Decoding of Space-Time Block Codes

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Part I

The MIMO Channel

The MIMO Channel

The quasi static fading channel

MIMO System

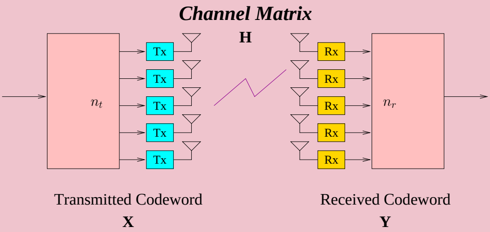

Figure: The Channel Model

The quasi static fading channel

MIMO System

Channel Matrix

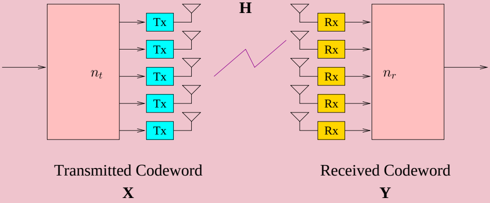

Figure: The Channel Model

- Received signal

$$\boxed{\boldsymbol{Y}_{n_r \times T} = \boldsymbol{H}_{n_r \times n_t} \cdot \boldsymbol{X}_{n_t \times T} + \boldsymbol{W}_{n_r \times T}} \quad (1)$$

  with $\boldsymbol{H}$ perfectly known at the receiver (coherent case).
- $\boldsymbol{H}$ is assumed constant during the transmission of one codeword.

Example of the Golden Code ($n_t = n_r = T = 2$)

Codeword

A codeword $\boldsymbol{X}$ of the Golden code is

$$\boldsymbol{X} = \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} \alpha(s_1 + \theta s_2) & \alpha(s_3 + \theta s_4) \\ i\bar{\alpha}(s_3 + \bar{\theta} s_4) & \bar{\alpha}(s_1 + \bar{\theta} s_2) \end{bmatrix}$$

with $\theta = \frac{1+\sqrt{5}}{2}$, $\bar{\theta} = \frac{1-\sqrt{5}}{2}$, $\alpha = 1 + i - i\theta$, $\overline{\alpha} = 1 + i - i\overline{\theta}$ and s_l, $l = 1 \ldots 4$ are the information symbols carved from a q-QAM.

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- Vectorization

$$\boldsymbol{y}_{n\cdot T\times 1} = \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} \boldsymbol{H} & 0 \\ 0 & \boldsymbol{H} \end{bmatrix} \cdot \begin{bmatrix} \alpha & \alpha\theta & 0 & 0 \\ 0 & 0 & i\bar{\alpha} & i\bar{\alpha}\bar{\theta} \\ 0 & 0 & \alpha & \alpha\theta \\ \bar{\alpha} & \bar{\alpha}\bar{\theta} & 0 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} + \boldsymbol{w}$$

$$= \frac{1}{\sqrt{5}} \cdot \boldsymbol{H}_{n\cdot T} \cdot \boldsymbol{\Phi}_{n\cdot T} \cdot \boldsymbol{s}_{n\cdot T\times 1} + \boldsymbol{w}_{n\cdot T\times 1}$$

where $\boldsymbol{\Phi}_{n\cdot T}$ is a unitary matrix.

Example of the Golden code (cont'd)

- Separation of the real and the imaginary parts

$$\begin{aligned} \boldsymbol{y}_{\mathbb{R}} &= \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} \Re(\boldsymbol{H}) & 0 & -\Im(\boldsymbol{H}) & 0 \\ 0 & \Re(\boldsymbol{H}) & 0 & -\Im(\boldsymbol{H}) \\ \Im(\boldsymbol{H}) & 0 & \Re(\boldsymbol{H}) & 0 \\ 0 & \Im(\boldsymbol{H}) & 0 & \Re(\boldsymbol{H}) \end{bmatrix} \cdot \begin{bmatrix} \Re(\boldsymbol{\Phi}) & -\Im(\boldsymbol{\Phi}) \\ \Im(\boldsymbol{\Phi}) & \Re(\boldsymbol{\Phi}) \end{bmatrix} \begin{bmatrix} \Re(\boldsymbol{s}) \\ \Im(\boldsymbol{s}) \end{bmatrix} + \begin{bmatrix} \Re(\boldsymbol{w}) \\ \Im(\boldsymbol{w}) \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \cdot \boldsymbol{H}_{\mathbb{R}} \cdot \boldsymbol{\Phi}_{\mathbb{R}} \cdot \boldsymbol{s}_{\mathbb{R}} + \boldsymbol{w}_{\mathbb{R}} \end{aligned}$$

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Equivalent Channel

Let $\boldsymbol{M} = \frac{1}{\sqrt{5}} \cdot \boldsymbol{H}_{\mathbb{R}} \cdot \boldsymbol{\Phi}_{\mathbb{R}}$, we get

$$\boxed{\boldsymbol{y}_{\mathbb{R}} = \boldsymbol{M} \cdot \boldsymbol{s}_{\mathbb{R}} + \boldsymbol{w}_{\mathbb{R}}}$$

where vectors $\boldsymbol{y}_{\mathbb{R}}$, $\boldsymbol{s}_{\mathbb{R}}$ and $\boldsymbol{w}_{\mathbb{R}}$ are 8-dimensional vectors and $\boldsymbol{s}_{\mathbb{R}}$ is a vector with integer components.

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where vectors $\boldsymbol{y}_{\mathbb{R}}$, $\boldsymbol{s}_{\mathbb{R}}$ and $\boldsymbol{w}_{\mathbb{R}}$ are 8-dimensional vectors and $\boldsymbol{s}_{\mathbb{R}}$ is a vector with integer components.

- More generally, the (real) dimension of the vectors is equal to $2 \cdot n_t \cdot T$.

The MIMO Channel

Lattice : Definition

Definition

A ***Euclidean lattice*** is a discrete additive subgroup with rank p, $p \leq n$ of the Euclidean space $\mathbb{R}^n$. We assume $p = n$ in the sequel.

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- A lattice $\boldsymbol{\Lambda}$ is a set generated by vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ of $\mathbb{R}^n$.
- An element $\boldsymbol{v}$ of $\boldsymbol{\Lambda}$ can be written as :

$$\boldsymbol{v} = a_1 \boldsymbol{v}_1 + a_2 \boldsymbol{v}_2 + \ldots + a_n \boldsymbol{v}_n, \; a_1, a_2, \ldots, a_n \in \mathbb{Z}$$

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- The lattice $\boldsymbol{\Lambda}$ can be defined as :

$$\boldsymbol{\Lambda} = \left\{ \sum_{i=1}^{n} a_i \boldsymbol{v}_i \mid a_i \in \mathbb{Z} \right\}$$

Lattices : Parameters (1)

- The set of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ is a **lattice basis**, with **dimension** n

Definition

Matrix $\boldsymbol{M}$ whose columns are vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ is a **generator matrix** of the lattice denoted $\Lambda_{\boldsymbol{M}}$.

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- Each vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $\Lambda_{\boldsymbol{M}}$, can be written as,

$$\boldsymbol{x} = \boldsymbol{M} \cdot \boldsymbol{z}$$

where $\boldsymbol{z} = \left(z_1, z_2, \ldots, z_p\right)^\top \in \mathbb{Z}^p$.

- Lattice $\Lambda_{\boldsymbol{M}}$ may be seen as the result of a linear transform applied to lattice $\mathbb{Z}^n$.

Lattices : Properties (2)

- Let $\boldsymbol{Q} \in \mathscr{M}_n(\mathbb{R})$, such that $\boldsymbol{Q} \cdot \boldsymbol{Q}^\top = I_n$ and $\det \boldsymbol{Q} = \pm 1$. $\boldsymbol{Q}$ is an isometry. The two lattices $\Lambda_{\boldsymbol{M}}$ and $\Lambda_{\boldsymbol{Q} \cdot \boldsymbol{M}}$ are **equivalent**.
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- Lattice $\Lambda_{\boldsymbol{Q}\cdot\boldsymbol{M}}$ is a rotated version of $\Lambda_{\boldsymbol{M}}$.
- If $\boldsymbol{Q} \in \mathscr{M}_n(\mathbb{Z})$ and $\det \boldsymbol{Q} \neq \pm 1$, then lattice $\Lambda_{\boldsymbol{M}\cdot\boldsymbol{Q}}$ is a **sublattice** of $\Lambda_{\boldsymbol{M}}$.
- A sublattice of $\Lambda_{\boldsymbol{M}}$ is a subgroup of $\Lambda_{\boldsymbol{M}}$.
- An integer lattice is a sublattice of $\mathbb{Z}^n$.

Lattices : Properties (3)

- The generator matrix $\boldsymbol{M}$ describes the lattice $\Lambda_{\boldsymbol{M}}$, but it is not unique. All matrices $\boldsymbol{M} \cdot \boldsymbol{T}$ with $\boldsymbol{T} \in \mathscr{M}_n(\mathbb{Z})$ and $\det \boldsymbol{T} = \pm 1$ are generator matrices of $\Lambda_{\boldsymbol{M}}$. $\boldsymbol{T}$ is called a unimodular matrix.
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Definitions

- The **fundamental parallelotope** of $\Lambda_{\boldsymbol{M}}$ is the region,

$$\mathscr{P} = \left\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} = a_1 \boldsymbol{v}_1 + a_2 \boldsymbol{v}_2 + \ldots + a_n \boldsymbol{v}_n,\ 0 \le a_i < 1,\ i = 1 \ldots n\right\}$$

- The ***fundamental volume*** is the volume of the fundamental parallelotope. It is denoted $\mathrm{vol}(\Lambda_{\boldsymbol{M}})$.
- $\boldsymbol{G} = \boldsymbol{M}^\top \cdot \boldsymbol{M}$ is the ***Gram matrix*** of the lattice (not invariant).
- The fundamental volume of the lattice is $|\det(\boldsymbol{M})|$, which is $\sqrt{|\det(\boldsymbol{G})|}$ either.

Lattices : Properties (4)

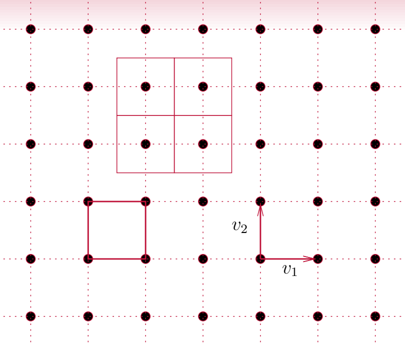
Definition

The ***Voronoï cell*** of a point u belonging to the lattice Λ is the region

$$\mathcal{V}(\boldsymbol{u}) = \left\{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}-\boldsymbol{u}\| \leq \left\|\boldsymbol{x}-\boldsymbol{y}\right\|, \boldsymbol{y} \in \Lambda\right\}$$

- Since a lattice is geometrically uniform, all Voronoï cells of a lattice are translated versions of the Voronoï cell of the zero point. This cell is called **Voronoï cell of the lattice**.
- The fundamental volume of a lattice is equal to the volume of its Voronoï cell.

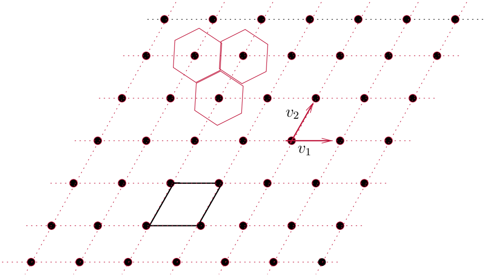
The $\mathbb{Z}^2$ -lattice



$\mathbb{Z}[i]$ lattice

●	Lattice Point
(v_1, v_2)	Lattice Basis
◇	Fundamental Parallelotope
◇	Voronoi region

The A_2 lattice



The A_2 lattice

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Constellations defined from $\mathbb{Z}[j]$

- Perfect STBCs of dimension 3 and 6 use symbols carved from $q-$HEX constellations.
- The lattice representation of a MIMO system using such codes needs some additional procedure. Simply, note that $\mathbb{Z}[j]$ is the hexagonal lattice A_2 with generator matrix.

$$\boldsymbol{B} = \begin{bmatrix} 1 & -0.5 \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{y}_{\mathbb{R}} &= \begin{bmatrix} \Re(\boldsymbol{H}) & \cdots & 0 & -\Im(\boldsymbol{H}) & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \Re(\boldsymbol{H}) & 0 & \cdots & -\Im(\boldsymbol{H}) \\ \Im(\boldsymbol{H}) & \cdots & 0 & \Re(\boldsymbol{H}) & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \Im(\boldsymbol{H}) & 0 & \cdots & \Re(\boldsymbol{H}) \end{bmatrix} \cdot \begin{bmatrix} 1 & \cdots & 0 & -0.5 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & -0.5 \\ 0 & \cdots & 0 & \frac{\sqrt{3}}{2} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \frac{\sqrt{3}}{2} \end{bmatrix} \\ &\cdot \begin{bmatrix} \Re(\boldsymbol{\Phi}) & -\Im(\boldsymbol{\Phi}) \\ \Im(\boldsymbol{\Phi}) & \Re(\boldsymbol{\Phi}) \end{bmatrix} \begin{bmatrix} \Re(\boldsymbol{s}) \\ \Im(\boldsymbol{s}) \end{bmatrix} + \begin{bmatrix} \Re(\boldsymbol{w}) \\ \Im(\boldsymbol{w}) \end{bmatrix} \\ &= \boldsymbol{H}_{\mathbb{R}} \cdot \boldsymbol{B}_{\otimes} \cdot \boldsymbol{\Phi}_{\mathbb{R}} \cdot \boldsymbol{s}_{\mathbb{R}} + \boldsymbol{w}_{\mathbb{R}} \end{aligned}$$

Part II

Lattice Decoding

Lattice Decoding

Introduction

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- Traditional constellations (QAM, HEX) are carved from lattices ($\mathbb{Z}^2$, A_2). Labelling and shaping is easier to perform.
- Another motivation is their decoding which can be derived from lattice decoding algorithms.
- Lattice decoding algorithms are now well-known, let's cite "sphere decoder, Schnorr-Euchner algorithm, sequential decoding,..."

Closest Point

- The **closest point** to $\boldsymbol{y}$ is the lattice point $\hat{\boldsymbol{z}}$ from Λ_M satisfying

$$\boxed{\|\boldsymbol{y}-\hat{\boldsymbol{z}}\|^2 \le \|\boldsymbol{y}-\boldsymbol{z}\|^2} \text{ for all } \boldsymbol{z} \in \Lambda_M$$

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  - Kannan's strategy : the region is a parallelotope
  - Pohst's strategy : the region is a sphere
- Pohst's strategy is the more practical method. Lattice decoders have been inspired by him : **Sphere decoder** and **Schnorr-Euchner** algorithm.

Lattice Decoding

Sphere Decoding (1)

- Decoding consists in searching the lattice point

$$\hat{z} = \arg\min_{z \in \Lambda} \|y - z\|^2$$

which is equivalent to the minimization

$$\min_{w \in y - \Lambda} \|w\|^2$$

- We need to work in the translated lattice $y - \Lambda$.

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Change of coordinates

Let's define

$$\begin{aligned}
\mathbf{z} &= \mathbf{M}\cdot\mathbf{u}, \quad \mathbf{u}\in\mathbb{Z}^n \\
\mathbf{y} &= \mathbf{M}\cdot\boldsymbol{\rho}, \quad \boldsymbol{\rho}=(\rho_1,\ldots,\rho_n)^\top\in\mathbb{R}^n \Rightarrow \text{The ZF point} \\
\mathbf{w} &= \mathbf{y}-\mathbf{z}=\mathbf{M}\cdot(\boldsymbol{\rho}-\mathbf{u})=\mathbf{M}\cdot\boldsymbol{\xi}, \quad \boldsymbol{\xi}=(\xi_1,\cdots,\xi_n)^\top\in\mathbb{R}^n
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- Components ξ_i are those of vector $\boldsymbol{u}$ of $\mathbb{Z}^n$ in the new reference.

Sphere Decoder (2)

- The aim is to find the lattice points in the sphere centered on the received signal and of radius $\sqrt{C}$. So,

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$$\|\boldsymbol{w}\|^2 = Q(\boldsymbol{\xi}) = \boldsymbol{\xi}^\top \cdot \boldsymbol{M}^\top \cdot \boldsymbol{M} \cdot \boldsymbol{\xi} = \boldsymbol{\xi}^\top \cdot \boldsymbol{G} \cdot \boldsymbol{\xi} = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} \xi_i \xi_j \leq C$$

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- The Cholesky factorization of the Gram matrix $\boldsymbol{G} = \boldsymbol{M}^\top \cdot \boldsymbol{M}$, gives $\boldsymbol{G} = \boldsymbol{R} \cdot \boldsymbol{R}^\top$, where $\boldsymbol{R}^\top = (r_{ji})_{i,j=1\ldots n}$ is an upper triangular matrix.

$$\boxed{Q(\boldsymbol{\xi}) = \boldsymbol{\xi}^\top \boldsymbol{R} \cdot \boldsymbol{R}^\top \boldsymbol{\xi} = \left\| \boldsymbol{R}^\top \cdot \boldsymbol{\xi} \right\|^2 = \sum_{i=1}^{n} \left(r_{ii} \xi_i + \sum_{j=i}^{n} r_{ij} \xi_j \right)^2 \leq C}$$

Sphere Decoding (3)

- Let

$$
\begin{aligned}
q_{ii} &= r_{ii}^2,\ i=1,\ldots,n \\
q_{ij} &= \frac{r_{ij}}{r_{ii}},\ i=1,\ldots,n,\ j=i+1,\ldots,n
\end{aligned}
$$

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Ellipsoid

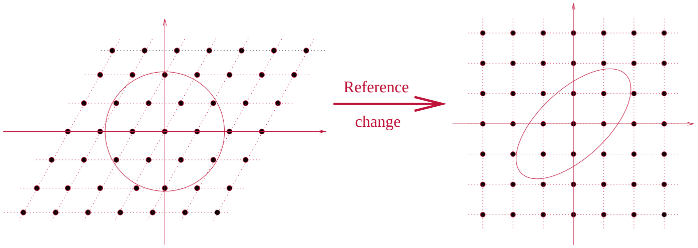
- We get

$$Q(\boldsymbol{\xi}) = \sum_{i=1}^{n} q_{ii} \left(\xi_i + \sum_{j=i+1}^{n} q_{ij} \xi_j \right)^2 \leq C$$

$$Q(\boldsymbol{\xi}) = \sum_{i=1}^{n} q_{ii} U_i^2 \leq C \ \Rightarrow \text{Interior of an ellipsoid}$$

Sphere Decoding (4)

- In the new system defined by $\boldsymbol{\xi}$, the sphere with radius $\sqrt{C}$, centered on the received point, is transformed into an ellipsoid centered on zero and defined by the bilinear form $Q(\boldsymbol{\xi})$.



Sphere Decoding (5)

- In order to determine the ellipsoid boundaries, let do some processing on ξ_n

$$q_{nn}\xi_n^2 \leq C$$

- We have $\xi_n = \rho_n - u_n$

$$\boxed{\left\lceil -\sqrt{\frac{C}{q_{nn}}} + \rho_n \right\rceil \leq u_n \leq \left\lfloor \sqrt{\frac{C}{q_{nn}}} + \rho_n \right\rfloor}$$

where $\lceil x \rceil$ is the smallest integer larger than x and $\lfloor x \rfloor$ is the largest integer smaller than x.

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- Now the ξ_i, $i = n-1, \ldots, 1$.

$$q_{n-1,n-1}(\xi_{n-1} + q_{n,n-1}\xi_n)^2 + q_{nn}\xi_n^2 \leq C$$

Sphere Decoding (6)

- We get

$$\left\lceil -\sqrt{\frac{C - q_{nn}\xi_n^2}{q_{n-1,n-1}}} + \rho_{n-1} + q_{n-1,n}\xi_n \right\rceil \leq u_{n-1} \leq \left\lfloor \sqrt{\frac{C - q_{nn}\xi_n^2}{q_{n-1,n-1}}} + \rho_{n-1} + q_{n-1,n}\xi_n \right\rfloor$$

Sphere Decoding (6)

- We get

$$\left\lceil -\sqrt{\frac{C-q_{nn}\xi_n^2}{q_{n-1,n-1}}}+\rho_{n-1}+q_{n-1,n}\xi_n \right\rceil \le u_{n-1} \le \left\lfloor \sqrt{\frac{C-q_{nn}\xi_n^2}{q_{n-1,n-1}}}+\rho_{n-1}+q_{n-1,n}\xi_n \right\rfloor$$

- This gives, for the i^{th} component u_i,

$$\left\lceil -\sqrt{\frac{1}{q_{ii}}\left(C-\sum_{l=i+1}^{n} q_{ll}\left(\xi_l+\sum_{j=l+1}^{n} q_{lj}\xi_j\right)^2\right)}+\rho_i+\sum_{j=i+1}^{n} q_{ij}\xi_j \right\rceil \le u_i$$

$$\left\lfloor \sqrt{\frac{1}{q_{ii}}\left(C-\sum_{l=i+1}^{n} q_{ll}\left(\xi_l+\sum_{j=l+1}^{n} q_{lj}\xi_j\right)^2\right)}+\rho_i+\sum_{j=i+1}^{n} q_{ij}\xi_j \right\rfloor \ge u_i$$

Sphere Decoding (7)

- In order to simplify the decoding expressions, we define

$$S_i = \rho_i + \sum_{l=i+1}^{n} q_{il}\xi_l, \; i = 1,\ldots,n$$

$$T_{i-1} = C - \sum_{l=i}^{n} q_{ll}\left(\xi_l + \sum_{j=l+1}^{n} q_{lj}\xi_j\right)^2 = T_i - q_{ii}(S_i - u_i)^2$$

- We get

$$\boxed{b_{\inf,i} = \left\lceil -\sqrt{\frac{T_i}{q_{ii}}} + S_i \right\rceil \le u_i \le \left\lfloor \sqrt{\frac{T_i}{q_{ii}}} + S_i \right\rfloor = b_{\sup,i}}$$

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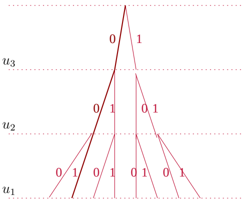
$$

- For each component of vector $\boldsymbol{u}$, we define an interval $I_i = [b_{\inf,i}, b_{\sup,i}]$ which contains it.

Sphere Decoding (8)

- The Closest point search consists in descending a tree.



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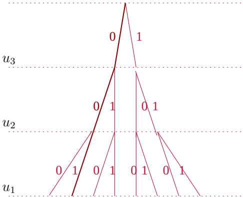

- When a lattice point is found, its squared distance from the received point is given by,

$$\boxed{\hat{d}^2 = C - T_1 + q_{11}(S_1 - u_1)^2}$$

If $\boxed{\hat{d}^2 \le C}$, the point is recorded.

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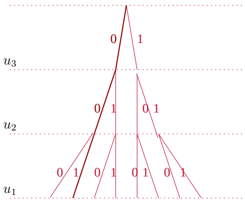

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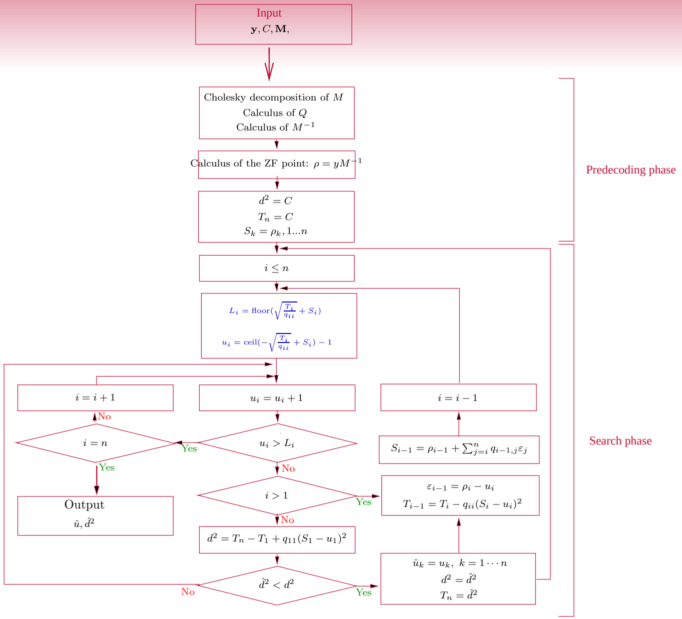
$$\hat{d}^2 = C - T_1 + q_{11}(S_1 - u_1)^2$$

If $\boxed{\hat{d}^2 \leq C}$, the point is recorded.

- The search algorithm makes the sphere radius as well as bounds $b_{\inf,i}$ and $b_{\sup,i}$ for, $i = 1 \cdots n$, release dynamically along the research process when a point is found, i.e. $C \geq \hat{d}^2$.

Flow Chart



Input
$\mathbf{y}, C, \mathbf{M},$
Cholesky decomposition of M
Calculus of Q
Calculus of M^{-1}
Calculus of the ZF point: $\rho = yM^{-1}$
$d^2 = C$
$T_n = C$
$S_k = \rho_k, 1...n$
$i \leq n$
$L_i = \text{floor}(\sqrt{\frac{T_i}{q_{ii}}} + S_i)$
$u_i = \text{ceil}(-\sqrt{\frac{T_i}{q_{ii}}} + S_i) - 1$
$i = i + 1$
$u_i = u_i + 1$
$i = i - 1$
No
$i = n$
Yes
$u_i > L_i$
$S_{i-1} = \rho_{i-1} + \sum_{j=i}^{n} q_{i-1,j}\varepsilon_j$
Yes
No
$i > 1$
Yes
$\varepsilon_{i-1} = \rho_i - u_i$
$T_{i-1} = T_i - q_{ii}(S_i - u_i)^2$
Output
$\hat{u}, \hat{d}^2$
No
$d^2 = T_n - T_1 + q_{11}(S_1 - u_1)^2$
$\hat{d}^2 < d^2$
No
Yes
$\hat{u}_k = u_k,\ k = 1 \cdots n$
$d^2 = \hat{d}^2$
$T_n = \hat{d}^2$
Predecoding phase
Search phase

Choice of the sphere radius

- The radius is a critical parameter for the complexity of the algorithm
    - **A too small radius** : no point inside the sphere
    - **A too large radius** : too many points inside the sphere, which increases the algorithm complexity

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  - **A too small radius** : no point inside the sphere
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- A good solution is to have the sphere radius equal to the covering radius of the lattice (**too complex**)
- An easier solution is to choose

$$C = \min\left(\min_i\left(\left(\mathrm{diag}\boldsymbol{M}\cdot\boldsymbol{M}^\top\right)_i\right), 2n\sigma^2\right)$$

Decoding of a finite part of a lattice

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16 – QAM

redefine intervals

$$I = I_i \cap I_C = \left[\sup(b_{inf,i}, c_{min}), \inf(b_{sup,i}, c_{max})\right]$$

where $I_C = \left[c_{min}, c_{max}\right] = [0,3]$ is the set of the in phase and quadrature components of the constellation.

Lattice Decoding

Schnorr-Euchner algorithm (1)

- It is a variant of the Sphere Decoder (SD)
- Same principle than SD applies, that is, search the closest point inside a sphere centered on the received point.

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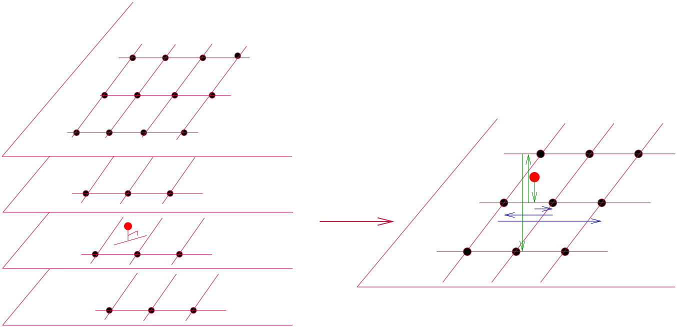
- Same principle than SD applies, that is, search the closest point inside a sphere centered on the received point.
- The main idea of SE is to see the set of $n-$dimensional points ($n-$dimensional lattice) as a superposition of $(n-1)-$dimensional points (in hyperplans).
- The closest point is found by successive projections on hyperplans.
- We need a starting point in the lattice.

Example of a 3 dimensional lattice



Schnorr-Euchner algorithm (2)

- The starting point is called "**Babai point**". It results from a suboptimal decoding.
- Starting from the Babai point, the algorithm visits the other lattice points inside the sphere centered on the received point, and whose radius is given by the distance between the **Babai point** and the received point.

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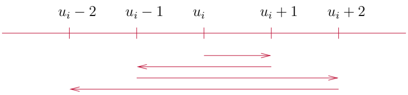
- Starting from the Babai point, the algorithm visits the other lattice points inside the sphere centered on the received point, and whose radius is given by the distance between the **Babai point** and the received point.
- We visit all points inside the sphere, zigzaging around each component of the Babai point



Comparison SD/SE

Similarities

- same principle : search the closest point inside a sphere
- same performance : ML

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Similarities

- same principle : search the closest point inside a sphere
- same performance : ML

Differences

- **Strategies are different**
  - SD : points are visited from the boundary of the sphere towards its center
  - SE : points are visited from the center of the sphere towards its boundaries
- **Sphere radius**
  - SD : needs to initialize the radius
  - SE : no initial radius to choose

Part III

Preprocessing

Preprocessing

A more general problem formulation

Definition

A lattice code $\mathscr{C}(\Lambda, \boldsymbol{t}, \mathscr{S})$ is the set of points of $\Lambda + \boldsymbol{t}$ inside the shaping region $\mathscr{S}$ that is,

$$\mathscr{C}(\Lambda, \boldsymbol{t}, \mathscr{S}) = \{\Lambda + \boldsymbol{t}\} \cap \mathscr{S}$$

- The considered communication model is

$$\boxed{\boldsymbol{y} = \boldsymbol{H} \cdot (\boldsymbol{x} + \boldsymbol{t}) + \boldsymbol{w}}$$

where $\boldsymbol{x} = \boldsymbol{\Phi} \cdot \boldsymbol{u}$, $\boldsymbol{u} \in \mathbb{Z}^m$ and $\boldsymbol{H} \in \mathbb{R}^{n \times m}$.

- $\boldsymbol{\Phi}$ is the precoding matrix.

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Decoding problem

Find

$$\boxed{\hat{\boldsymbol{u}} = \arg \min_{\boldsymbol{u} \in \mathscr{U} \subset \mathbb{Z}^m} \left\| \boldsymbol{y} - \boldsymbol{H} \cdot \boldsymbol{t} - \boldsymbol{H} \cdot \boldsymbol{\Phi} \cdot \boldsymbol{u} \right\|^2} \quad (2)$$

Why preprocessing?

- Applications of sphere decoding suffers from two inconveniences
  1. When $\text{rank}(\boldsymbol{H}\cdot\boldsymbol{\Phi}) < m$ or $\boldsymbol{H}\cdot\boldsymbol{\Phi}$ is ill-conditioned the spread of the diagonal elements of $\boldsymbol{H}\cdot\boldsymbol{\Phi}$ is large and the search can be very complex.
  2. Enforcing $\boldsymbol{u}$ is very difficult when constellation $\mathscr{U}$ has a complicated shape
  3. Lattice decoding can solve this problem by searching over $\mathbb{Z}^m$ (instead of $\mathscr{U}$) but it is far from ML in general.

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- **Solution:** Preprocessing!
- In addition, preprocessing $\boldsymbol{H}$ and $\boldsymbol{\Phi}$ can have a great effect on the complexity of the search stage to make the tree more "friendly" (improving the quality of the ZF-DFE).

The preprocessing stage

- ***Left Preprocessing*** ($\rightarrow \times \boldsymbol{H}$): Modifies $\boldsymbol{H}$ and $\boldsymbol{w}$ such that the resulting **CL**osest **P**oint **S**earch (CLPS) is not equivalent to **ML** but has a much better conditioned "channel" matrix and makes lattice decoding near-optimal.

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Preprocessing

Left preprocessing applied only on the channel matrix; right preprocessing applied on the whole. **Important:** any preprocessing should not destruct the code structure

Preprocessing

The QR decomposition

- QR decomposition applies to $\boldsymbol{H}$.

$$\mathrm{H} = \mathrm{Q} \cdot \mathrm{R}$$

Unitary Matrix (Q) Triangular Matrix (R)

- It can be seen as ZF-DFE with
  - Feedforward matrix $\boldsymbol{Q}$
  - Backward matrix $\boldsymbol{R}$

- When $\boldsymbol{y} = \boldsymbol{H} \cdot \boldsymbol{x} + \boldsymbol{w}$, CLPS is $\boxed{\min_{\boldsymbol{x}} \left\| \boldsymbol{y} - \boldsymbol{H} \cdot \boldsymbol{x} \right\|^2}$ equivalent to $\boxed{\min_{\boldsymbol{X}} \left\| \boldsymbol{Q}^{\dagger} \cdot \boldsymbol{y} - \boldsymbol{R} \cdot \boldsymbol{x} \right\|^2}$.

  Hence, a tree of the channel can be constructed.

The MMSE-DFE (1)

- **MMSE-DFE** outperforms **ZF-DFE** in terms of SINR

$$\tilde{H} \triangleq \begin{bmatrix} H \\ I \end{bmatrix} = \tilde{Q} \cdot R_1$$

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$$\boxed{\tilde{\boldsymbol{H}} \triangleq \begin{bmatrix} \boldsymbol{H} \\ \boldsymbol{I} \end{bmatrix} = \tilde{\boldsymbol{Q}} \cdot \boldsymbol{R}_1}$$

- Let $\boldsymbol{Q}_1$ be the upper $n \times m$ part of $\tilde{\boldsymbol{Q}}$. Transformed CLPS is

$$\min_{\boldsymbol{u} \in \mathcal{U}} \left\| \boldsymbol{Q}_1^\dagger \cdot \boldsymbol{r} - \boldsymbol{R}_1 \cdot \boldsymbol{\Phi} \cdot \boldsymbol{u} \right\|^2$$

which is not equivalent to (2) with $\boldsymbol{r} = \boldsymbol{y} - \boldsymbol{H} \cdot \boldsymbol{t}$ since $\boldsymbol{Q}_1$ is not unitary.

The MMSE-DFE (2)

- We have

$$\begin{aligned} \boldsymbol{Q}_1^{\dagger} \cdot \boldsymbol{r} &= \boldsymbol{Q}_1^{\dagger} \cdot \boldsymbol{H} \cdot \boldsymbol{\Phi} \cdot \boldsymbol{u} + \boldsymbol{Q}_1^{\dagger} \cdot \boldsymbol{w} \\ &= \boldsymbol{R}_1 \cdot \boldsymbol{\Phi} \cdot \boldsymbol{u} + \boldsymbol{z} \end{aligned}$$

- The additive noise $\boldsymbol{z} = \boldsymbol{Q}_1^{\dagger} \cdot \boldsymbol{r} - \boldsymbol{R}_1 \cdot \boldsymbol{\Phi} \cdot \boldsymbol{u}$ has a Gaussian component $\boldsymbol{Q}_1^{\dagger} \cdot \boldsymbol{w}$ and a non-Gaussian (signal dependent) component $\left(\boldsymbol{Q}_1^{\dagger} \cdot \boldsymbol{H} - \boldsymbol{R}_1\right) \cdot \boldsymbol{x}$.

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The noise is white!!

$$\boxed{\left[\boldsymbol{R}_1 - \boldsymbol{Q}_1^\dagger \cdot \boldsymbol{H}\right]\left[\boldsymbol{R}_1 - \boldsymbol{Q}_1^\dagger \cdot \boldsymbol{H}\right]^\dagger + \boldsymbol{Q}_1^\dagger \cdot \boldsymbol{Q}_1 = \boldsymbol{I}}$$

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- We can solve under-determined linear systems since matrix $\boldsymbol{R}_1$ is always full rank with eigenvalues ≥ 1.
- MMSE-DFE atenuates the problem of **boundary control** in the next steps.

Preprocessing

Right preprocessing (1)

- When left preprocessing has been done, we need to QR-decompose matrix

$$\boldsymbol{R}_1 \cdot \Phi = \boldsymbol{Q} \cdot \boldsymbol{R}$$

We want $\boldsymbol{R}$ to be as sparse as possible (e.g. $\boldsymbol{R} \to \boldsymbol{I}$)

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Problem

Find a unimodular matrix $\boldsymbol{T}$ such that QR decomposition $\boldsymbol{R}_1 \cdot \boldsymbol{\Phi} \cdot \boldsymbol{T}^{-1}$ minimizes the sparsity index of R.

$$\varsigma(\boldsymbol{R}) \triangleq \max_{k=1,2,\ldots,m} \frac{\sum_{i=k+1}^{m} r_{i,j}^2}{r_{i,i}^2}$$

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- Good approximations to the solutions of this problem exist

Right Preprocessing (2)

- **Lattice reduction:** **L**enstra, **L**enstra and **L**ovász (**LLL**) algorithm (possibly with deep insertion [Schnorr-Euchner]). Find a new lattice basis with reduced vectors $\boldsymbol{H} \cdot \boldsymbol{\Phi} \cdot \boldsymbol{T}_1^{-1}$ (i.e., small norms and/or as orthogonal as possible).
- Column permutation $\boldsymbol{\Pi}$ of $\boldsymbol{H} \cdot \boldsymbol{\Phi} \cdot \boldsymbol{T}_1^{-1}$ such that $\min_i r_{i,i}$ is maximized.
- Right multiply by

$$\boldsymbol{T}^{-1} = \boldsymbol{T}_1^{-1} \cdot \boldsymbol{\Pi}^{-1}$$

- Right multiplication by unimodular matrices does not alter lattice decoding.

Form the tree of the system

- We give $\boldsymbol{H}$ the channel matrix and $\boldsymbol{\Phi}$ the precoding matrix (after vectorization).
- Perform left and right preprocessing
- QR-decompose $\boxed{\boldsymbol{Q}_1^{\dagger} \cdot \boldsymbol{H} \cdot \boldsymbol{\Phi} \cdot \boldsymbol{T}^{-1} = \boldsymbol{Q} \cdot \boldsymbol{R}}$

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Equivalent System

With convenient notations, we get,

$$\boxed{\begin{pmatrix} y_m \\ \vdots \\ \vdots \\ y_1 \end{pmatrix} = \begin{pmatrix} r_{m,m} & \cdots & \cdots & r_{m,1} \\ 0 & r_{m-1,m-1} & \cdots & r_{m-1,1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{1,1} \end{pmatrix} \cdot \begin{pmatrix} x_m \\ \vdots \\ \vdots \\ x_1 \end{pmatrix} + \begin{pmatrix} w_m \\ \vdots \\ \vdots \\ w_1 \end{pmatrix}} \tag{3}$$

Preprocessing

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Assumptions

- Rayleigh Flat Fading Channel
- MISO system
- DAST Codes used

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Received signal and aim of this section

$$\boldsymbol{y} = \boldsymbol{H} \cdot \boldsymbol{\Phi} \cdot \boldsymbol{x} + \boldsymbol{n} \tag{4}$$

where

$$\boldsymbol{H} = \operatorname{diag}[h_1, h_2, \ldots, h_n]$$

$\boldsymbol{n}$ is the i.i.d. Gaussian noise and $\boldsymbol{\Phi}$ is a unitary transform bringing modulation diversity to the system. The aim is to design a “not too complex” detector by doing some new lattice reduction.

Assumptions on the unitary transform

Background

- We use $\mathbb{F} = \mathbb{Q}(i)$ as the base field with ring of integer $\mathbb{Z}[i]$ (QAM).
- $\mathbb{K} = \mathbb{F}(\theta)$ is the smallest field containing $\mathbb{F}$ and θ, an element of order n. Its ring of integers is $\mathcal{O}_{\mathbb{K}}$.
- $Gal_{\mathbb{K}/\mathbb{F}}$ is the Galois group of automorphisms on $\mathbb{K}$ with elements denoted $\sigma_i, i = 1, \ldots, n$.

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The Φ matrix (diversity of modulation)

The structure of $\mathbf{\Phi}$ is the following,

$$\mathbf{\Phi} = \mathbf{\Delta} \cdot \begin{bmatrix} \sigma_1(\omega_1) & \sigma_1(\omega_2) & \cdots & \sigma_1(\omega_n) \\ \sigma_2(\omega_1) & \sigma_2(\omega_2) & \cdots & \sigma_2(\omega_n) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_n(\omega_1) & \sigma_n(\omega_2) & \cdots & \sigma_n(\omega_n) \end{bmatrix} \tag{5}$$

where $\omega_1, \omega_2, \ldots, \omega_n \in \mathcal{O}_{\mathbb{K}}$ are linearly independent on $\mathbb{F}$. $\mathbf{\Delta}$ is diagonal.

A 2 dimensional example

Background

$\mathbb{K} = \mathbb{F}(\theta)$ with $\theta = \frac{1+\sqrt{5}}{2}$, an element of order 2. Its ring of integers is $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[i, \frac{1+\sqrt{5}}{2}]$.
Minimal polynomial of θ is $\mu_\theta(X) = X^2 - X - 1$.

- $Gal_{\mathbb{K}/\mathbb{F}}$ is the Galois group of $\mathbb{K}$ with elements $\{1, \sigma\}$ such that

$$\sigma : \theta \longmapsto \bar{\theta} = \frac{1-\sqrt{5}}{2}$$

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$$\sigma : \theta \longmapsto \bar{\theta} = \frac{1-\sqrt{5}}{2}$$

The Φ matrix ("Golden Field")

Take $\omega_1 = 1 + i(1-\theta)$ and $\omega_2 = \theta - i$. Then,

$$\mathbf{\Phi} = \frac{1}{\sqrt{5}} \begin{bmatrix} \omega_1 & \omega_2 \\ \sigma(\omega_1) & \sigma(\omega_2) \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 + i(1-\theta) & \theta - i \\ 1 + i(1-\bar{\theta}) & \bar{\theta} - i \end{bmatrix}$$

Matrix representation of an algebraic number

Example 1: $\mathbb{C} \longrightarrow \mathcal{M}_2(\mathbb{R})$

$$z = x + iy \longmapsto \mathbf{T}_z = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

$$N_{\mathbb{C}/\mathbb{R}}(z) = x^2 + y^2 = \det \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

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Example 2: $\mathbb{Q}\left(e^{\frac{i\pi}{4}}\right) \longrightarrow \mathcal{M}_2(\mathbb{Q}(i))$

$$z = x + y\theta \longmapsto \mathbf{T}_z = \begin{pmatrix} x & iy \\ y & x \end{pmatrix}$$

$$N_{\mathbb{Q}\left(e^{\frac{i\pi}{4}}\right)/\mathbb{Q}(i)}(z) = x^2 - iy^2 = \det \begin{pmatrix} x & iy \\ y & x \end{pmatrix}$$

Transforming Fadings into a Basis Change (1)

- Matrix H can be expressed as

$$\boldsymbol{H} = \left|\prod_{i=1}^{n} h_i\right|^{\frac{1}{n}} \cdot \operatorname{diag}\left[a_1, a_2, \ldots, a_n\right]$$

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- **Assume** that the vector $(|a_1|, |a_2|, \ldots, |a_n|)$ is composed by the magnitudes of the conjugates of some unit u in $\mathcal{O}_{\mathbb{K}}$, i.e., $a_k = e^{i\beta_k}\sigma_k(u), \forall k$ with $\beta_k = \arg a_k - \arg\sigma_k(u)$. The received signal can then be expressed as

$$\boldsymbol{y} = \left|\prod_{i=1}^{n} h_i\right|^{\frac{1}{n}} \cdot \mathrm{diag}\left[e^{i\beta_1}, \ldots, e^{i\beta_n}\right] \cdot \mathrm{diag}\left[\sigma_1(u), \ldots, \sigma_n(u)\right] \cdot \boldsymbol{\Phi} \cdot \boldsymbol{x} + \boldsymbol{n} \qquad (6)$$

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- So,

$$\boldsymbol{y} = \left| \prod_{i=1}^{n} h_i \right|^{\frac{1}{n}} \cdot \boldsymbol{\Psi} \cdot \boldsymbol{\Phi} \cdot \boldsymbol{T}_u \cdot \boldsymbol{x} + \boldsymbol{n} \quad (7)$$

with $\boldsymbol{\Psi} = \operatorname{diag}\left[e^{i\beta_1}, e^{i\beta_2}, \ldots, e^{i\beta_n}\right]$ and $\boldsymbol{T}_u$ (**unimodular**) being **the matrix representation** of the unit u.

Transforming Fadings into a Basis Change (2)

- Denote $\boldsymbol{z} = \left|1/\prod_{i=1}^{n} h_i\right|^{\frac{1}{n}} \cdot \boldsymbol{\Phi}^{\dagger} \cdot \boldsymbol{\Psi}^{\dagger} \cdot \boldsymbol{y}$, then

$$\boldsymbol{z} = \boldsymbol{T}_u \cdot \boldsymbol{x} + \boldsymbol{w}$$

where $\boxed{\boldsymbol{w} = \left|1/\prod_{i=1}^{n} h_i\right|^{\frac{1}{n}} \cdot \boldsymbol{\Phi}^{\dagger} \cdot \boldsymbol{\Psi}^{\dagger} \cdot \boldsymbol{n}}$ remains an *i.i.d.* noise vector.

- Now, since $\boxed{|\det \boldsymbol{T}_u| = 1}$, ($u$ is a unit), then a ML lattice decoder is obvious as it is a slicer followed by the product with matrix $\boldsymbol{T}_u^{-1}$.

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Approximation

What happens if $(|a_1|, |a_2|, \ldots, |a_n|)$ is not composed by the modules of conjugates of some unit u?

The Logarithmic Lattice

Theorem (Dirichlet)

Let $\mathbb{K}$ be an extension of $\mathbb{Q}$ with signature (r,s) (with degree $r+2s$). Then there exists $r+s-1$ units named "***fundamental units***" $u_1, u_2, \ldots, u_{r+s-1}$ such that any unit u can be expressed as

$$u = \epsilon \cdot \prod_{i=1}^{r+s-1} u_i^{k_i}$$

where ϵ is a complex number with module equal to 1 and $k_i \in \mathbb{Z}$.

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- Now from a unit u, construct the vector

$$\boldsymbol{u}_{\log} = \left(\log|\sigma_1(u)|, \ldots, \log|\sigma_{r+s}(u)|\right)^\top$$

Then vector $\boldsymbol{u}_{\log}$ lies in a hyperplane with equation

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$$\sum_{i=1}^{r+s} x_i = 0$$
- All vectors of type $\boldsymbol{u}_{\log}$ are in a lattice named the logarithmic lattice, with generator matrix,
$$\begin{bmatrix} \log|\sigma_1(u_1)| & \log|\sigma_2(u_1)| & \cdots & \log|\sigma_{r+s}(u_1)| \\ \log|\sigma_1(u_2)| & \log|\sigma_2(u_2)| & \cdots & \log|\sigma_{r+s}(u_2)| \\ \vdots & \vdots & \ddots & \vdots \\ \log|\sigma_1(u_{r+s-1})| & \log|\sigma_2(u_{r+s-1})| & \cdots & \log|\sigma_{r+s}(u_{r+s-1})| \end{bmatrix}$$
and fundamental volume R, the regulator.

Golden Field

- Let $\mathbb{K} = \mathbb{Q}(i, \sqrt{5})$ with $\mathbf{\Phi} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1+i(1-\theta) & \theta - i \\ 1+i(1-\bar{\theta}) & \bar{\theta} - i \end{bmatrix}$ and $\theta = \frac{1+\sqrt{5}}{2}$. The logarithmic lattice ($\cong \mathbb{Z}$) has generator matrix $\begin{bmatrix} 0.481 & -0.481 \end{bmatrix}$.

$$\boldsymbol{H} = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$$

- Assume that fadings are $\boxed{h_1 = 1.271 e^{i\eta_1}}$ and $\boxed{h_2 = 0.071 e^{i\eta_2}}$. We get

$$\boldsymbol{H} = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} = 0.3 \begin{pmatrix} e^{i\eta_1} & 0 \\ 0 & -e^{i\eta_2} \end{pmatrix} \begin{pmatrix} \theta^3 & 0 \\ 0 & \bar{\theta}^3 \end{pmatrix}$$

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Reduction

Equations (6) and (7) give

$$\boldsymbol{y} = \boldsymbol{H}\mathbf{\Phi}\boldsymbol{x} + \boldsymbol{n} = 0.3 \begin{pmatrix} e^{i\eta_1} & 0 \\ 0 & -e^{i\eta_2} \end{pmatrix} \begin{pmatrix} \theta^3 & 0 \\ 0 & \bar{\theta}^3 \end{pmatrix} \mathbf{\Phi}\boldsymbol{x} + \boldsymbol{n}$$

$$= 0.3 \begin{pmatrix} e^{i\eta_1} & 0 \\ 0 & -e^{i\eta_2} \end{pmatrix} \mathbf{\Phi} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \boldsymbol{x} + \boldsymbol{n}$$

The Reduction

In fact, the true received signal is (after reduction)

$$\boldsymbol{y} = \left| \prod_{i=1}^{n} h_i \right|^{\frac{1}{n}} \cdot \boldsymbol{\Psi\Lambda\Phi T}_u \boldsymbol{x} + \boldsymbol{n}$$

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- If $\mathbf{H}$ is proportional to a unitary transform, then $\boldsymbol{\Lambda} = \mathbf{I}$, else, the nearest unit (in the logarithmic lattice) is chosen and $\boldsymbol{\Lambda}$ is a diagonal matrix whose dynamic is bounded (covering radius) and controlled by the logarithmic lattice.

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- $\boldsymbol{T}_u$ is the unimodular basis change matrix; it is **the** reduction matrix.

The diversity property (with ZF)

Theorem

The asymptotic ($\gamma \to \infty$) expression of the codeword error probability for the zero forcing detection is

$$P_e(\gamma) \leq O\left(\frac{\log^{n-1}\gamma}{\gamma^n}\right)$$

where n is the dimension

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Sketch of the proof

- In fact, $P_e(\gamma) \leq \int_0^\infty p_X(x)e^{-\gamma x}dx$ where $X = \sqrt[n]{\prod_{i=1}^n X_i}$ (by using the covering radius of the logarithmic lattice as an upperbound)
- X_i are i.i.d. random variables with an exponential distribution
- Recursion on n gives the result.

Part IV

Tree Search Strategy

Tree Search Strategy

General Branch and Bound (1)

- We keep notations from (3). We have

$$\boldsymbol{y} = \boldsymbol{R} \cdot \boldsymbol{x} + \boldsymbol{z}$$

  with $\boldsymbol{x} \in \mathbb{Z}^m$.

- The node at level k is denoted $\boldsymbol{x}_1^k = (x_1, x_2, \ldots, x_k)$. Every node is associated to the metric

$$w_k\left(\boldsymbol{x}_1^k\right) = \left|\boldsymbol{y}_k - \sum_{i=1}^{k} r_{k,i} x_i\right|^2$$

- **B**ranch and **B**ound (**BB**) reduces the complexity of tree search by determining if an intermediate node $\boldsymbol{x}_1^k$ has any chance of giving the optimum leaf node, when extended.

General Branch and Bound (2)

- The decision is taken by comparing a cost function (namely $f\left(\boldsymbol{x}_1^k\right)$) against a bounding function t_k.
- BB maintains a list of valid nodes that can be extended, $\mathcal{N}$. BB ends when $\mathcal{N}$ is empty.
- Different BB algorithms differ in their cost functions, bounding functions and the rules to generate and sort the nodes.

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A unified framework

BB brings a unified framework for many searching algorithms, considered as special cases. Sphere Decoder, Schnorr-Euchner, Sequential decoding, ...

Tree Search Strategy

10 Branch and Bound
- General Branch and Bound

11 Classification
- Breadth First Search
- Depth First Search
- Best First Search
- The best tradeoff

Breadth First Search (1)

- Bounding function is fixed and cost function is never updated.
- **Pohst Enumeration**: Bounding function is

$$t_k = C_0 \text{ (sphere radius)}.$$

  Cost function is

$$f\left(\boldsymbol{x}_1^k\right) = \sum_{i=1}^{k} w_i\left(\boldsymbol{x}_1^i\right) \leq C_0.$$

- All nodes satisfying $\sum_{i=1}^{k} w_i\left(\boldsymbol{x}_1^i\right) \leq C_0$ are generated before termination.

Breadth First Search (2)

- Generating the child nodes is simplified. For any parent node $\boldsymbol{x}_1^k$, the condition $\sum_{i=1}^{k} w_i\left(\boldsymbol{x}_1^i\right) \leq C_0$ implies for the child nodes that component k of the generated nodes lies in some interval (see SD).
- We can apply some heuristic statistical pruning, e.g. increased radii ($t_k < t_{k+1}$).
- Variants can be found. $M-$ and $T-$algorithms
  - $M-$algorithm only keeps the M best survivors whereas the $T-$algorithm adjusts the bounding function by the best cost function at the level k combined with a predefined threshold (use of the Fano metric, for instance).
- This algorithm remains complex.

Depth First Search

Principle

Order nodes in $\mathcal{N}$ in reverse order of generation. The final bound vector is

$$\boldsymbol{t} = \left[\min\left(t_1, f\left(\mathbf{x}_1^m\right)\right), \min\left(t_2, f\left(\mathbf{x}_1^m\right)\right), \ldots, \min\left(t_m, f\left(\mathbf{x}_1^m\right)\right)\right].$$

Different DFS depends on the way children are generated and on the partial bounds and the ordering of the generated child nodes.

- **Viterbo-Boutros algorithm:** We have $f\left(\boldsymbol{x}_1^k\right) = \sum_{i=1}^k w_i\left(\boldsymbol{x}_1^i\right)$ and for any node $\boldsymbol{x}_1^{k-1}$, its valid children (verifying $\sum_{i=1}^k w_i\left(\boldsymbol{x}_1^i\right) \leq C_0$) are generated lexicographically.
- **Schnorr-Euchner algorithm:** Same properties, but the child nodes are generated *w.r.t.* the accumulated squared distance $\sum_{i=1}^k w_i\left(\boldsymbol{x}_1^i\right)$.

Best First Search (1)

- Sort nodes in $\mathcal{N}$ such that their cost functions are increasing.
- Search can be terminated once a leaf node reaches the top of $\mathcal{N}$.
- Stack algorithm (sequential decoding) is BeFS with cost function

$$\boxed{f\left(\boldsymbol{x}_1^k\right) = \sum_{i=1}^{k+1} w_i\left(\boldsymbol{x}_{1,b}^{k+1}\right) - b(k+1)}$$

where $\boldsymbol{x}_{1,b}^{k+1}$ is the best child of $\boldsymbol{x}_1^k$ not generated yet and $f\left(\boldsymbol{x}_1^m\right) = -\infty$. b is the *bias*.

Theorem

The stack algorithm with $b = 0$ generates the least number of nodes among all optimal tree search algorithms.

Best First Search (2)

- The stack algorithm offers a natural solution for the problem of choosing the initial radius (or radii): $t_k = \infty$.
- The stack allows for a systematic approach for trading-off performance for complexity: $b = 0 \Rightarrow$ Optimal CLPS. $b = \infty \Rightarrow$ MMSE-Babai point decoder.
- In general, for systems with small dimension m, and/or high SNRs/"friendly" channels, one can obtain near-optimal performance with relatively large values of b (i.e., reduced complexity).
- **Disadvantage:** The required memory to maintain the active list $\mathcal{N}$ can be prohibitive.

The best tradeoff

- In general, the choice of the algorithm depends on the dimensions, codes, ...
- For a large variety of MIMO channel, the best tradeoff complexity/performance is given by

Best strategy

Left preprocessing (MMSE-DFE) + **Right** preprocessing (Lattice reduction and reordering), followed by a stack search stage in the lattice.

Conclusion and perspectives

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- Sphere decoding has been the first ML decoding algorithm for MIMO (encoded) channels
- Sequential decoding has also been proposed as a near optimal decoder but with much less complexity than SD.
- Preprocessing can decrease a lot the searching complexity
- Now, do processing at the matrix form level and not at the vector form level.

Thank You