HOMOTOPY THEORY OF SPACES OF REPRESENTATIONS

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I Background

Let Γ denote a finitely generated discrete group, and G a compact Lie group.

Here we will be interested in spaces of the form

$\mathrm{Hom}(\Gamma, G)$ = space of all homomorphisms $f : \Gamma \to G$

We have a diagram

$$\begin{tikzcd} \mathrm{Hom}(\Gamma, G) \arrow[rr] \arrow[dr] & & {[B\Gamma, BG]} \\ & \pi_0 \mathrm{Hom}(\Gamma, G) \arrow[ur, "E"] & \end{tikzcd}$$

flat bundles

Thus we can use <u>bundle theory</u> to understand aspects of the geometry of $\mathrm{Hom}(\Gamma, G)$, more precisely

- what can we say about the number and structure of the path components of $\mathrm{Hom}(\Gamma, G)$?

- how far is E from being a bijection?

- what is the cohomology or the homotopy type of $\mathrm{Hom}(\Gamma, G)$ for specific choices of Γ and G?

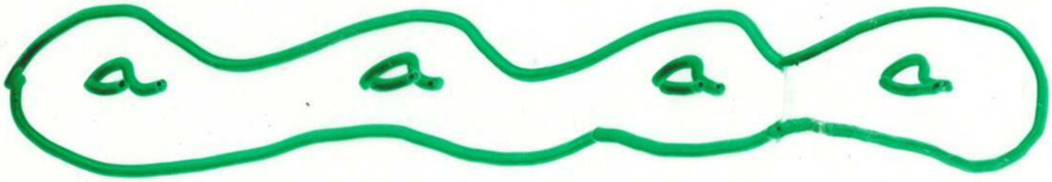
There is a G-action via conjugation and the quotient is denoted

$$\mathrm{Rep}(\Gamma, G) = \mathrm{Hom}(\Gamma, G) / G$$

If M is a manifold, then

$\mathrm{Rep}(\pi_1(M), G) \longleftrightarrow$ Moduli space of isomorphism classes of flat connections on principal G-bundles over M

Example: Let M_g denote an orientable Riemann surface of genus $g>1$



Then, if G is a compact, connected semi-simple Lie group, we have that

$$\underbrace{\pi_0 \mathrm{Hom}(\pi_1 M_g, G)}_{\text{path components}} \overset{E}{\cong} \left\{ \begin{array}{c} \text{iso classes} \\ \text{of} \\ \text{principal } G\text{-bundles} \\ \text{over } M_g \end{array} \right\} \cong \pi_1 G$$

$$\underset{\text{path-connected}}{\mathrm{Hom}(\pi_1 M_g, G)} \iff G \text{ simply connected}$$

Theorem:

If $\mathcal{M}$ is the complement of an arrangement of complex hyperplanes in $\mathbb{C}^n$ which has contractible universal cover, then every unitary representation of $\pi_1(\mathcal{M})$ induces a trivial bundle over $\mathcal{M}$. ///

Example:

$$\mathcal{M} = \{(z_1, \dots, z_m) \in \mathbb{C}^m \,/\, z_i \neq z_j \text{ if } i \neq j\}$$

Then $\pi_1(\mathcal{M}) = P_m$ pure braid group

$$\mathrm{Hom}(P_m, U(n)) \longrightarrow \text{Bundles}$$

is trivial, even though we do not know if $\pi_0 \mathrm{Hom}(P_m, U(n)) = 1$.

II. COMMUNTING ELEMENTS ③

Consider the case $\Gamma = \mathbb{Z}^n$

$\mathrm{Hom}(\mathbb{Z}^n, G) =$ ordered commuting n-tuples in G

• if $G = U(m), SU(m), Sp(m)$

$\mathrm{Hom}(\mathbb{Z}^n, G)$ is path connected &

$$\mathrm{Hom}(\mathbb{Z}^n, G) \underset{\mathbb{Q}}{\sim} G \times_{NT} T^n$$

$T \subset G$ maximal torus

(T. Baird)

- if $G = PU(p)$, then

$\mathrm{Hom}(\mathbb{Z}^n, G)$ has

$$\frac{(p^n-1)(p^{n-1}-1)}{p^2-1} + 1$$

connected components; the one corresponding to $(1, \ldots, 1)$ has the rational cohomology of $G \times_{NT} T^n$, whereas all the rest are homeomorphic to

$SU(p)/E_p$ (Q_8 for $p=2$)

$E_p \subset SU(p)$ extraspecial of order p^3 and exp p

Remarks

- a key map

$$G \times T^n \longrightarrow \mathrm{Hom}(\mathbb{Z}^n, G)$$

$$(g, t_1, \ldots, t_n) \longmapsto (g t_1 g^{-1}, \ldots, g t_n g^{-1})$$

- issues arise at $p \mid |W(G)|$

- instead of considering $\mathrm{Hom}(\mathbb{Z}^n, G)$ one at a time, let's assemble them into

$$\{\mathrm{Hom}(\mathbb{Z}^n, G)\}_{n \geq 0}$$

III SIMPLICIAL STRUCTURE

Consider

$$E_{n+1}(q,G) = G \times \mathrm{Hom}(F_n/\Gamma^q, G)$$

with

$$d_i(g_0,\dots,g_n) = \begin{cases} (g_0,\dots,g_i g_{i+1},\dots,g_n) & 0 \le i < n \\ (g_0,\dots,g_{n-1}) & i = n \end{cases}$$

$$s_j(g_0,\dots,g_n) = (g_0,\dots,g_j,1,g_{j+1},\dots,g_n)$$

Here $\Gamma^i = \Gamma^i(F_n)$ $\quad F_n$ free group on n generators

$\Gamma^1(Q) = Q, \quad \Gamma^{i+1}(Q) = [\Gamma^i(Q), Q]$

descending central series

Similarly we define

$$B_n(q, G) = \mathrm{Hom}(F_n/\Gamma^q, G)$$

with maps d_i and s_j defined in the same way, except that the first coordinate g_0 is omitted and d_0 takes the form $d_0(g_1, \ldots, g_n) = (g_2, \ldots, g_n)$.

The maps d_i, s_j on the spaces $E_n(q, G)$ and $B_n(q, G)$ are well-defined and equip them with the structure of simplicial spaces.

Here G is a locally compact Hausdorff topolgical group with $1 \in G$ a non degenerate basepoint.

The projection $G^{n+1} \to G^n$ defines a simplicial map

$$p : E_*(q, G) \to B_*(q, G)$$

Moreover, G acts on the first coordinate of $E_n(q, G)$ making $E_*(q, G)$ into a G-simplicial space, with orbit space $B_*(q, G)$.

- the natural surjection $F_n/\Gamma^{q+1} \to F_n/\Gamma^q$ induces a map of simplicial spaces compatible with the projections:

$$\begin{tikzcd}
E_n(q, G) \arrow[r] \arrow[d] & E_n(q+1, G) \arrow[d] \\
B(q, G) \arrow[r] & B_n(q+1, G)
\end{tikzcd}$$

- There are natural morphisms of principal G-bundles

$$\begin{array}{ccccc} |E_*(q,G)| & \longrightarrow & |E_*(q+1,G)| & \rightarrow & EG \\ \downarrow & & \downarrow & & \downarrow \\ |B_*(q,G)| & \longrightarrow & |B_*(q+1,G)| & \rightarrow & BG \end{array}$$

Here $|X_*|$ = geometric realization

<u>Definition</u>:

$$E(q,G) = |E_*(q,G)|$$

$$B(q,G) = |B_*(q,G)| \quad ///$$

Note that $B(2,G)$ is built out of the commuting n-tuples in G.

Properties :

① The $E(q,G)$, $B(q,G)$ are functors on topological groups

② There is a filtration

$$B(2,G) \to B(3,G) \to \cdots \to BG$$

If G is finite $\exists\, N \geq 2$ such that $B(q,G) = B(N,G)$ for all $q \geq N$.

③ If H is nilpotent of class $< q$, then $B(q,H) = BH$ and any $f: H \to G$ leads to a factorization

$$\begin{tikzcd} BH \arrow[rr, "Bf"] \arrow[dr] & & BG \\ & B(q,G) \arrow[ur] & \end{tikzcd}$$

4. The map $B(q,G) \to BG$ can be identified with the classifying map of the principal G-bundle

$$\begin{tikzcd} E(q,G) \arrow[r] & B(q,G) \arrow[d] \\ & BG \end{tikzcd}$$

Note that $E(q,G)$ is not contractible in general:

$$E(2,S_3) \simeq \bigvee_{1}^{8} S^1$$

$$B(2,S_3) = K(\pi,1)$$

$$1 \to F_8 \to \pi \to S_3 \to 1$$

IV Cohomology & Homotopy

We now consider the case when G is a compact, connected Lie group

Theorem :

(1) There is a homotopy equivalence $\forall q \geq 2$

$$G \times \Omega(E(q,G)) \simeq \Omega B(q,G)$$

(2) If $|W(G)|$ is invertible in R

$$H^*(B(2,G),R) \cong H^*(G/T \times BT, R)^{W(G)}$$

where $T \subset G$ maximal torus with Weyl group $W(G)$. ///

For G a finite group we have

- the map
$$H^*(BG, \mathbb{F}_p) \to H^*(B(q,G), \mathbb{F}_p)$$
has nilpotent kernel $\forall\, q \geq 2$

- if G has mod p cohomology detected on subgroups of nilpotence class less than q, then
$$H^*(BG, \mathbb{F}_p) \hookrightarrow H^*(B(q,G), \mathbb{F}_p)$$
is a monomorphism

- the $\overline{H}_i(B(q,G), \mathbb{Z})$ are finite abelian groups with torsion only at $p \mid |G|$ //

Note that $H_*(E(g,G), \mathbb{Z})$ are $\mathbb{Z}G$-modules defined in a natural way. These rep.'s contain important information about the group.

Proposition :

The Feit-Thompson Theorem is equivalent to the following result: for G of odd order, the homomorphism

$$H_1(E(g,G), \mathbb{Z}) \to H_1(B(g,G), \mathbb{Z})$$

Cannot be surjective. ////

For G finite, let

$$N_q(G) = \{ A \subset G \mid \Gamma^q A = \{1\} \}$$

$$G(q) = \operatorname*{colim}_{A \in N_q(G)} A$$

<u>Theorem</u>:

(a) there is a fibration with simply connected finite dim. fiber

$$K_q \longrightarrow B(q,G) \longrightarrow BG(q)$$

(b) $\pi_1(E(q,G))$ is torsion free and $E(q,G) \simeq$ finite complex

///

<u>Question</u> : Are the spaces $B(q, G)$ for G finite aspherical? [a $K(\pi, 1)$]

This can be verified in some cases :

If G is a transitively commutative group, then $B(2, G) \simeq BG(2)$, where $G(2)$ is the amalgamated product of the maximal abelian subgroups of G along the centre $Z(G)$. In particular $B(2, G)$ is a $K(\pi, 1)$.

For G a TC group we have

$$\begin{tikzcd}[column sep=small]
C_G(a_1) \arrow[dr, dash] & C_G(a_2) \arrow[d, dash] & \cdots & C_G(a_k) \arrow[dll, dash] \\
& Z(G) & &
\end{tikzcd}$$

$$B(2,G) \simeq B\Big(\mathop{\ast}_{Z(G)} C_G(a_i) \Big)$$

$$E(2,G) \simeq \bigvee^{N_G} S^1$$

where

$$N_G = 1 - [G:ZG] + \Big(\sum_{1 \le i \le k} [G:ZG] - [G:C_G(a_i)] \Big)$$

- if $Z(G) = \{1\}$ then

$$B(2,G) \underset{st}{\simeq} \bigvee_{p \mid |G|} \bigvee_{P \in Syl_p(G)} BP$$

VI STABLE SPLITTINGS

It turns out that the simplicial spaces $B_n(q, G)$ have natural filtrations which can be useful for understanding spaces of homomorphisms

$S_n(j, q, G) =$ n-tuples in $\mathrm{Hom}(F_n/\Gamma^q, G)$ such that at least j of them are 1

then

$$S_n(j, q, G) \subset S_n(j-1, q, G)$$

and they filter $B_n(q, G)$

Special case $q=2$:

<u>Technical Lemma</u> :

The $(S_n(j-1,2,G), S_n(j,2,G))$ are NDR pairs if G is compact Lie.

$$\Downarrow$$

$S_n(j,2,G) \hookrightarrow S_n(j-1,2,G)$ are cofibrations

" $B_*(2,G)$ is simplicially NDR as well as proper."

Notation: $S_K(G) = S_K(1,2,G)$

Theorem :

For G compact Lie there are homotopy equivalences

$$\Sigma \operatorname{Hom}(\mathbb{Z}^n, G) \simeq \bigvee_{1 \le k \le n} \Sigma \bigvee^{\binom{n}{k}} \operatorname{Hom}(\mathbb{Z}^k, G) / S_k(G)$$

$$\Sigma^k \operatorname{Hom}(\mathbb{Z}^k, G) / S_k(G) \simeq \frac{F_k B(2,G)}{F_{k-1} B(2,G)}$$

These decompositions descend to spaces of representations.
(use equivariant methods)

Some examples:

$$\frac{\mathrm{Hom}(\mathbb{Z}^n, SO(3))}{S_n(SO(3))} \simeq \begin{cases} \mathbb{RP}^3 & n=1 \\ (\mathbb{RP}^2)^{n\lambda_2} \vee \left(\bigvee_{c(n)} (\mathbb{S}^3/Q_{8})\right) & n \geq 2 \end{cases}$$

$$c(n) = \frac{1}{2}(3^{n-1} - 1)$$

$$\frac{\mathrm{Hom}(\mathbb{Z}^n, SU(2))}{S_n(SU(2))} \simeq \begin{cases} \mathbb{S}^3 & n=1 \\ (\mathbb{RP}^3)^{n\lambda_2} / S_n(\mathbb{RP}^3) & n \geq 2 \end{cases}$$

[Crabb]

If G is a compact connected Lie group such that $\mathrm{Rep}(\mathbb{Z}^r, G)$ is connected for $1 \le r \le n$ and $T \subset G$ is a maximal torus with Weyl group W, then

$$\mathrm{Rep}(\mathbb{Z}^n, G) \cong T^n / W$$

$$\Sigma \mathrm{Rep}(\mathbb{Z}^n, G) \simeq \bigvee_{1 \le r \le n} \Sigma \left(\bigvee^{\binom{n}{r}} T^{\wedge r} / W \right)$$

$$\mathrm{Rep}(\mathbb{Z}^n, U(m)) \cong SP^m((S^1)^n)$$

$$\mathrm{Rep}(\mathbb{Z}^n, U) \simeq \prod_{1 \le r \le n} K(\mathbb{Z}^{\binom{n}{r}}, r)$$

$$\mathrm{Rep}(\mathbb{Z}^n, Sp(m)) \cong SP^m\left((S^1)^n/\mathbb{Z}/2\right)$$

Complex conjugation

$$\mathrm{Rep}(\mathbb{Z}^n, Sp) \simeq \prod_{1 \le i \le [n/2]} K\left(\mathbb{Z}^{\binom{n}{2i}} \oplus (\mathbb{Z}/2)^{r_{2i}}, 2i\right)$$

$$r_j = \begin{cases} \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n-j-1} & \text{if } 1 \le j \le n \\ 0 & \text{if } i > n \end{cases}$$

Questions:

- are the $(S_n(j-1, q, G), S_n(j, q, G))$ NDR pairs for $q > 2$?

- what is the geometry of $\mathrm{Hom}(\mathbb{Z}^k, G)/S_k(G)$?

- extend calculations done by Borel-Friedman-Morgan for $n = 2, 3$ to larger values (use almost commuting elements).