u - 2\nabla_k\nabla_i u \nabla_k \nabla_j u + O(H).$$

Similarly, for the gradient term we get

$$\begin{aligned}
\nabla_i\nabla_j|\nabla u|^2 &= 2\nabla_k u \nabla_k[\nabla_i\nabla_j u] \\
&\quad + 2\nabla_i\nabla_k u \nabla_j \nabla_k u + O(|\nabla u|^2) \\
&= 2\nabla_k u \nabla_k\{W_{ij} + \frac{1}{2}|\nabla u|^2 g_{ij} - \nabla_i u \nabla_j u\} \\
&\quad + 2\nabla_i\nabla_k u \nabla_j \nabla_k u + O(H) \\
&= 2\nabla_k u \nabla_k W_{ij} + \langle \nabla u, \nabla|\nabla u|^2\rangle g_{ij} \\
&\quad - \nabla_i|\nabla u|^2 \nabla_j u - \nabla_j |\nabla u|^2 \nabla_i u \\
&\quad + 2\nabla_i\nabla_k u \nabla_j \nabla_k u + O(H).
\end{aligned}$$

Adding these formulas together, we get the remarkably simple identity

$$\begin{aligned}\nabla_i\nabla_j H &= \nabla_i\nabla_j(\Delta u) + \nabla_i\nabla_j|\nabla u|^2\\ &= \Delta W_{ij} + 2\nabla_k u\nabla_k W_{ij} + |\nabla^2 u|^2 g_{ij}\\ &\quad + \langle\nabla u, \nabla H\rangle g_{ij} - \nabla_i H\nabla_j u - \nabla_j H\nabla_i u\\ &\quad + O(H).\end{aligned}$$

In addition, from our remarks at the beginning of the proof we know that

$$\begin{aligned}|\nabla^2 u|^2 &\geq \frac{1}{n}(\Delta u)^2\\ &\geq \delta_0 H^2\end{aligned}$$

for some dimensional constant $\delta_0 > 0$. Also, at the maximum point of H the gradient vanishes, so we get

$$\begin{aligned}\nabla_i\nabla_j H \geq \Delta W_{ij} + 2\nabla_k u\nabla_k W_{ij} + \delta_0 H^2 g_{ij}\\ + O(H).\end{aligned}$$

Let $F^{ij} = \frac{\partial F}{\partial W_{ij}}$; then $F^{ij} > 0$ (since $W \in \Gamma_k^+$). At the maximum point of H, we therefore have

$$0 \geq F^{ij}\nabla_i\nabla_j H$$
$$\geq F^{ij}\{\Delta W_{ij} + 2\nabla_k u \nabla_k W_{ij} + \delta_0 H^2 g_{ij} + O(H)\} \quad (16)$$

Claim 1. *We have*

$$F^{ij}\nabla_k W_{ij}\nabla_k u \geq F^{ij}\{-C[1+e^{-2u}]Hg_{ij}\},$$

and

$$F^{ij}\Delta W_{ij} \geq F^{ij}\{-C[1+e^{-2u}]Hg_{ij}\}.$$

Proof. To prove the Claim, we will need two basic properties of the symmetric functions (see ref #3 from Lecture II). The first is that

$$F^{kk} = \sum_{i,j} F^{ij} \geq 1.$$

This follows from the Newton-Maclaurin inequality and properties of the Newton transforms.

The second property we already mentioned in connection with the Evans-Krylov result: $F : \Gamma_k^+ \subset \mathbb{R}^{n\times n} \to \mathbb{R}$ is concave, i.e.,

$$F^{ij,\alpha\beta}\xi_{ij}\xi_{\alpha\beta} \leq 0$$

for any $\xi \in \mathbb{R}^{n\times n}$.

To begin the proof of the Claim, by the definition of F^{ij} we have

$$\nabla_k F(W) = F^{ij}\nabla_k W_{ij}$$

Therefore, from the equation

$$\begin{aligned} F^{ij}\nabla_k W_{ij}\nabla_k u &= \nabla_k F(W)\nabla_k u \\ &= \nabla_k[f(x)e^{-2u}]\nabla_k u \\ &\geq -C[1+e^{-2u}]H. \end{aligned}$$

Using the fact that $F^{kk} = F^{ij}g_{ij} \geq 1$, we conclude

$$F^{ij}\nabla_k W_{ij}\nabla_k u \geq F^{ij}\{-C[1+e^{-2u}]Hg_{ij}\}.$$

This proves the first part of the claim.

To prove the second part, we use the concavity of F to get

$$\nabla_k F(W) = F^{ij} \nabla_k W_{ij},$$
$$\Rightarrow$$
$$\begin{aligned} \Delta F(W) &= F^{ij,\alpha\beta} \nabla_k W_{ij} \nabla_k W_{\alpha\beta} + F^{ij} \Delta W_{ij} \\ &\leq F^{ij} \Delta W_{ij}. \end{aligned}$$

Therefore, using the equation once again and arguing as before we get

$$\begin{aligned} F^{ij} \Delta W_{ij} &\geq \Delta[f(x)e^{-2u}] \\ &\geq F^{ij}\{-C[1 + e^{-2u}]H g_{ij}\}. \end{aligned}$$

□

• Applying the Claim to inequality (16), at the maximum point of H we have

$$0 \geq F^{ij}\{\delta_0 H^2 g_{ij} - C[1 + e^{-2u}]H g_{ij} + O(H)\},$$

and this clearly implies

$$H \leq C[1 + e^{-2u}].$$

QED

Homework. Carry out the C^2-estimate for the case $k = 1$ to conclude

$$\max_{B(1/2)} \left[|\nabla u|^2 + |\Delta u| \right] \leq C(1 + \max_{B(1)} e^{-2u}).$$

Hint: Recall when $k = 1$, the equation is

$$\sigma_1(A) + \Delta u - \frac{n-2}{2}|\nabla u|^2 = f(x)e^{-2u}.$$

(What is F^{ij} in this case?)